Stable Distributions

Lecture Notes

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Ulm
2018

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Foreword

These lecture notes are based on the undergraduate course “Stable distributions” which originally took place at Ulm University during the summer term 2016.

In modern applications, there is a need to model phenomena that attain very high numerical values occuring rarely. In probability theory, one talks about distributions with heavy tails. One class of such distributions are stable laws which (apart from the Gaussian one) do not have a finite variance. So, the aim of this course is to give an introduction into the theory of stable distributions, its basic facts and properties.

The choice of material of the course is selective and is mainly dictated by its introductory nature and limited lecture scope. The main topics of these lecture notes are
1) Stability with respect to convolution
2) Characteristic functions and densities
3) Non-Gaussian limit theorem for i.i.d. random summands
4) Representations and tail properties, symmetry and skewness
5) Simulation.

For each topic, several exercises are included for deeper understanding of the subject. Since the target audience are undergraduate students of mathematics, no prerequisites other than basic probability course are assumed.

The author hopes you find these notes helpful. If you notice an error, please do not hesitate to contact him at evgeny.spodarev@uni-ulm.de.

The author is also grateful to Dr. Vitalii Makogin for typesetting this lectures in LaTeX, making illustrations, and the selection of exercises.

02.11.2016 Prof. Dr. Evgeny Spodarev

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space. The property of stability of random variables with respect to (w.r.t.) convolution is known for you from the basic course of probability. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. One can rewrite this property as follows. Let $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} X \sim N(0,1)$ and X, X_1, X_2 be independent. Then $\forall a, b \in \mathbb{R}$ $aX_1 + bX_2 \sim N(0, a^2 + b^2)$, and so

$$aX_1 + bX_2 \stackrel{d}{=} \underbrace{\sqrt{a^2 + b^2}}_{c \geq 0} X. \tag{1.0.1}$$

Additionally, for any random variables $X_1, \dots, X_n$ i.i.d., $X_i \stackrel{d}{=} X, \forall i = 1, \dots, n$, it holds $\sum_{i=1}^n X_i \stackrel{d}{=} \sqrt{n} X$. Property (1.0.1) rewrites in terms of cumulative distribution functions of X_1, X_2, X as $\Phi\left(\frac{x}{a}\right) \star \Phi\left(\frac{x}{b}\right) = \Phi\left(\frac{x}{c}\right), x \in \mathbb{R}$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, x \in \mathbb{R}$, and $\star$ is the convolution operation.

It turns out that the normal law is not unique satisfying (1.0.1). Hence, it motivates the following definition.

Definition 1.0.1
A random variable X is *stable* if $\forall a, b \in \mathbb{R}_+ \exists c, d \in \mathbb{R}, c > 0$ s.t.

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{1.0.2}$$

where X_1, X_2 are independent copies of X. X as above is called *strictly stable* if $d = 0$.

Remark 1.0.1
Let F_X be the cumulative distribution function (c.d.f.) of X, i.e., $F_X(y) = \mathbb{P}(X \leq y), y \in \mathbb{R}$. Then the property (1.0.2) rewrites as $F_X\left(\frac{y}{a}\right) \star F_X\left(\frac{y}{b}\right) = F_X\left(\frac{y-d}{c}\right), y \in \mathbb{R}$, if $a, b, c \neq 0$. The case $c = 0$ corresponds to $X \equiv$ const a.s., which is a degenerate case. Obviously, a constant random variable is always stable. The property (1.0.1) shows that $X \sim N(0,1)$ is strictly stable.

Exercise 1.0.1
Show that $X \sim N(\mu, \sigma^2)$ is stable for any $\mu \in \mathbb{R}, \sigma^2 > 0$. Find the parameters c and d in (1.0.2) for it. Prove that $X \sim N(\mu, \sigma^2)$ is strictly stable if and only if (iff) $\mu = 0$.

The notion of *(strict) stability* was first introduced by Paul Lévy in his book *Calcul des probabilités (1925)*. However, stable distributions (different from the Gaussian one) were known long before. Thus, French mathematicians Poisson and Cauchy some 150 years before Lévy found the distribution with density

$$f_\lambda(x) = \frac{\lambda}{\pi(x^2 + \lambda^2)}, \quad x \in \mathbb{R}, \tag{1.0.3}$$

depending on parameter $\lambda > 0$. Now this distribution bears the name of Cauchy, and it is known to be strictly stable. Its characteristic function $\varphi_\lambda(t) = \int_{\mathbb{R}} e^{itx} f_\lambda(x) dx$, $t \in \mathbb{R}$ has the form $\varphi_\lambda(t) = e^{-\lambda|t|}$.

In 1919 the Danish astronomer J. Holtsmark found a law of random fluctuation of the gravitational field of some stars in space, which had characteristic function $\varphi(t) = e^{-\lambda \|t\|^{3/2}}, t \in \mathbb{R}^3$, leading to the family of characteristic functions

$$\varphi(t) = e^{-\lambda |t|^{\alpha}}, \quad t \in \mathbb{R}, \quad \lambda > 0. \tag{1.0.4}$$

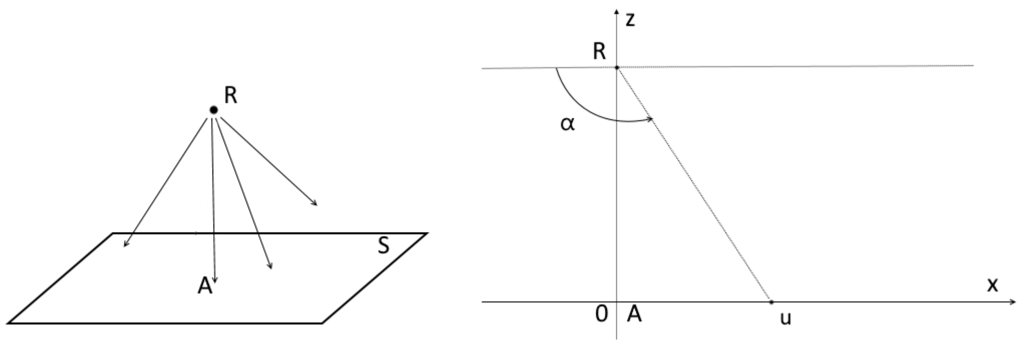
For $\alpha = 3/2$, it appeared to be strictly stable and now bears the name of Holtsmark. It needed some time till it was proven by P. Lévy in 1927 that (1.0.4) is a valid characteristic function of some (strictly stable) distribution only for $\alpha \in (0, 2]$. The theory of stable random variables took its modern form after 1938 when the books by P. Lévy and A. Khinchin were published.

Let us give further examples of stable laws and their applications.

Example 1.0.1 (Constants):
Any constant c is evidently a stable random variable.

Example 1.0.2 (Cauchy distribution in nuclear physics):
Let a point source of radiation $\mathbf{R}$ be located at $(0, 0, 1)$ and radiate its elementary particles onto a screen $S = \{(x, y, 0), x, y \in \mathbb{R}\}$. The screen S is covered by a thin layer of metal so that it yields light flashes as the emitted particles reach it. Let $(U, V, 0)$ be the coordinates of one

Figure 1.1:



of these (random) flashes. Due to the symmetry of this picture (the whole process of radiation is rotationally symmetric around axis RA cf. Fig. 1.1) it is sufficient to find the distribution of one coordinate of (u, v), say, $U \stackrel{d}{=} V$. Project the whole picture onto the plane (x, z). Let $F_U(x) = \mathbb{P}(U \leq x)$ be the c.d.f. of U. The angle α to the ray RU varies in $(0, \pi)$ if it arrives at S. It is logic to assume that $\alpha \sim U[0, \pi]$. Since tg $(\alpha - \frac{\pi}{2}) = \frac{U}{1} = U$, it follows $\alpha = \pi/2 + \arctan U$. Then for any $x > 0$ $\{U \leq x\} = \{\text{tg}(\alpha - \pi/2) \leq x\} = \{\alpha \leq \pi/2 + \arctan x\}$. So,

$$\begin{aligned} F_U(x) = \mathbb{P}(\alpha \leq \pi/2 + \arctan x) &= \frac{\pi/2 + \arctan x}{\pi} = \frac{1}{2} + \frac{1}{\pi} \arctan x \\ &= \int_{-\infty}^{x} \frac{1}{\pi} \frac{dy}{1 + y^2} = \int_{-\infty}^{x} f_1(y) dy, \end{aligned}$$

with $f_1(\cdot)$ as in (1.0.3), $\lambda = 1$. So, $U \sim Cauchy(0, 1)$. For instance, it describes the distribution of energy of unstable states in nuclear reactions (Lorenz law).

Example 1.0.3 (Theory of random matrices):
Let $A_n Y_n = B_n$ be a random system of n linear equations, where $A_n = \left(X_{ij}^{(n)}\right)_{i,j=1}^{n}$ be a random

$(n \times n)-$matrix, and $B_n = \left(B_i^{(n)}\right)_{i=1}^n$ be a random $n-\dim$ vector in $\mathbb{R}^n$. If $\det(A_n) \neq 0$, its solution is $Y_n = A_n^{-1} B_n$ (for $\det(A_n) = 0$, put $Y_n = 0$ a.s.). As $n \to \infty$, the solution Y_n is numerically very hard to compute. Then the following approximation (as $n \to \infty$) is helpful. Assume that for each $n \in \mathbb{N}$ A_n and B_n are mutually independent, $\mathbb{E}X_{ij}^{(n)} = \mathbb{E}B_i^{(n)} = 0$, $\text{Var}X_{ij}^{(n)} = \text{Var}B_i^{(n)} = 1$ $\forall i,j = 1,\dots,n$. If $\sup_{n,i,j} \mathbb{E}\left(|X_{ij}^{(n)}|^5 + |B_i^{(n)}|^5\right) < \infty$ then for any $1 \le i,j \le n$, $i \neq j$ $\lim_{n\to\infty} \mathbb{P}(Y_i^{(n)} \le x) = \frac{1}{2} + \frac{1}{\pi}\arctan x, x > 0$, where $Y_n = (Y_i^{(n)})_{i=1\dots n}$. Hence, here again, $Y_i^{(n)} \sim \mathit{Cauchy}(0,1), i = 1\dots n$, compare Exercise 1.0.1

Exercise 1.0.2
Show that if $X \in \mathit{Cauchy}(0,1)$ then $X \stackrel{d}{=} \frac{Y_1}{Y_2}$, where Y_1, Y_2 are i.i.d. $N(0,1)-$distributed random variables.

Exercise 1.0.3
1) Prove that Cauchy distribution is stable. If it is centered, i.e., $X \sim \mathit{Cauchy}(0,\lambda)$, then it is strictly stable.
2) Show that if $X \sim \mathit{Cauchy}(0,\lambda)$ then $X \stackrel{d}{=} \frac{1}{X}$.

In fact, it can be shown that for $X \sim \mathit{Cauchy}(0,1)$, $X_1,\dots,X_n$ i.i.d. and $X_i \stackrel{d}{=} X$, $\sum_{i=1}^n X_i \stackrel{d}{=} nX$, i.e., the constant c in (1.0.2) is equal to $a+b$ here. The property $\sum_{i=1}^n X_i \stackrel{d}{=} nX$ rewrites $\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i \stackrel{d}{=} X$, i.e., the arithmetic mean of X_i is distributed exactly as one of X_i.

Example 1.0.4 (Lévy distribution in branching processes):
Consider the following branching process in discrete time. A population of particles evolves in time as follows: at each time step, each particle (independently from others) dies with probability $p > 0$, doubles (i.e., is divided into two new similar particles) with probability $p > 0$, or simply stays untouched (with complimentary probability $1-2p$). Let $G(s) = p + (1-2p)s + ps^2, |s| \le 1$ be the generating function describing this evolution in one step. Let $\nu_0(k)$ be the number of particles in generation $k-1$, which died in k-th step. Let $\nu = \sum_{k=1}^\infty \nu_0(k)$ be the total number of died particles during the whole evolution of the process. Assuming that there is only one particle at time $k=0$, put $\nu_0(0) = 0$, and denote $q_n = \mathbb{P}(\nu = n), n \in \mathbb{N}_0$. Let

$$\varphi(s) = \sum_{n=0}^\infty q_n s^n, |s| < 1 \tag{1.0.5}$$

be the generating probability function of ν.

Exercise 1.0.4
Show that $\varphi(s) = G(\varphi(s)) + p(s-1), |s| < 1$.

From this evolution, it follows $\varphi(s) = p + (1-2p)\varphi(s) + p\varphi^2(s) + p(s-1)$, or $\varphi^2(s) - 2\varphi(s) + s = 0 \Longrightarrow \varphi(s) - 1 = \pm\sqrt{1-s}, |s| < 1$. Since $|\varphi(s)| \le 1$ $\forall s : |s| < 1$, then $\varphi(s) = 1 - \sqrt{1-s} > 1$ is not a solution $\Longrightarrow \varphi(s) = 1 - \sqrt{1-s}, |s| < 1$. Expanding it in the Taylor series, we get

$$\varphi(s) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^\infty \frac{\Gamma(n-1/2)}{n!} s^n, |s| < 1, \tag{1.0.6}$$

which follows from $\varphi(0) = 0, \varphi'(0) = \frac{1}{2\sqrt{1-0}} = \frac{1}{2}$, $\varphi''(0) = \frac{3}{2}$, and so on: $\varphi^{(n)}(0) = \frac{\Gamma(n-1/2)}{2\Gamma(1/2)}, n \in \mathbb{N}$.

Exercise 1.0.5
Prove it inductively.

Recall the Stirling's formula for Gamma function: $\forall x > 0\ \Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x e^{\mu(x)}$, where $0 < \mu < \frac{1}{12x}$. Comparing the form (1.0.5) and (1.0.6), we get

$$
\begin{aligned}
q_n &= \frac{\Gamma(n-1/2)}{2\sqrt{\pi}\Gamma(n+1)} = \frac{\sqrt{\frac{2\pi}{n-1/2}} \left(\frac{n-1/2}{e}\right)^{n-1/2} e^{\mu(n-1/2)}}{2\sqrt{\pi}\sqrt{\frac{2\pi}{n}} n \left(\frac{n}{e}\right)^n e^{\mu(n)}} \\
&= \frac{1}{2\sqrt{\pi}} \left(\frac{n-1/2}{n}\right)^{n-1} \frac{\sqrt{e}}{n^{3/2}} e^{\mu(n-1/2)-\mu(n)} \sim \frac{1}{2\sqrt{\pi}} \left(1 - \frac{1}{2n}\right)^{n-1} \frac{1}{n^{3/2}} e^{1/2+\mu(n-1/2)-\mu(n)} \\
&\sim \frac{n^{-3/2}}{2\sqrt{\pi}} \exp\left(\frac{1}{2} + (n-1)\log\left(1 - \frac{1}{2n}\right) + o(1)\right) \\
&\sim \frac{n^{-3/2}}{2\sqrt{\pi}} \exp\left(\frac{1}{2} + (n-1)\left(-\frac{1}{2n}\right) + o(1)\right) \sim \frac{n^{-3/2}}{2\sqrt{\pi}}, n \to \infty.
\end{aligned}
$$

Summarizing, $q_n \sim \frac{n^{-3/2}}{2\sqrt{\pi}}, n \to \infty$.

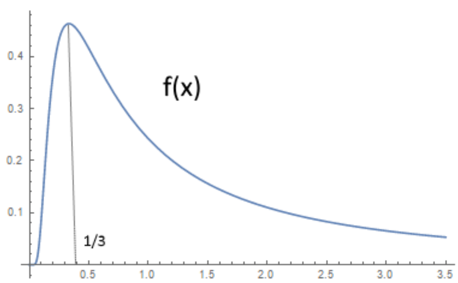
Now assume that the whole process starts with n particles at the initial moment of time $t = 0$. Then, the total number of died particles is a sum $\sum_{i=1}^n \nu_i$ of i.i.d. r.v.'s $\nu_i \overset{d}{=} \nu$. It is possible to show $\frac{1}{n^2}\sum_{i=1}^n \nu_i \overset{d}{\to} X, n \to \infty$, where X is a standard Lévy distributed random variable with density

$$
f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{1}{2x}\right), x > 0. \tag{1.0.7}
$$

Exercise 1.0.6
Let X be as above. Then

1. $X \overset{d}{=} Y^{-2}$, where $Y \sim N(0,1)$.
2. $f_X(s) \sim \frac{1}{\sqrt{2\pi}} x^{-3/2}, x \to +\infty$.
3. $\mathbb{E}X = \mathrm{Var}X = \infty$.
4. The standard Lévy distribution is strictly stable, i.e., for independent $X_1 \overset{d}{=} X_2 \overset{d}{=} X$: $X_1 + X_2 \overset{d}{=} 4X$.

Figure 1.2: Graph of f_X



The graph of $f_X(\cdot)$ looks like it has its mode at $x = 1/3$, and $f(0) = 0$ by continuity, since $\lim_{x\to+0} f(x) = 0$. Relation from Exercise 1.0.6(4) can be interpreted as $\frac{X_1+X_2}{2} \stackrel{d}{=} 2X$, the arithmetic mean of $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} X$ is distributed as $2X$. Compare it with the same property of Cauchy distribution.

2 Properties of stable laws

2.1 Equivalent definitions of stability

Now we would like to give a number of further definitions of stability which appear to be equivalent. At the same time, they give important properties of stable laws.

Definition 2.1.1
A random variable X is *stable* if there exists a family of i.i.d. r.v.'s $\{X_i\}_{i=1}^{\infty}$ and number sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$, $b_n > 0\ \forall n \in \mathbb{N}$ s.t.

$$\frac{1}{b_n}\sum_{i=1}^{n} X_i - a_n \xrightarrow{d} X, n \to \infty. \tag{2.1.1}$$

Remark 2.1.1
Notice that this definition does not require the r.v. X_1 to have a finite variance or even a finite mean. But if $\sigma^2 = \text{Var}X_1 \in (0, +\infty)$ then $X \sim N(0,1)$ according to the central limit theorem with $b_n = \sqrt{n}\sigma, a_n = \frac{n\mu}{\sqrt{n}\sigma} = \sqrt{n}\frac{\mu}{\sigma}$, where $\mu = \mathbb{E}X_1$.

Definition 2.1.2
A non-constant random variable X is *stable* if its characteristic function has the form $\varphi_X(s) = e^{\eta(s)}, s \in \mathbb{R}$, where $\eta(s) = \lambda(is\gamma - |s|^\alpha + is\omega(s,\alpha,\beta)), s \in \mathbb{R}$ with

$$\omega(s,\alpha,\beta) = \begin{cases} |s|^{\alpha-1}\beta \text{tg}\left(\frac{\pi}{2}\alpha\right), & \alpha \neq 1, \\ -\beta\frac{2}{\pi}\log|s|, & \alpha = 1, \end{cases} \tag{2.1.2}$$

$\alpha \in (0,2], \beta \in [-1,1], \gamma \in \mathbb{R}, \lambda > 0$. Here α is called *stability index*, β is the *coefficient of skewness*, λ is the *scale parameter*, and $\mu = \lambda\gamma$ is the *shift parameter*.

We denote the class of all stable distributions with given above parameters $(\alpha,\beta,\lambda,\gamma)$ by $S_\alpha(\lambda,\beta,\gamma)$. Sometimes, the shift parameter μ is used instead of $\gamma : S_\alpha(\lambda,\beta,\mu)$. $X \in S_\alpha(\lambda,\beta,\gamma)$ means that X is a stable r.v. with parameters $(\alpha,\beta,\lambda,\gamma)$.

Unfortunately, the parametrisation of $\eta(s)$ in Definition 2.1.2 is not a continuous function of parameters $(\alpha,\beta,\lambda,\gamma)$. It can be easily seen that $\omega(s,\alpha,\beta) \to \infty$ as $\alpha \to 1$ for any $\beta \neq 0$, instead of tending to $-\beta\frac{\pi}{2}\log|s|$. To remedy this, we can introduce an additive shift $+\lambda\beta\text{tg}\left(\frac{\pi}{2}\alpha\right)$ to get $\eta(s) = \lambda(is\gamma_M - |s|^\alpha + is\omega_M(s,\alpha,\beta)), s \in \mathbb{R}$, where

$$\gamma_M = \begin{cases} \gamma + \beta\text{tg}\left(\frac{\pi}{2}\alpha\right), & \alpha \neq 1 \\ \gamma, & \alpha = 1 \end{cases}, \quad \omega_M(s,\alpha,\beta) = \begin{cases} (|s|^{\alpha-1} - 1)\beta\text{tg}\left(\frac{\pi}{2}\alpha\right), & \alpha \neq 1, \\ -\beta\frac{2}{\pi}\log|s|, & \alpha = 1. \end{cases} \tag{2.1.3}$$

(M stands for "modified")

Exercise 2.1.1
Check that this modified parametrisation is a continuous function of all parameters.

Another possibility to parametrise $\eta(s)$ is given as follows:
$\eta(s) = \lambda_B(is\gamma_B - |s|^\alpha + is\omega_B(s,\alpha,\beta_B)), s \in \mathbb{R}$, where

$$\omega_B(s,\alpha,\beta_B) = \begin{cases} \exp\left(-i\frac{\pi}{2}\beta_B K(\alpha)\mathrm{sign}(s)\right), & \alpha \neq 1, \\ \frac{\pi}{2} + i\beta_B \log|s|\,\mathrm{sign}(s), & \alpha = 1, \end{cases} \quad K(\alpha) = \alpha - 1 + \mathrm{sign}(1-\alpha),$$

$$\text{and for } \alpha \neq 1 : \lambda = \lambda_B \cos\left(\frac{\pi}{2}\beta_B K(\alpha)\right), \gamma = \gamma_B \Big/ \cos\left(\frac{\pi}{2}\beta_B K(\alpha)\right),$$

$$\beta = \mathrm{ctg}\left(\frac{\pi}{2}\alpha\right)\mathrm{tg}\left(\frac{\pi}{2}\beta_B K(\alpha)\right);$$

$$\text{for } \alpha = 1 : \lambda = \frac{\pi}{2}\lambda_B, \gamma = \frac{2}{\pi}\gamma_B, \beta = \beta_B.$$

(B stays for "bounded" representation). In this form $\eta(s)$ is again not continuous at $\alpha = 1$, but for $\alpha \to 1, \alpha \neq 1$ the whole function $\eta(s)$ does not go to $+\infty$ as in (2.1.2), but has a limiting finite form which corresponds to a characteristic function of a stable law with $\eta(s) = \lambda_B\left(is\left(\gamma_B \pm \sin\left(\frac{\pi}{2}\beta_B\right)\right) - |s|\cos\left(\frac{\pi}{2}\beta_B\right)\right)$. Here, the "+"sign is chosen for $\alpha \to 1+0$, and "−" for $\alpha \to 1-0$.

Exercise 2.1.2
Show this convergence for $\alpha \to 1 \pm 0$.

Let us give two more definitions of stability.

Definition 2.1.3
A random variable X is *stable* if for the sequence of i.i.d. r.v.'s $\{X_i\}_{i\in\mathbb{N}}$, $X_i \stackrel{d}{=} X, \forall i \in \mathbb{N}$, for any $n \geq 2$ $\exists c_n > 0$ and $d_n \in \mathbb{R}$ s.t.

$$\sum_{i=1}^n X_i \stackrel{d}{=} c_n X + d_n. \tag{2.1.4}$$

It turns out that this definition can be weakened.

Definition 2.1.4
It is sufficient for stability of X to require (2.1.4) to hold only for $n = 2, 3$.

Now let us formulate the equivalence statement.

Theorem 2.1.1
Definitions 1.0.1,2.1.1-2.1.4 are all equivalent for a non-degenerate random variable X (i.e., $X \not\equiv \mathrm{const}$).

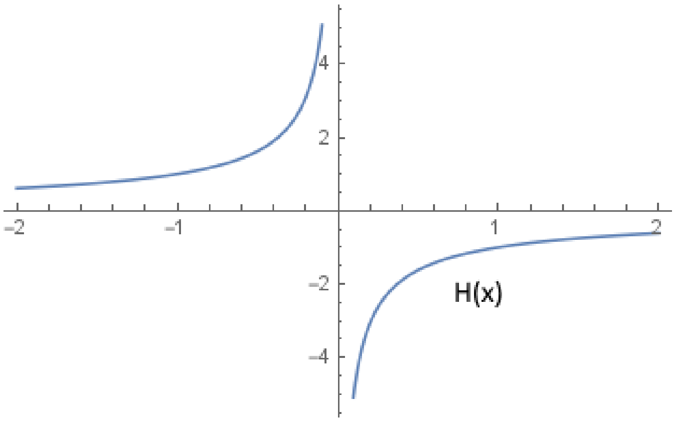
The proof of this result will require a number of auxiliary statements which are now to be formulated. The first of them is a limit theorem describing domains of attraction of infinitely divisible laws.

Theorem 2.1.2 (Khinchin):
Let $\{X_{n_j}, j = 1\ldots k_n, n \in \mathbb{N}\}$ be the sequence of series of independent random variables with the property

$$\lim_{n\to\infty} \max_{j=1\ldots k_n} \mathbb{P}(|X_{n_j}| > \varepsilon) = 0, \forall\varepsilon > 0 \tag{2.1.5}$$

and with c.d.f. F_{n_j}. Let $S_n = \sum_{j=1}^{k_n} X_{n_j} - a_n, n \in \mathbb{N}$ for some sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$. Then a random variable X with c.d.f. F_X is a weak limit of S_n ($S_n \stackrel{d}{\to} X, n \to \infty$) iff the characteristic

Figure 2.1: Graph of H



function φ_X of X has the form

$$\varphi_X(s) = \exp\left(isa - bs^2 + \int_{\{x\neq 0\}} \left(e^{isx} - 1 - is\sin x\right) dH(x)\right), s \in \mathbb{R}, \tag{2.1.6}$$

where $a \in \mathbb{R}, b \geq 0$, $H : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is non-decreasing on $\mathbb{R}_+$ and $\mathbb{R}_-$, $H(s) \to 0$, as $|x| \to +\infty$, and $\int_{0<|x|<1} x^2 dH(s) < \infty$.

This theorem will be given without proof.

Remark 2.1.2

1. The condition (2.1.5) is called the *asymptotic smallness condition* of X_{n_j}.
2. Representation (2.1.6) is called the *canonic representation of Lévy-Kninchin.*
3. Laws of X with ch.f. φ_X as in (2.1.6) are called *infinitely divisible*. For more properties of those, see lectures "Stochastics II".
4. The function H is called a *spectral function* of X.

Exercise 2.1.3
Show that CLT is a special case of Theorem (2.1.2): find X_{n_j} and a_n.

Another important result was obtained by B.V. Gnedenko.

Theorem 2.1.3 (Gnedenko):
Consider $A_n(y) = \sum_{j=1}^{k_n} \mathbb{E}(X_{n_j}\mathbb{I}(|X_{n_j}| < y)), n \in \mathbb{N}$, where $y \in \mathbb{R}$ is a number s.t. y and $-y$ are continuity points of H in (2.1.6). Introduce $\sigma_n^\varepsilon = \sum_{j=1}^{k_n} \operatorname{Var}(X_{n_j}\mathbb{I}(|X_{n_j}| < \varepsilon))$, $\varepsilon > 0$. Let F_X be a c.d.f. with ch.f. φ_X as in (2.1.6). Take

$$a_n = A_n(y) - a - \int_{|u|<y} u dH(u) + \int_{|u|\geq y} \frac{1}{u} dH(u), \quad n \in \mathbb{N}.$$

Then, $S_n \xrightarrow{d} X, n \to \infty$ (or $F_n \to F, n \to \infty$ weakly) iff

1) For each point x of continuity of H it holds

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \left(F_{n_j}(x) - \frac{1}{2}(1 + \operatorname{sign}(x))\right) = H(x).$$

2) $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sigma_n^\varepsilon = \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\sigma_n^\varepsilon = 2b$.

Without proof.

Remark 2.1.3 1. In order $S_n = \frac{1}{b_n}\sum_{i=1}^n X_i - a_n$ from Definition 2.1.1 to fulfill condition (2.1.5), it is sufficient to require $b_n \to \infty, n \to \infty$. Indeed, in this case $X_{n_j} = X_j/b_n$, and, since X_j are i.i.d., $\lim_{\to\infty}\max_{j=1\dots k_n}\mathbb{P}(|X_{n_j}| > \varepsilon) = \lim_{n\to\infty}\mathbb{P}(|X_1| > \varepsilon b_n) = 0$ if $b_n \to \infty$.

2. In the above setting, property (2.1.5) holds whenever $F_n \to F_X$ weakly, where F_X is non-degenerate, i.e., $X \not\equiv$ const a.s. Indeed, let (2.1.5) does not hold, i.e.,

$$\lim_{\to\infty}\max_{j=1\dots k_n}\mathbb{P}(|X_{n_j}| > \varepsilon) \neq 0$$

for some $\varepsilon > 0$. Then $\exists$ a subsequence $n_k \to \infty$ as $n \to \infty$ s.t. $b_{n_k} = O(1)$. Since, $S_n \xrightarrow{d} X, n \to \infty$, s.t. $\varphi_{S_{n_k}}(s) \to \varphi_X(s), k \to \infty$, where

$$\varphi_{S_{n_k}}(s) = \mathbb{E}e^{is\sum_{j=1}^{n_k} X_j/b_{n_k} - isa_{n_k}} = e^{-isa_{n_k}}\left(\varphi_{X_1}\left(\frac{s}{b_{n_k}}\right)\right)^{n_k}, s \in \mathbb{R},$$

so, $|\varphi_X(s)| = \left|\varphi_{X_1}\left(\frac{s}{b_{n_k}}\right)\right|^{n_k}(1+o(1)), k \to \infty$. Then for each $s \in B_\delta(0)$ $|\varphi_{X_1}(s)| = |\varphi_X(sb_{n_k})|^{1/n_k}(1+o(1)) \to 1, k \to \infty$ for some $\delta > 0$, which can be only if $|\varphi_{X_1}(s)| \equiv 1, \forall s \in \mathbb{R}$, and hence $|\varphi_X(s)| \equiv 1$, which means $X \equiv$ const a.s. This contradicts with our assumption $X \not\equiv$ const.

Definition 2.1.5 1) A function $L : (0,+\infty) \to (0,+\infty)$ is called *slowly varying at infinity* if for any $x > 0$

$$\frac{L(tx)}{L(t)} \to 1, t \to +\infty.$$

2) A function $U : (0,+\infty) \to (0,+\infty)$ is called *regularly varying at infinity* if $U(x) = x^\rho L(x), \forall x > 0$, for some $\rho \in \mathbb{R}$ and some slowly varying (at infinity) function L.

Example 2.1.1 1. $L(x) = |\log(x)|^p, x > 0$ is slowly varying for each $p \in \mathbb{R}$.

2. If $\lim_{x\to+\infty} L(x) = p > 0$ then L is slowly varying.

3. $U(x) = (1+x^2)^p, x > 0$ is regularly varying for each $p \in \mathbb{R}$ with $\rho = 2p$.

Lemma 2.1.1
A monotone function $U : (0,+\infty) \to (0,+\infty)$ is regularly varying at ∞ iff $\frac{U(tx)}{U(t)} \to \psi(x), t \to +\infty$ on a dense subset A of $(0,+\infty)$, and $\psi(x) \in (0,+\infty)$ for all $x \in \mathbb{R}_+$.

Proof Let $x_1, x_2 \in A$. For $t \to +\infty$ we get

$$\psi(x_1x_2) \leftarrow \frac{U(tx_1x_2)}{U(t)} = \frac{U(tx_1x_2)}{U(tx_2)}\frac{U(tx_2)}{U(t)} \to \psi(x_1)\psi(x_2).$$

Hence, $\psi(x_1x_2) = \psi(x_1)\psi(x_2)$. Since U is monotone, so is ψ. By monotonicity, define ψ anywhere on $\mathbb{R}_+$ by continuity from the right. Then $\psi(x_1x_2) = \psi(x_1)\psi(x_2)$ holds for any $x_1, x_2 \in \mathbb{R}_+$. Set $x = e^y, \psi(e^y) = \varphi(y)$. The above equation transforms to $\varphi(y_1+y_2) = \varphi(y_1)\varphi(y_2)$. One can easily show that if has a unique (up to a constant ρ) solution bounded on any finite interval, and it is $\varphi(y) = e^{\rho y} \Leftrightarrow \psi(x) = x^\rho$. □

The proof of Theorem 2.1.1 will make use of the following important statement which is interesting on its own right.

Theorem 2.1.4
Let X be a stable r.v. in the sense of Definition 2.1.1 with characteristic function φ_X as in (2.1.6). Then its spectral function H has the form

$$H(x) = \begin{cases} -c_1 x^{-\alpha}, & x > 0 \\ c_2(-x)^{-\alpha}, & x < 0, \end{cases} \quad \text{where } \alpha \in (0,2), c_1, c_2 \geq 0.$$

Proof Consider the non-trivial case of a non-degenerate distribution of X (otherwise $c_1 = c_2 = 0$). Denote by $\mathcal{X}_H$ the set of all continuity points of the spectral function H.

Exercise 2.1.4
Prove that $\mathbb{R} \setminus \mathcal{X}_H$ is at most countable.

Since X is stable in the sense of Definition 2.1.1, $\exists$ an i.i.d. sequence of r.v.'s $\{X_i\}_{i\in\mathbb{N}}$ and number sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $b_n > 0$ $\forall n \in \mathbb{N}$ s.t. $S_n = \frac{1}{b_n}\sum_{i=1}^n X_i - a_n \xrightarrow{d} X, n \to \infty$. Using Theorem 2.1.3, condition 1), it means that $\forall x \in \mathcal{X}_H$ $n(F(b_n x) - \frac{1}{2}(1 + \text{sign}x)) \to H(x), n \to \infty$, where $F(y) = \mathbb{P}(X_i \leq y), y \in \mathbb{R}$.

Consider the case $x > 0$. If $H(x) \not\equiv 0$ on $\mathbb{R}_+$, so $\exists x_0 \in \mathcal{X}_H, x_0 > 0$ with $q := -H(x_0) > 0$, compare Fig. 2.1 For each $t > 0$, find an $n = n(t) \in \mathbb{N}$ s.t. $n(t) = \min\{k : b_k x_0 \leq t < b_{k+1}x_0\}$. Since $\overline{F}(x) = 1 - F(x) \downarrow$ on $\mathbb{R}_+$, we get

$$\frac{\overline{F}(b_{n+1}x_0 x)}{\overline{F}(b_n x_0)} \leq \frac{\overline{F}(tx)}{\overline{F}(t)} \leq \frac{\overline{F}(b_n x_0 x)}{\overline{F}(b_{n+1}x_0)}, \quad \forall x > 0. \tag{2.1.7}$$

Since $n(t) \to \infty, t \to \infty$, $-n\overline{F}(b_n x) = n(F(b_n x) - 1) \to H(x), x \to \infty$, we get for $x_0 x \in \mathcal{X}_H$

$$\frac{\overline{F}(b_{n+1}x_0 x)}{\overline{F}(b_n x_0)} = \frac{-n\overline{F}(b_{n+1}x_0 x)}{-n\overline{F}(b_n x_0)} \to \frac{H(x_0 x)}{H(x_0)} = -\frac{H(x_0 x)}{q} := L(x).$$

The same holds for the right-hand side of (2.1.7). Hence, for any $x, y > 0$ s.t. $x_0 x, x_0 y, x_0 xy \in \mathcal{X}_H$ we have $\frac{\overline{F}(txy)}{\overline{F}(t)} \to L(xy), \to +\infty$. Otherwise,

$$\frac{\overline{F}(txy)}{\overline{F}(t)} = \frac{\overline{F}(txy)}{\overline{F}(ty)}\frac{\overline{F}(ty)}{\overline{F}(t)} \to L(x)L(y), t \to \infty$$

by the same reasoning. As a result, we get the separation $L(xy) = L(x)L(y)$ which holds for all $x, y > 0$. (may be except for a countable number of points since $\mathcal{X}_H$ is at most countable.)

By definition of $L(x) := -\frac{H(x_0 x)}{q}$, $L : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, $L(1) = 1, L(\infty) = 0$. It can be shown (cf. the proof of Lemma 2.1.1) that the solution of the equation

$$\begin{cases} L(xy) = L(x)L(y), \\ L(1) = 1, L(\infty) = 0 \end{cases}$$

is $L(x) = 1/x^\alpha, \alpha > 0$. Hence, for $x > 0$ $H(x) = -qL(x/x_0) = H(x_0)x^{-\alpha}/x_0^{-\alpha} = x_0^\alpha H(x_0)x^{-\alpha} = -c_1 x^{-\alpha}, c_1 \geq 0$. Since $\int_{0<|x|<1} x^2 dH(x) < \infty$ (cf. Theorem 2.1.2), it holds $\int_{0<|x|<1} x^{2-\alpha-1}dx < \infty \iff 2 - \alpha > 0 \iff \alpha < 2$. Hence, $0 < \alpha < 2$, $c_1 \geq 0$ can be arbitrary.

The case $x < 0$ is treated analogously and leads to the representation $H(x) = c_2(-x)^{-\delta}, c_2 \geq 0, 0 < \delta < 2$.

Show that $\alpha = \delta$. Since $\frac{\overline{F}(tx)}{\overline{F}(t)} \sim x^{-\alpha}, t \to \infty$ for $x > 0$, it means that $\overline{F}(s)$ is regularly varying by Lemma 2.1.1. Hence, exists a slowly varying function $h_1 : (0, +\infty) \to (0, +\infty)$ s.t. $\overline{F}(x) = x^{-\alpha} h_1(x), x > 0$. By property 1) of Theorem 2.1.3, $n\overline{F}(b_n x) = nb_n^{-\alpha} x^{-\alpha} h_1(b_n x) \to H(x) = -c_1 x^{-\alpha}, n \to \infty$. Since $\frac{h_1(b_n x)}{h_1(b_n)} \to 1, n \to \infty$, it holds

$$c_1 \leftarrow nb_n^{-\alpha} h_1(b_n x) = nb_n^{-\alpha} h_1(b_n) \frac{h_1(b_n x)}{h_1(b_n)} \sim nb_n^{-\alpha} h_1(b_n), n \to \infty. \tag{2.1.8}$$

Analogously, we get $F(x) = (-x)^{-\delta} h_2(-x), x < 0$, where $h_2 : (0, +\infty) \to (0, +\infty)$ is slowly varying, and $nb_n^{-\delta} h_1(b_n) \sim c_2$. Assuming $c_1, c_2 > 0$ (otherwise the statement gets trivial since either α or δ can be chosen arbitrary), we get $b_n^{-\alpha+\delta} \frac{h_1(b_n)}{h_2(b_n)} \to \frac{c_1}{c_2} > 0, n \to \infty$, where h_1/h_2 is slowly varying at $+\infty$, which is possible only if $\alpha = \delta$. □

Corollary 2.1.1
Under the conditions of Theorem 2.1.4, assume that $c_1 + c_2 > 0$. Then the normalizing sequence b_n in Definition 2.1.1 behaves as $b_n \sim n^{1/\alpha} h(n)$, where $h : (0, +\infty) \to (0, +\infty)$ is slowly varying at $+\infty$.

Proof Assume, for simplicity, $c_1 > 0$. Then, formula (2.1.8) yields $n \sim c_1 b_n^{\alpha} h_1^{-1}(b_n), \alpha \in (0, 2)$. Hence, $b_n \sim n^{1/\alpha} c_1^{-1/\alpha} (h_1(b_n))^{1/\alpha} = n^{1/\alpha} h(n)$, where $h(n) = (c_1^{-1} h_1(b_n))^{1/\alpha}$ is slowly varying at $+\infty$ due to the properties of h_1. □

Proof of Theorem 2.1.1. 1) *Show the equivalence of Definitions 2.1.1 and 2.1.2.*

Let X be a non-constant r.v. with characteristic function φ_X as in (2.1.6). Assume that X is stable in the sence of Definition 2.1.1. By Theorem 2.1.4, its spectral function H has the form $H(x) = \begin{cases} -c_1/|x|^{\alpha} & x > 0, \\ c_2/|x|^{\alpha}, & x < 0 \end{cases}, \alpha \in (0, 2), c_1, c_2 \geq 0$. Put it into the formula (2.1.6): $\log \varphi_X(x) = isa - bs^2 + c_1 Q_{\alpha}(s) + c_2 \overline{Q_{\alpha}(s)}, s \in \mathbb{R}$, where

$$Q_{\alpha}(s) = -\int_0^{\infty} \left(e^{isx} - 1 - is \sin x\right) dx^{-\alpha} = \operatorname{Re}\left(\psi_{\alpha}(i, t)\right)|_{t=-is},$$

and $\psi_{\alpha}(z, t) = t \int_0^{\infty} \left(e^{-zx} - e^{-tx}\right) x^{-\alpha} dx$ for $z, t \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} t > 0, \alpha \in (0, 2)$. Here the real part of $\psi_{\alpha}(i, t)$ is then under the assumption $t > 0$, and after that, $t = -is$ is plugged in. Integrating by parts, we get

$$\begin{aligned}
\psi_{\alpha}(z, t) &= \frac{t}{1-\alpha} \int_0^{+\infty} (ze^{-zx} - te^{-tx}) x^{1-\alpha} dx \\
&= \frac{t}{1-\alpha} \left(z^{\alpha-1} \int_0^{+\infty} e^{-zx} (zx)^{1-\alpha} d(zx) - t^{\alpha-1} \int_0^{+\infty} (e^{-tx})(tx)^{1-\alpha} d(tx) \right) = \begin{vmatrix} xz = y \\ xt = y \end{vmatrix} \\
&= \frac{t}{1-\alpha} \left(z^{\alpha-1} \int_0^{+\infty} e^{-y} y^{2-\alpha-1} dy - t^{\alpha-1} \int_0^{+\infty} e^{-y} y^{2-\alpha-1} dy \right) \\
&= \frac{t\Gamma(2-\alpha)}{1-\alpha} \left(z^{\alpha-1} - t^{\alpha-1} \right), \text{ for any } \alpha \neq 1, \operatorname{Re} z, \operatorname{Re} t > 0.
\end{aligned}$$

For fixed $z, t \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} t > 0$ the function $\psi_\alpha(z,t) : (0,2) \to \mathbb{C}$ as a function of α is continuous on $(0,2)$. Hence,

$$\begin{aligned}\psi_1(z,t) &= \lim_{\alpha\to 1}\psi_\alpha(z,t) = \lim_{\alpha\to 1}\frac{t\Gamma(2-\alpha)}{1-\alpha}(z^{\alpha-1}-t^{\alpha-1})\\ &= \lim_{1-\alpha\to 0}\frac{t}{1-\alpha}(e^{(\alpha-1)\log z}-e^{(\alpha-1)\log t}) = |1-\alpha = x|\\ &= \lim_{x\to 0}\frac{t}{x}(1-x\log z-1+x\log t+o(x)) = t(\log t-\log z) = t\log(t/z).\end{aligned}$$

Then for $\alpha \neq 1$ we get

$$\begin{aligned}Q_\alpha(s) &= \frac{-is\Gamma(2-\alpha)}{1-\alpha}\left(\operatorname{Re}(e^{i(\pi/2)(\alpha-1)}) - (-is)^{\alpha-1}\right)\\ &= \frac{-is\Gamma(2-\alpha)}{1-\alpha}\left(\operatorname{Re}(e^{i(\pi/2)(\alpha-1)}) - e^{(\alpha-1)i(-\pi/2)\mathrm{sign}s}|s|^{\alpha-1}\right)\\ &= -is\Gamma(1-\alpha)\left(\cos\left(\frac{\pi}{2}(\alpha-1)\right) - \left(\cos\left(\frac{\pi}{2}(\alpha-1)\right) - i(\mathrm{sign}s)\sin\left(\frac{\pi}{2}(\alpha-1)\right)\right)|s|^{\alpha-1}\right)\\ &= -is\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1-\alpha) + \sin\left(\frac{\pi\alpha}{2}\right)i(\mathrm{sign}s)|s|^\alpha\Gamma(1-\alpha) + i^2|s|^\alpha\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\\ &= -\Gamma(1-\alpha)\cos(\pi\alpha/2)|s|^\alpha - is(1-|s|^{\alpha-1})\Gamma(1-\alpha)\sin(\pi\alpha/2).\end{aligned}$$

For $\alpha = 1$

$$\begin{aligned}Q_\alpha(s) &= -is\operatorname{Re}(\log(t/i))|_{t=-is} = -is\log(t)|_{t=-is} = -is\log(-is)\\ &= -is(\log|s| + i(-\pi/2)\mathrm{sign}s) = -|s|\frac{\pi}{2} - is\log|s|.\end{aligned}$$

Then

$$|\varphi_X(s)| = \exp\{-bs^2 - d|s|^\alpha\}, \tag{2.1.9}$$

where $d = (c_1+c_2)\frac{\Gamma(2-\alpha)}{1-\alpha}\sin\left(\frac{\pi}{2}(1-\alpha)\right), \alpha \neq 1$. For $\alpha = 1$ get limit as $\alpha \to 1$ as a value of d: $(c_1+c_2)\pi/2$. Show that $bd = 0$.

If, for instance, $d > 0$, then show that $b = 0$. By Definition 2.1.1, $\exists$ sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$: $b_n \to \infty$ as $n \to \infty$ and a characteristic function $\varphi_{X_1}(s)$ s.t. $e^{-isa_n}\varphi^n_{X_1}(s/b_n) \to \varphi_X(s)$, $n \to \infty, s \in \mathbb{R}$. Hence, $|\varphi_{X_1}(s/b_n)|^n \to |\varphi_X(s)|, n \to \infty$ where $b_n = n^{1/\alpha}h(n)$ by Corollary 2.1.1. Since, h is slowly varying, $\frac{b_n}{b_{nk}} \to k^{-1/\alpha}, n \to \infty$ for any $k \in \mathbb{N}$. Then

$$|\varphi_X(s)| \underset{n\to\infty}{\leftarrow} \left|\varphi_{X_1}\left(\frac{s}{b_{nk}}\right)\right|^{nk} = \left|\varphi_{X_1}\left(s\frac{b_n}{b_{nk}}b_n^{-1}\right)\right|^{nk} \underset{n\to\infty}{\rightarrow} \left|\varphi_X\left(sk^{-1/\alpha}\right)\right|^k, \forall k \in N,$$

i.e., by (2.1.9), $\exp\{-bs^2 - d|s|^\alpha\} = \exp\{-bs^2k^{1-2/\alpha} - d|s|^\alpha\}$, which is only possible if $b = 0$.

Now set

$$\begin{cases}\lambda = \begin{cases} d, & \text{if } c_1+c_2 > 0,\\ b, & \text{if } c_1+c_2 = 0 \text{ (Gaussian case)}\end{cases},\\ \beta = \begin{cases}(c_1-c_2)/\lambda, & \text{if } c_1+c_2>0,\\ 0, & \text{if } c_1+c_2 = 0 \text{ (Gaussian case)}\end{cases},\\ \gamma = \frac{1}{\lambda}(a+\bar{a}), \text{ where } \bar{a} = \begin{cases}(c_2-c_1)\Gamma(1-\alpha)\sin(\pi\alpha/2) & \text{if } \alpha\neq 1,\\ 0, & \text{if } \alpha = 1.\end{cases}\end{cases} \tag{2.1.10}$$

Then φ_X satisfies representation in Definition 2.1.2 with the above parameters $\lambda, \beta, \gamma, \alpha$.

Vice versa, if φ_X satisfies Definition 2.1.2, then it can be represented as in (2.1.6) with spectral function H as in Theorem 2.1.4, see the formula (2.1.10), where c_1, c_2 can be restored from λ, β, γ uniquely. By Theorem 2.1.2, the limit theorem $S_n \xrightarrow{d} X, n \to \infty$ takes place.

Exercise 2.1.5

Show that $\{X_{n_j}\}$ can be chosen here as in Definition 2.1.1 (since $b_n = n^{1/\alpha}h(n)$ is clear, $b_n \to \infty$, one has only to fix a_n, cf. Remark 2.1.3)

2) Show the equivalence of Definitions 2.1.1 and 1.0.1.

Let X be stable in the sense of Definition 1.0.1. By induction, if follows from the relation $aX_1 + bX_2 \stackrel{d}{=} cX + d$ of Definition 1.0.1 (with $a = b = 1$) that for any $n \geq 2$ $\exists$ constants $b_n > 0$, a_n s.t. for independent copies $X_i, i = 1 \dots n$ of X : $X_1 + \dots + X_n \stackrel{d}{=} b_n X + a_n$, or $\frac{1}{b_n}\sum_{i=1}^n X_i - \frac{a_n}{b_n} \stackrel{d}{=} X$. So, for $n \to \infty$, the limiting distribution of the left-hand side coincides with that of X, and Definition 2.1.1 holds.

Vice versa, we show that from Definition 2.1.2 (which is equivalent to Definition 2.1.1) it follows Definition 1.0.1. Definition 1.0.1 can be rewritten in terms of characteristic function as

$$\varphi_X(as)\varphi_X(bs) = \varphi_X(cs)e^{isd}, \tag{2.1.11}$$

where $a > 0$ and $b > 0$ are arbitrary constants, and $c > 0, d \in \mathbb{R}$ are chosen as in Definition 1.0.1, $\varphi_X(s) = \mathbb{E}e^{is\lambda}$. By Definition 2.1.2, $\varphi_X(s) = \exp\{\lambda(is\gamma - |s|^2 + is\omega(s, \alpha, \beta))\}, s \in \mathbb{R}$ with $\omega(s, \alpha, \beta)$ as in (2.1.2). It is quite easy to see that (2.1.2) follows with $c = (a^\alpha + b^\alpha)^{1/\alpha}$, $d = \begin{cases} \lambda\gamma(a + b - c), & \alpha \neq 1, \\ \lambda\beta\frac{2}{\pi}(a\log(a/c) + b\log(b/c)), & \alpha = 1. \end{cases}$

3) Show the equivalence of Definition 2.1.3 and Definition 1.0.1. Definition 2.1.3 follows from Definition 1.0.1 as it was shown in 2). Vise versa, from Definition 2.1.3 it follows Definition 2.1.1 (see 2)), which is equivalent to Definition 1.0.1.

4) Show the equivalence of Definitions 2.1.3 and 2.1.4. In one direction (Definition 2.1.3 $\Rightarrow$ Definition 2.1.4) it is evident, in the other direction, assume that $X_1 + X_2 \stackrel{d}{=} c_2 X + d_2$, $X_1 + X_2 + X_3 \stackrel{d}{=} c_3 X + d_3$ for some $c_2, c_3 > 0, d_2, d_3 \in \mathbb{R}$. In order to show Definition 2.1.3, it is sufficient to check that

$$n\eta(s) = \eta(c_n s) + isd_n \tag{2.1.12}$$

for any $n \geq 4$, some $c_n > 0$ and $d_n \in \mathbb{R}$, where $\eta(s) = \log\varphi_X(s), s \in \mathbb{R}$. Since (by assumption) (2.1.12) holds for $n = 2, 3$, it holds (by induction) for any $n = \begin{cases} 2^m \\ 3^m \end{cases}$ with $c_n = \begin{cases} c_2^m, \\ c_3^m \end{cases}$, $d_n = \begin{cases} d_2(1 + c_2 + \dots + c_2^{m-1}), \\ d_3(1 + c_3 + \dots + c_3^{m-1}). \end{cases}$ $m \in \mathbb{N}$. Hence, the distribution of X is infinitely divisible, and then $|\varphi(s)| \neq 0, \forall s \in \mathbb{R}$.

From the said above, it holds

$$2^j 3^k \eta(s) = \eta(c_2^j c_3^k s) + ia_{jk}s \tag{2.1.13}$$

for some $c_2, c_3 > 0, a_{jk} \in \mathbb{R}, j, k \in \mathbb{Z}$ [1]. The set $\{2^j 3^k, j, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+$, since $2^j 3^k = \exp\{j\log 2 + k\log 3\}$, and the set $\{j + \omega k, j, k \in \mathbb{Z}\}, \omega \notin \mathbb{Q}$ is dense in $\mathbb{R}$. Hence, for any n $\exists$

[1] Let $t = s/c_2$ then it follows from (2.1.12) that $\frac{1}{2}\eta(t) = \eta(c_2^{-1}t) - is\frac{d_2}{c_2}$. Similarly we get $\frac{1}{3}\eta(t) = \eta(c_3^{-1}t) - is\frac{d_3}{c_3}$. So, formula (2.1.13) also holds for negative $j, k \in \mathbb{Z}$.

sequence $\{r_m\}_{m\in\mathbb{N}}, r_m \to n$ as $m \to \infty$, and $r_m = 2^{j_m}3^{k_m}$. Let $c_n(m) = c_2^{j_m}c_3^{k_m}, m \in \mathbb{N}$. Show that $\{c_n(m)\}_{m\in\mathbb{N}}$ is bounded. It follows from (2.1.13) that $r_m \mathrm{Re}\,(\eta(s)) = \mathrm{Re}\,(\eta(c_n(m)s))$.

Assume that $c_n(m)$ is unbounded, then $\exists$ subsequence $\{c_n(m')\}$ such that $|c_n(m')| \to \infty, m' \to \infty$. Set $s' = sc_n(m')$ in the last equation. Since $r_{m'} \to n, m' \to \infty$, we get $\mathrm{Re}\,\eta(s') = r_{m'}\mathrm{Re}\,\eta(\frac{s'}{c_n(m')}) \to 0, m' \to \infty$. Hence, $|\eta(s)| \equiv 1$, which can not be due to the assumption that $X \not\equiv const$.

Then $\{c_n(m)\}_{m\in N}$ is bounded, and $\exists$ a subsequence $\{c_n(m')\}_{m'\in\mathbb{N}}$ such that $|c_n(m')| \to c_n, m' \to \infty$. Then $a_{j_{m'}k_{m'}} = \frac{i}{s}(\eta(c_n(m')) - r_{m'}\eta(s)) \to \frac{i}{s}(\eta(c_n s) - n\eta(s)) := d_n$. Hence, $\forall n \in \mathbb{N}$ and $s \in \mathbb{R}$ it holds $n\eta(s) = \eta(c_n s) + is\eta(d_n)$, which is the statement of equation (2.1.12), so we are done. □

Remark 2.1.4
It follows from the proof of Theorem 2.1.1 1) that the parameter $\beta = \frac{c_1 - c_2}{c_1 + c_2}$, if $c_1 + c_2 > 0$ in non-Gaussian case. Consider the extremal values of $\beta = \pm 1$. It is easy to see that for $\beta = 1$ $c_2 = 0$, for $\beta = -1$ $c_1 = 0$. This corresponds to the following situation in Definition 2.1.1:

a) Consider $\{X_n\}_{n\in\mathbb{N}}$ to be i.i.d. and positive a.s., i.e., $X_1 > 0$ a.s. By Theorem 2.1.3,1) it follows that $H(x) = 0, x < 0 \implies c_2 = 0 \implies \beta = 1$.

b) Consider $\{X_n\}_{n\in\mathbb{N}}$ to be i.i.d. and negative a.s. As above, we conclude $H(x) = 0, x > 0$, and $c_1 = 0 \implies \beta = -1$.

Although this relation can not be inverted (from $\beta = \pm 1$ it does not follow that $X > (<) 0$ a.s.), it explains the situation of total skewness of a non-Gaussian X as a limit of sums of positive or negative i.i.d. random variables $S_n = \frac{1}{b_n}\sum_{i=1}^n X_i - a_n$.

Remark 2.1.5
One can show that $c_n = n^{1/\alpha}$ in Definition 2.1.3, formula (2.1.4), for $\alpha \in (0,2]$.

Proof We prove it only for strictly stable laws. First, for $\alpha = 2$ (Gaussian case $X, X_i \sim N(0,1)$) it holds $\sum_{i=1}^n X_i \sim N(0,n) \stackrel{d}{=} \sqrt{n}X \implies c_n = n^{1/\alpha}$ with $\alpha = 2$.

Now let $\alpha \in (0,2)$. Let X be strictly stable, s.t. $\sum_{i=1}^n X_i \stackrel{d}{=} c_n X$. Take $n = 2^k$, then

$$S_n = \underbrace{(X_1 + X_2)}_{X_1'} + \underbrace{(X_3 + X_4)}_{X_2'} + \cdots + \underbrace{(X_{n-1} + X_n)}_{X_{n/2}'} \stackrel{d}{=} c_2(X_1' + X_2' + \cdots + X_{n/2}') \stackrel{d}{=} \cdots \stackrel{d}{=} c_2^k X,$$

from which it follows $c_n = c_{2^k} = c_2^k = c_2^{\log n/\log 2}$, so

$$\log c_n = \left(\frac{\log n}{\log 2}\right)\log c_2 = \log\left(n^{\log c_2/\log 2}\right), \quad c_n = n^{1/\alpha_2}, \tag{2.1.14}$$

where $\alpha_2 = \log 2/\log c_2$, for $n = 2^k, k \in \mathbb{N}$. Generalizing the above approach to $n = m^k$ turns, we get

$$c_n = n^{1/\alpha_m}, \alpha_m = \frac{\log m}{\log c_m}, n = m^k, k \in \mathbb{N}. \tag{2.1.15}$$

To prove that $c_n = n^{1/\alpha_0}$ it suffices to show that if $c_r = r^{1/\beta}$ then $\beta = \alpha_0$. Now by (2.1.15) $c_{r^j} = r^{j/\alpha_r}$ and $c_{\rho^k} = \rho^{k/\alpha_\rho}$. But for each k there exists a j such that $r^j < \rho^k \le r^{j+1}$. Then

$$(c_{r^j})^{\alpha_r/\alpha_\rho} < c_{\rho^k} = \rho^{k/\alpha_\rho} \le r^{1/\alpha_\rho}(c_{r^j})^{\alpha_r/\alpha_\rho}. \tag{2.1.16}$$

Note that S_{m+n} is the sum of the independent variables S_m and $S_{m+n} - S_m$ distributed, respectively, as $c_m X$ and $c_n X$. Thus for symmetric stable distributions $c_{m+n}X \stackrel{d}{=} c_m X_1 + c_n X_2$. Next put $\eta = m+n$ and notice that due to the symmetry of the variables X, X_1, X_2 we have for $t > 0$ $\mathbb{P}(X > t) \le 2\mathbb{P}(X_2 > \frac{tc_\eta}{2c_n})$, if $n > m$. It follows that for $\eta > n$ that ratios c_n/c_η remain bounded. So, it follows from (2.1.16) that

$$r \ge (c_{\rho^k})^{\alpha_\rho - \alpha_r} \left(\frac{c_{\rho^k}}{c_{r^j}}\right)^{\alpha_r}$$

and hence $\alpha_r \ge \alpha_\rho$. Interchanging the roles of r and ρ we find similarly that $\alpha_r \le \alpha_\rho$ and hence $\alpha_r = \alpha_\rho \equiv \alpha_0$ for any $r, \rho \in \mathbb{N}$.

We get the conclusion that $c_n = n^{1/\alpha_0}, n \in \mathbb{N}$. It can be further shown that $\alpha_0 = \alpha$. □

Definition 2.1.6
A random variable X (or its distribution $\mathbb{P}_X$) is said to be *symmetric* if $X \stackrel{d}{=} -X$. X is *symmetric about* $\mu \in \mathbb{R}$ if $X - \mu$ is symmetric. If X is $\alpha-$stable and symmetric, we write $X \sim S\alpha S$. This definition is justified by the property $X \sim S_\alpha(\lambda, \beta, \gamma)$, $X-$symmetric $\Leftrightarrow \gamma = \beta = 0$, which will be proven later.

2.2 Strictly stable laws

As it is clear from the definition of strict stability (Definition 1.0.1) X is stable iff for any $a, b \in \mathbb{R}_+$ $\exists c > 0$ s.t. $\varphi_X(as)\varphi_X(bs) = \varphi_X(cs), s \in \mathbb{R}$, where $\varphi_X(s) = \mathbb{E}e^{isX}, s \in \mathbb{R}$.

Theorem 2.2.1
Let $X \not\equiv const$ a.s. It is strictly stable if its characteristic function admits one of the following representations: $\forall s \in \mathbb{R}$

1. $$\log \varphi_X(s) = \begin{cases} \lambda(-|s|^\alpha + is\omega(s,\alpha,\beta)) & \alpha \ne 1, \\ \lambda(is\gamma - |s|) & \alpha = 1, \end{cases} \quad \text{i.e.} \quad \begin{cases} \gamma = 0, & \alpha \ne 1, \\ \beta = 0, & \alpha = 1 \end{cases}$$

   with $\omega(s, \alpha, \beta)$ as in (2.1.2).

2. (form C) $\log \varphi_X(s) = -\lambda_C |s|^\alpha \exp(-i\frac{\pi}{2}\theta\alpha \mathrm{sign} s)$, where $\alpha \in (0, 2]$, $\lambda_C > 0$, $\theta \le \theta_\alpha = \min\{1, \frac{2}{\alpha} - 1\}$.

Proof 1. In the proof of Theorem 2.1.1, 2) it is shown that

$$d = \left.\begin{cases} \lambda\gamma(a+b-c), & \alpha \ne 1 \\ \lambda\beta\frac{2}{\pi}(a\log(a/c) + b\log(b/c)), & \alpha = 1 \end{cases}\right| = 0 \Leftrightarrow \begin{cases} \gamma = 0, & \alpha \ne 1 \\ \beta = 0, & \alpha = 1. \end{cases}$$

2. Take the parametrisation (B) of φ_X with parameters γ, β as in 1, and left α unchanged,

$$\begin{cases} \theta = \beta_B \frac{K(\alpha)}{\alpha}, \lambda_C = \lambda_B, & \alpha \ne 1, \\ \theta = \frac{2}{\pi}\mathrm{arctg}\left(\frac{2}{\pi}\gamma_B\right), \lambda_C = \lambda_B\left(\frac{\pi^2}{4} + \gamma_B^2\right)^{1/2}, & \alpha = 1. \end{cases}$$

□

2.3 Properties of stable laws

Here we consider further basic properties of $\alpha-$stable distributions.

Theorem 2.3.1
Let $X_i, i = 1, 2$ be $S_\alpha(\lambda_i, \beta_i, \gamma_i)$-distributed independent random variables, $X \sim S_\alpha(\lambda, \beta, \gamma)$. Then

1) X has a density (i.e, has absolutely continuous distribution), which is bounded with all its derivatives.

2) $X_1 + X_2 \sim S_\alpha(\lambda, \beta, \gamma)$ with

$$\lambda = \lambda_1 + \lambda_2, \quad \beta = \frac{\beta_1\lambda_1 + \beta_2\lambda_2}{\lambda_1 + \lambda_2}, \gamma = \frac{\lambda_1\gamma_1 + \lambda_2\gamma_2}{\lambda_1 + \lambda_2}.$$

3) $X + a \sim S_\alpha(\lambda, \beta, \gamma + a/\lambda)$, where $a \in \mathbb{R}$ is a constant.

4) For a real constant $a \neq 0$ it holds

$$aX \sim \begin{cases} S_\alpha\left(|a|^\alpha\lambda, \text{sign}(a)\beta, \gamma|a|^{1-\alpha}\text{sign}(a)\right), & \alpha \neq 1, \\ S_1\left(|a|\lambda, \text{sign}(a)\beta, \text{sign}(a)\left(\gamma - \frac{2}{\pi}(\log|a|)\beta\right)\right), & \alpha = 1. \end{cases}$$

5) For $\alpha \in (0, 2)$, $X \sim S_\alpha(\lambda, \beta, 0) \Leftrightarrow -X \sim S_\alpha(\lambda, -\beta, 0)$.

6) X is symmetric iff $\beta = \gamma = 0$. It is symmetric about $\lambda\gamma$ iff $\beta = 0$.

7) Let $\alpha \neq 1$. X is strictly stable iff $\gamma = 0$.

Proof 1) Let φ_X, φ_{X_i} be the characteristic function of $X, X_i, i = 1, 2$. It follows from Definition 2.1.2 that $|\varphi_X(s)| = e^{-\lambda|s|^\alpha}, s \in \mathbb{R}$. Take the inversion formula for the characteristic function. If $|\varphi_X|$ is integrable on $\mathbb{R}$ (which is here the case) then the density f_X of X exists and $f_X(s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-isx}\varphi_X(s)ds, x \in \mathbb{R}$. Additionally, the $n-$th derivative of f_X is

$$\left|f_X^{(n)}(x)\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}} |s|^n \underbrace{|\varphi_X(s)|}_{\exp(-\lambda|s|^\alpha)} ds = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi\alpha}\lambda^{-\frac{n+1}{2}} < \infty, x \in \mathbb{R}, n \in \mathbb{N}.$$

2) Prove it for the case $\alpha \neq 1$, the case $\alpha = 1$ is treated similarly. Consider the characteristic function of $X_1 + X_2$, and take its logarithms:

$$\begin{aligned} \log\varphi_{X_1+X_2}(s) &= \log(\varphi_{X_1}(s)\varphi_{X_2}(s)) = \log\varphi_{X_1}(s) + \log\varphi_{X_2}(s) \\ &= \sum_{j=1}^{2}\lambda_j\left(is\gamma_j - |s|^\alpha + s|s|^{\alpha-1}i\beta_j\text{tg}(\pi\alpha/2)\right) \\ &= -|s|^\alpha(\lambda_1 + \lambda_2) + is(\lambda_1\gamma_1 + \lambda_2\gamma_2) + is|s|^{\alpha-1}(\lambda_1\beta_1 + \lambda_2\beta_2)\text{tg}(\pi\alpha/2) \\ &= (\lambda_1 + \lambda_2)\left(is\frac{\lambda_1\gamma_1 + \lambda_2\gamma_2}{\lambda_1 + \lambda_2} - |s|^\alpha + is|s|^{\alpha-1}\frac{\lambda_1\beta_1 + \lambda_2\beta_2}{\lambda_1 + \lambda_2}\text{tg}(\pi\alpha/2)\right) \\ &= \lambda(is\gamma - |s|^\alpha + is\omega(s, \alpha, \beta)), \end{aligned}$$

with

$$\lambda = \lambda_1 + \lambda_2, \gamma = \frac{\lambda_1\gamma_1 + \lambda_2\gamma_2}{\lambda_1 + \lambda_2}, \beta = \frac{\lambda_1\beta_1 + \lambda_2\beta_2}{\lambda_1 + \lambda_2}.$$

So, $X_1 + X_2 \sim S_\alpha(\lambda, \beta, \gamma)$ by Definition 2.1.2.

3) $\log \varphi_{X+a}(s) = isa + \lambda is\gamma - \lambda|s|^\alpha + \lambda is\omega(s, \alpha, \beta) = \lambda(is(\gamma + a/\lambda) - |s|^\alpha + is\omega(s, \lambda, \beta))$, hence $X + a \sim S_\alpha(\lambda, \beta, \gamma + a/\lambda)$.

4) Consider the case $\alpha \neq 1$.

$$\begin{aligned} \log \varphi_{aX}(s) = \log \varphi_X(as) &= \lambda(ias\gamma - |as|^\alpha + ias\omega(as, \alpha, \beta)) \\ &= \lambda|a|^\alpha \left(is\gamma \frac{a}{|a|^\alpha} - |s|^\alpha + is\frac{a|a|^{\alpha-1}}{|a|^\alpha}|s|^{\alpha-1}\beta \mathrm{tg}(\pi\alpha/2) \right) \\ &= \lambda|a|^\alpha \left(is\gamma|a|^{1-\alpha}\mathrm{sign}(a) - |s|^\alpha + is\mathrm{sign}(a)\beta|s|^{\alpha-1}\mathrm{tg}(\pi\alpha/2) \right), \end{aligned}$$

hence $aX \sim S_\alpha(\lambda|a|^\alpha, \mathrm{sign}(a)\beta, \gamma|a|^{1-\alpha}\mathrm{sign}(a))$.

For $\alpha = 1$, we have

$$\begin{aligned} \log \varphi_{aX}(s) = \log \varphi_X(as) &= \lambda \left(ias\gamma - |as| - ias\beta\frac{2}{\pi}\log|as| \right) \\ &= \lambda|a| \left(is\gamma\frac{a}{|a|} - is\frac{a}{|a|}\beta\frac{2}{\pi}\log|a| - |s| - is\frac{a}{|a|}\beta\frac{2}{\pi}\log|s| \right) \\ &= \lambda|a| \left(i\mathrm{sign}(a)s\left(\gamma - \beta\frac{2}{\pi}\log|a|\right) - |s| - i\mathrm{sign}(a)s\beta\frac{2}{\pi}\log|s| \right), \end{aligned}$$

hence $aX \sim S_1\left(\lambda|a|, \mathrm{sign}(a)\beta, \mathrm{sign}(a)\left(\gamma - \beta\frac{2}{\pi}\log|a|\right)\right)$.

5) follows from 4) with $a = -1$.

6) X is symmetric by definition iff $X \stackrel{d}{=} -X$, i.e., $\varphi_X(s) = \varphi_{-X}(s) = \varphi_X(-s), \forall s \in \mathbb{R}$, which is only possible if $\varphi_X(s) \in \mathbb{R}, s \in \mathbb{R}$. Indeed, $\mathbb{E}e^{isX} = \mathbb{E}\cos(sX) + i\mathbb{E}\sin(sX) = \mathbb{E}\cos(-sX) + i\mathbb{E}\sin(-sX) = \mathbb{E}\cos(sX) - i\mathbb{E}\sin(sX)$ iff $2i\mathbb{E}\sin(sX) = 0, \forall s \in \mathbb{R}$. Using Definition 2.1.2, $\varphi_X(s)$ is real only if $\gamma = 0$ and $\omega(s, \alpha, \beta) = 0$, i.e., $\beta = 0$.

X is symmetric around $\lambda\gamma$ by definition iff $X - \lambda\gamma \stackrel{d}{=} -(X - \lambda\gamma) = -X + \lambda\gamma$. By property 3) and 4), $X - \lambda\gamma \sim S_\alpha(\lambda, \beta, \gamma - \gamma), -X + \lambda\gamma \sim S_\alpha(\lambda, -\beta, -\gamma + \gamma)$. So, $X - \lambda\gamma \stackrel{d}{=} -X + \lambda\gamma$ iff $\beta = 0$.

7) Is already proven in Theorem 2.2.1. □

Remark 2.3.1

1) The analytic form of the density of a stable law $S_\alpha(\lambda, \beta, \gamma)$ is explicitly known only in the cases $\alpha = 2$ (Gaussian law), $\alpha = 1$ (Cauchy law), $\alpha = 1/2$ (Lévy law).

2) Due to Property 3) of Theorem 2.3.1, the parameter γ (or sometimes $\lambda\gamma$) is called *shift parameter.*

3) Due to Property 4) of Theorem 2.3.1, the parameter λ (or sometimes $\lambda^{1/\alpha}$) is called *shape* or *scale parameter.* Notice that this name is natural for $\alpha \neq 1$ or $\alpha = 1, \beta = 0$. In case $\alpha = 1, \beta \neq 0$, scaling of X by a results in a non-zero shift of the law of X by $\frac{2}{\pi}\beta\log|a|$, hence the use of this name in this particular case can hardly be recommended.

4) Due to properties 5)-6) of Theorem 2.3.1, parameter β is called *skewness parameter.* If $\beta > 0 (\beta < 0)$ then $S_\alpha(\lambda, \beta, \gamma)$ is said to be *skewed to the right (left).* $S_\alpha(\lambda, \pm 1, \gamma)$ is said to be *totally skewed to the right* (for $\beta = 1$) or *left* (for $\beta = -1$).

5) It follows from Theorem 2.2.1 and Theorem 2.3.1, 3) that if $X \sim S_\alpha(\lambda, \beta, \gamma), \alpha \neq 1$, then $X - \lambda\gamma \sim S_\alpha(\lambda, \beta, 0)$ is strictly stable.

6) It follows from Theorem 2.2.1 and Definition 2.1.2 that no non-strictly 1-stable random variable can be made strictly stable by shifting. Indeed, if $X \sim S_1(\lambda, \beta, \gamma)$ is not strictly stable then $\beta \neq 0$, which can not be eliminated due to $\log|s|$ in $\omega(s, \alpha, \beta)$. Analogously, every strictly 1-stable random variable can be made symmetric by shifting.

Corollary 2.3.1
Let $X_i, i = 1, \dots, n$ be i.i.d. $S_\alpha(\lambda, \beta, \gamma)-$distributed random variables, $\alpha \in (0, 2]$. Then

$$X_1 + \dots + X_n \stackrel{d}{=} \begin{cases} n^{1/\alpha} X_1 + \lambda\gamma(n - n^{1/\alpha}), & \text{if } \alpha \neq 1, \\ nX_1 + \frac{2}{\pi}\lambda\beta n \log n, & \text{if } \alpha = 1. \end{cases}$$

This means, c_n and d_n in Definition 2.1.3 have values

$$c_n = n^{1/\alpha}, \alpha \in (0, 2], \quad d_n = \begin{cases} \lambda\gamma(n - n^{1/\alpha}), & \text{if } \alpha \neq 1, \\ \frac{2}{\pi}\lambda\beta n \log n, & \text{if } \alpha = 1. \end{cases}$$

Proof It follows by induction from the proof of Theorem 2.1.1 2). There, it is shown $aX_1 + bX_2 \stackrel{d}{=} cX + d$, with $c = (a^\alpha + b^\alpha)^{1/\alpha}$, $d = \begin{cases} \lambda\gamma(a + b - c), & \alpha \neq 1, \\ \lambda\beta\frac{2}{\pi}(a\log(a/c) + b\log(b/c)), & \alpha = 1. \end{cases}$ Take $n = 2, a = b = 1 \Rightarrow c_2 = 2^{1/\alpha}, d_2 = \begin{cases} \lambda\gamma(2 - 2^{1/\alpha}), & \alpha \neq 1, \\ \lambda\beta\frac{2}{\pi}2\log(2)), & \alpha = 1. \end{cases}$ The induction step is trivial. □

Corollary 2.3.2
It follows from Theorem 2.3.1, 2) and 3) that if $X_1, X_2 \sim S_\alpha(\lambda, \beta, \gamma)$ are independent then $X_1 - X_2 \sim S_\alpha(2\lambda, 0, 0)$ and $-X_1 \sim S_\alpha(\lambda, -\beta, -\gamma)$.

Proposition 2.3.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $X_n \sim S_{\alpha_n}(\lambda_n^M, \beta_n^M, \gamma_n^M), n \in \mathbb{N}$, where $\alpha_n \in (0, 2), \lambda_n^M > 0$, $\beta_n^M \in [-1, 1], \gamma_n^M \in \mathbb{R}$. Assume that $\alpha_n \to \alpha, \lambda_n^M \to \lambda^M, \beta_n^M \to \beta^M, \gamma_n^M \to \gamma^M$ as $n \to \infty$ for some $\alpha \in (0, 2), \lambda^M > 0, \beta^M \in [-1, 1], \gamma^M \in \mathbb{R}$. Then $X_n \xrightarrow{d} X \sim S_\alpha(\lambda^M, \beta^M, \gamma^M)$ as $n \to \infty$. Here the superscript "M" means the modified parametrisation, cf. formula (2.1.3) after Definition 2.1.2.*

Proof $X_n \xrightarrow{d} X$ as $n \to \infty$ is equivalent to $\varphi_{X_n}(s) \to \varphi_X(s), n \to \infty, s \in \mathbb{R}$, or, $\log \varphi_{X_n}(s) = \lambda_n^M(is\gamma_n^M - |s|^{\alpha_n} + is\omega_M(s, \alpha_n, \beta_n^M)) \underset{n\to\infty}{\to} \lambda^M(is\gamma^M - |s|^\alpha + is\omega_M(s, \alpha, \beta^M))$ which is straightforward by the continuity of the modified parametrisation w.r.t. its parameters. □

Our aim now is to prove the following result.

Proposition 2.3.2. *Let $X \sim S_\alpha(\lambda, 1, 0), \lambda > 0, \alpha \in (0, 1)$. Then $X \geq 0$ a.s.*

This property justifies again the use of β as skewness parameter and brings a random variable $X \sim S_\alpha(\lambda, 1, 0)$ the name of *stable subordinator*. The above proposition will easily follow from the next theorem.

Theorem 2.3.2
1) For $\alpha \in (0,1)$, consider $X_\delta = \sum_{k=1}^{N_\delta} U_{\delta,k}$ to be compound Poisson distributed, where N_δ is a $Poisson(\delta^{-\alpha})-$distributed random variable, $\delta > 0$, and $\{U_{\delta,k}\}_{k\in\mathbb{N}}$ are i.i.d. positive random variables, independent of N_δ, with $\mathbb{P}(U_{\delta,k} > x) = \begin{cases} \delta^\alpha/x^\alpha, & x > \delta, \\ 1, & x \le \delta. \end{cases}$

Then $X_\delta \xrightarrow{d} X, \delta \to 0$, where $X \sim S_\alpha(\lambda, 1, 0)$ with $\lambda = \Gamma(1-\alpha)\cos(\pi\alpha/2)$.
2) Let $X \sim S_\alpha(\lambda,1,0), \alpha \in (0,1)$. Then its Laplace transform $\hat{l}_X(s) := \mathbb{E}e^{-sX}$ is equal to

$$\hat{l}_X(s) = e^{-\Gamma(1-\alpha)s^\alpha}, s \ge 0. \tag{2.3.1}$$

Proof 1) Since the generating function of $N \sim Poisson(a)$ is equal to $\hat{g}_N(z) = \mathbb{E}z^N = \sum_{k=0}^\infty z^k \mathbb{P}(N=k) = \sum_{k=0}^\infty z^k e^{-a}\frac{a^k}{k!} = e^{-a}\sum_{k=0}^\infty \frac{(az)^k}{k!} = e^{-a}e^{az} = e^{a(z-1)}, z \in \mathbb{C}$, we have $\hat{g}_{N_\delta}(z) = e^{\delta^{-\alpha}(z-1)}, z \in \mathbb{C}$, and hence

$$\begin{aligned}\varphi_{X_\delta}(s) &= \mathbb{E}e^{isX_\delta} = \mathbb{E}\left(\mathbb{E}\left(e^{isX_\delta}|N_\delta\right)\right) = \mathbb{E}\left(\mathbb{E}\left(e^{is\sum_{k=0}^{N_\delta}U_{\delta,k}}|N_\delta\right)\right) \\ &= \mathbb{E}\left(\prod_{k=1}^{N_\delta}\mathbb{E}e^{isU_{\delta,1}}\right) = \hat{g}_{N_\delta}(\varphi_{U_{\delta,1}}(s)) = e^{\delta^{-\alpha}(\varphi_{U_{\delta,1}}(s)-1)},\end{aligned}$$

where $\varphi_{U_{\delta,1}}(s) = \int_0^\infty e^{isx} d\mathbb{P}(U_{\delta,1} \le x) = \alpha\int_\delta^\infty e^{isx}\delta^\alpha x^{-\alpha-1}dx$. So (since $\alpha\int_\delta^\infty x^{-\alpha-1}dx = -\delta^{-\alpha}$)

$$\varphi_{X_\delta}(s) = \exp\left\{\alpha\int_\delta^\infty (e^{isx}-1)x^{-\alpha-1}dx\right\} \underset{\delta\to+0}{\to} \exp\left\{\alpha\int_0^\infty (e^{isx}-1)x^{-\alpha-1}dx\right\},$$

which is of the form (2.1.6) with $H(x) = -c_1 x^{-\alpha}\mathbb{I}(x>0)$ as in Theorem 2.1.4 ($c_2 = 0$). Consider $\varphi_X(s) := \exp\left\{\alpha\int_0^\infty (e^{isx}-1)x^{-\alpha-1}dx\right\}, s \ge 0, \alpha \in (0,1)$. Show that

$$\int_0^\infty \frac{e^{isx}-1}{x^{\alpha+1}}dx = -s^\alpha\frac{\Gamma(1-\alpha)}{\alpha}e^{-i\alpha\pi/2}. \tag{2.3.2}$$

If it is true then $\log\varphi_X(s) = -|s|^\alpha\Gamma(1-\alpha)(\cos(\pi\alpha/2) - i\,\mathrm{sign}(s)\sin(\pi\alpha/2))$ since for $s<0$ we make the substitution $s \to -s$, $i \to -i$. Then, $\log\varphi_X(s) = -|s|^\alpha\Gamma(1-\alpha)\cos(\pi\alpha/2)(1 - i\,\mathrm{sign}(s)\mathrm{tg}(\pi\alpha/2)), s \in \mathbb{R}$, which means that, according to Definition 2.1.2, $X \sim S_\alpha(\lambda,1,0)$. Now prove relation (2.3.2). It holds

$$\begin{aligned}\int_0^\infty \frac{e^{isx}-1}{x^{\alpha+1}}dx &= \lim_{\theta\to+0}\int_0^\infty \frac{e^{isx-\theta x}-1}{x^{\alpha+1}}dx = \lim_{\theta\to+0} -\frac{1}{\alpha}\int_0^\infty (e^{-\theta x+isx}-1)d(x^{-\alpha}) \\ &= \lim_{\theta\to+0}\left(-\frac{1}{\alpha}(e^{-\theta x+isx}-1)\frac{1}{x^\alpha}\Big|_0^\infty + \frac{-\theta+is}{\alpha}\int_0^\infty\frac{e^{-\theta x+isx}}{x^\alpha}dx\right) \\ &= \lim_{\theta\to+0} -\frac{\theta-is}{\theta^{1-\alpha}\alpha}\Gamma(1-\alpha)\theta^{1-\alpha}\int_0^\infty \frac{e^{isx}x^{1-\alpha-1}e^{-\theta x}}{\Gamma(1-\alpha)}dx \\ &= -\lim_{\theta\to+0}\frac{\theta-is}{\theta^{1-\alpha}\alpha}\Gamma(1-\alpha)\frac{1}{(1-is/\theta)^{1-\alpha}} = -\lim_{\theta\to+0}\frac{(\theta-is)^{1-1+\alpha}}{\theta^{1-\alpha}\alpha/\theta^{1-\alpha}}\Gamma(1-\alpha) \\ &= -\lim_{\theta\to+0}\frac{(\theta-is)^\alpha\Gamma(1-\alpha)}{\alpha} = -\frac{\Gamma(1-\alpha)}{\alpha}\lim_{\theta\to+0}\left(\sqrt{\theta^2+s^2}e^{i\xi}\right)^\alpha \\ &= -\frac{\Gamma(1-\alpha)}{\alpha}s^\alpha e^{-i\frac{\pi}{2}\alpha},\end{aligned}$$

where $\xi = \arg(\theta - is) \underset{\theta \to +0}{\to} -\pi/2$.

2) Similarly to said above,

$$\hat{l}_{X_\delta}(s) = \mathbb{E}e^{-sX_\delta} = \exp\left\{\alpha \int_\delta^\infty (e^{isx} - 1)x^{-\alpha-1}dx\right\} \underset{\delta \to +0}{\to} \exp\left\{\alpha \int_0^\infty (e^{isx} - 1)x^{-\alpha-1}dx\right\}$$

$$(\text{sub. } y = sx) = \exp\left\{s^\alpha \int_0^\infty \alpha(e^{iy} - 1)y^{-\alpha-1}dy\right\} = \exp\left\{-s^\alpha \int_0^\infty x^{-\alpha}e^{-x}dx\right\}$$
$$= \exp\{-s^\alpha \Gamma(1-\alpha)\}, s \geq 0.$$

□

Proof of Proposition 2.3.2 Since $X_\delta \geq 0, X_\delta \underset{\delta \to +0}{\to} X$ as in Theorem 2.3.2,1) it holds $X \geq 0$. This means that the support of the density f of $X \sim S_\alpha(\lambda, 1, 0)$ is contained in $\mathbb{R}_+$. Moreover, one can show that $\text{supp} f := \{x \in \mathbb{R} : f(x) > 0\} = \mathbb{R}_+$ by showing that $\forall a, b > 0 : a^\alpha + b^\alpha = 1$ it holds $a \cdot \text{supp} f + b \cdot \text{supp} f = \text{supp} f$. It follows from this relation that $\text{supp} f = \mathbb{R}_+$ since it can not be $\mathbb{R}$.

□

Exercise 2.3.1
Show this!

Remark 2.3.2
Actually, formula (2.3.1) is valid for all $\alpha \neq 1, \alpha \in (0, 2]$: for $X \sim S_\alpha(\lambda, 1, 0)$,

$$\hat{l}_X(s) = \begin{cases} \exp\left\{-\frac{\lambda}{\cos(\pi\alpha/2)}s^\alpha\right\}, & \alpha \neq 1, \alpha \in (0, 2], \\ \exp\left\{-\lambda \frac{2}{\pi} s \log s\right\}, & \alpha = 1, \end{cases} \quad s \geq 0,$$

where $\Gamma(1-\alpha) = \frac{\lambda}{\cos(\pi\alpha/2)}$ for $\alpha \neq 1$. Here, $-\frac{\lambda}{\cos(\pi\alpha/2)} = \begin{cases} < 0, & \alpha \in (0,1), \\ > 0, & \alpha \in (1,2), \\ \lambda, & \alpha = 2. \end{cases}$

Proposition 2.3.3. *The support of* $S_\alpha(\lambda, \beta, 0)$ *is* $\mathbb{R}$, *if* $\beta \in (-1, 1), \alpha \in (0, 2)$.

Proof Let $X \sim S_\alpha(\lambda, \beta, 0), \alpha \in (0, 2), \beta \in (-1, 1)$ with density f. It follows from properties 2)-4) of Theorem 2.3.1 that $\exists$ i.i.d. random variables $Y_1, Y_2 \sim S_\alpha(\lambda, 1, 0)$ and constants $a, b > 0, c \in \mathbb{R}$ s.t. $X \overset{d}{=} \begin{cases} aY_1 - bY_2, & \alpha \neq 1, \\ aY_1 - bY_2 + c, & \alpha = 1. \end{cases}$ Since, $Y_1 \geq 0$ and $-Y_2 \leq 0$ a.s. by Proposition 2.3.2, and their support is the whole $\mathbb{R}_+$ ($\mathbb{R}_-$, resp.), it holds $\text{supp} f = \mathbb{R}$.

□

Remark 2.3.3
One can prove that the support of $S_\alpha(\lambda, \pm 1, 0)$ is $\mathbb{R}$ as well, if $\alpha \in [1, 2)$.

Now consider the tail behavior of stable random variables. In the Gaussian case ($\alpha = 2$), it is exponential:

Proposition 2.3.4. *Let* $X \sim N(0, 1)$. *Then,* $\mathbb{P}(X < -x) = \mathbb{P}(X > x) \sim \frac{\varphi(x)}{x}, x \to \infty$, *where* $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ *is the standard normal density.*

Proof Due to the symmetry of X, $\mathbb{P}(X < -x) = \mathbb{P}(X > x), \forall x > 0$. Prove the more accurate inequality

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) < \mathbb{P}(X > x) < \frac{\varphi(x)}{x}, \forall x > 0. \tag{2.3.3}$$

The asymptotic $\mathbb{P}(X > x) \sim \frac{\varphi(x)}{x}, x \to +\infty$ follows immediately from it.

First prove the left relation in (2.3.3). Since $e^{-t^2/2} < e^{-t^2/2}\left(1 + \frac{1}{t^2}\right), \forall t > 0$, it holds for $x > 0 : \mathbb{P}(X > x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt \leq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\left(1 + \frac{1}{t^2}\right)dt = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{x}$, where the last equality can be easily verified by differentiation w.r.t. $x : -\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left(1 + \frac{1}{x^2}\right) = \frac{1}{\sqrt{2\pi}}\left(-\frac{x}{x}e^{-x^2/2} - e^{-x^2/2}\frac{1}{x^2}\right) = \left(\frac{\varphi(x)}{x}\right)'$. Analogously, $e^{-t^2/2}\left(1 - \frac{3}{t^4}\right) < e^{-t^2/2}, \forall t > 0$, hence

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\left(1 - \frac{3}{t^4}\right)dt \leq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt = \mathbb{P}(X > x), \tag{2.3.4}$$

where again the left equality in (2.3.4) is proved by differentiation w.r.t. x. □

Remark 2.3.4
If $X \sim N(\mu, \sigma^2)$, then $\mathbb{P}(X > x) \sim \frac{\sigma}{x-\mu}\varphi\left(\frac{x-\mu}{\sigma}\right), x \to +\infty$ accordingly.

However, for $\lambda \in (0, 2)$, the behaviour of right and left hand side tail probabilities is polynomial in $\frac{1}{x}$:

Proposition 2.3.5. *Let* $X \sim S_\alpha(\lambda, \beta, \gamma), \alpha \in (0, 2)$. *Then*

$$x^\alpha \mathbb{P}(X > x) \to c_\alpha \frac{1+\beta}{2}\lambda, \quad x^\alpha \mathbb{P}(X < -x) \to c_\alpha \frac{1-\beta}{2}\lambda, \text{ as } x \to +\infty,$$

where

$$c_\alpha = \left(\int_0^\infty \frac{\sin x}{x^\alpha}dx\right)^{-1} = \begin{cases} \frac{1}{\Gamma(1-\alpha)\cos(\pi\alpha/2)}, & \alpha \neq 1 \\ \frac{2}{\pi}, & \alpha = 1. \end{cases}$$

Remark 2.3.5
1) The above proposition states, for $\beta = \pm 1$, that for

$$\begin{cases} X \sim S_\alpha(\lambda, -1, 0), & \text{it holds } \mathbb{P}(X > x)x^\alpha \to 0, x \to +\infty, \\ X \sim S_\alpha(\lambda, 1, 0), & \text{it holds } \mathbb{P}(X < -x)x^\alpha \to 0, x \to +\infty, \end{cases}$$

which means that the tails go to zero faster than $x^{-\alpha}$. But what is the correct asymptotic in this case? For $\alpha \in (0, 1)$ we know that X is totally skewed to the left (right) and hence $\mathbb{P}(X > x) = 0, \forall x > 0$ for $\beta = -1$ and $\mathbb{P}(X < -x) = 0, \forall x > 0$ for $\beta = 1$.

For $\alpha \geq 1$, this asymptotic is far from being trivial. Thus, it can be shown (see [6, Theorem 2.5.3]) that

$$\begin{cases} \mathbb{P}(X > x) \underset{x\to+\infty}{\sim} \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}}\left(\frac{x}{\alpha a_\alpha}\right)^{-\frac{\alpha}{2(\alpha-1)}} \exp\left(-(\alpha-1)\left(\frac{x}{\alpha a_\alpha}\right)^{\frac{\alpha}{(\alpha-1)}}\right), & \alpha > 1, \\ \mathbb{P}(X > x) \underset{x\to+\infty}{\sim} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\pi/2)\lambda x - 1}{2} - e^{(\pi/2)\lambda x - 1}\right), & \alpha = 1, \end{cases} \quad \beta = -1,$$

where $a_\alpha = (\lambda/\cos(\pi(2-\alpha)/2))^{1/\alpha}$.

For $\beta = 1$ and $\mathbb{P}(X < -x)$ the same asymptotic applies, since $\mathbb{P}(X < -x) = \mathbb{P}(-X > x)$, and $-X \sim S_\alpha(\lambda, -1, 0)$ with $X \sim S_\alpha(\lambda, 1, 0)$.

2) In the specific case of $S\alpha S$ X, i.e., $\beta = 0$, $X \sim S_\alpha(\lambda, 0, 0)$, Proposition 2.3.5 yields $\mathbb{P}(X < -x) = \mathbb{P}(X > x) \sim \frac{\lambda c_\alpha}{2} \frac{1}{x^\alpha}$, $x \to +\infty$.

Proposition 2.3.5 will be proved later after we have proven important results needed for it. Let us state now some corollaries.

Corollary 2.3.3
For any $X \sim S_\alpha(\lambda, \beta, \gamma), 0 < \alpha < 2$ it holds $\mathbb{E}|X|^p < \infty$ iff $p \in (0, \alpha)$. In particular, $\mathbb{E}|X|^\alpha = +\infty$.

Proof It follows immediately from the tail asymptotic of Proposition 2.3.5 and the formula $\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(|X|^p > x)dx$. □

Proposition 2.3.6. *Let $X \sim S_\alpha(\lambda, \beta, 0)$ for $0 < \alpha < 2$, and $\beta = 0$ if $\alpha = 1$. Then $(\mathbb{E}|X|^p)^{1/p} = c_{\alpha,\beta}(p)\lambda^{1/\alpha}$, where $p \in (0, \alpha)$ and $c_{\alpha,\beta}(p)$ is a constant s.t.*

$$c_{\alpha,\beta}(p) = \frac{2^{p-1}\Gamma(1 - p/\alpha)}{p \int_0^\infty u^{-p-1} \sin^2 u du} \left(1 + \beta^2 \mathrm{tg}^2 \left(\frac{\alpha\pi}{2}\right)\right)^{p/(2\alpha)} \cos\left(\frac{p}{\alpha} \mathrm{arctg}(\beta \mathrm{tg}(\alpha\pi/2))\right).$$

Proof We show only that $(\mathbb{E}|X|^p)^{1/p} = c_{\alpha,\beta}(p)\lambda^{1/\alpha}$, where $c_{\alpha,\beta}(p) = (\mathbb{E}|X_0|^p)^{1/p}$ with $X_0 \sim S_\alpha(1, \beta, 0)$. The exact calculation of $c_{\alpha,\beta}(p)$ will be left without proof. The first statement follows from Theorem 2.3.1,4), namely, since $X \stackrel{d}{=} \lambda^{1/\alpha} X_0$. Then $(\mathbb{E}|X|^p)^{1/p} = \lambda^{1/\alpha}(\mathbb{E}|X_0|^p)^{1/p} = \lambda^{1/\alpha} c_{\alpha,\beta}(p)$. □

2.4 Limit theorems

Let us reformulate Definition 2.1.1 as follows.

Definition 2.4.1
We say that the distribution fuction F belongs to the domain of attraction of distribution function G if for a sequence of i.i.d. r.v.'s $\{X_n\}_{n \in \mathbb{N}}$, $X_n \sim F$ $\exists$ sequences of constants $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} : a_n \in \mathbb{R}, b_n > 0, \forall n \in \mathbb{N}$ s.t.

$$\frac{1}{b_n} \sum_{i=1}^n X_i - a_n \xrightarrow{d} X \sim G, n \to \infty.$$

Let us state and prove the following result.

Theorem 2.4.1
1) G has a domain of attraction iff G is a distribution function of a stable law.
2) F belongs to the domain of attraction of $N(\mu, \sigma^2), \sigma > 0$ iff

$$\mu(x) := \int_{-x}^x y^2 F(dy), x > 0$$

is slowly varying at ∞. This holds, in particular, if F has a finite second moment (then $\exists \lim_{x \to +\infty} \mu(x) = \mathbb{E}X_1^2$).
3) F belongs to the domain of attraction of α-stable law, $\alpha \in (0, 2)$, iff

$$\mu(x) \sim x^{2-\alpha} L(x), \tag{2.4.1}$$

where $L:\mathbb{R}_+\to\mathbb{R}_+$ is slowly varying at $+\infty$ and it holds the tail balance condition

$$\begin{aligned}\frac{\mathbb{P}(X_1>x)}{\mathbb{P}(|X_1|>x)}&=\frac{1-F(x)}{1-F(x)+F(-x)}\underset{x\to+\infty}{\to}p,\\ \frac{\mathbb{P}(X_1<-x)}{\mathbb{P}(|X_1|>x)}&=\frac{F(-x)}{1-F(x)+F(-x)}\underset{x\to+\infty}{\to}q\end{aligned}\tag{2.4.2}$$

for some $p,q\geq 0: p+q=1$ with $X_1\sim F$. Condition (2.4.1) is equivalent to

$$\mathbb{P}(|X_1|>x)=1-F(x)+F(-x)\underset{x\to+\infty}{\sim}x^{-\alpha}L(x).\tag{2.4.3}$$

Remark 2.4.1
a) In Definition 2.4.1, one can choose $b_n=\inf\{x:\mathbb{P}(|X_1|>x)\leq n^{-1}\}$, $a_n=n\mathbb{E}(X_1\mathbb{I}(|X_1|\leq b_n))$.
b) It is quite clear that statements 2) and 3) are special cases of the following one:
4) F belongs to the domain of attraction of an α-stable law, $\alpha\in(0,2]$, iff (2.4.1) and (2.4.2) hold.
c) It can be shown that $\{b_n\}$ in Theorem 2.4.1 must satisfy the condition $\lim_{n\to\infty}\frac{nL(b_n)}{b_n^\alpha}=\lambda c_\alpha$, with c_α as in Proposition 2.3.5. Then $\{a_n\}$ can be chosen as

$$a_n=\begin{cases}0, & \alpha\in(0,1),\\ nb_n^2\int_{\mathbb{R}}\sin(x/b_n)dF(x), & \alpha=1,\\ nb_n^2\int_{\mathbb{R}}xdF(x), & \alpha\in(1,2).\end{cases}$$

Proof of Proposition 2.3.5 We just give the sketch of the proof. It is quite clear that $S_\alpha(\lambda,\beta,\gamma)$ belongs to the domain of attraction of $S_\alpha(\lambda,\beta,0)$ with $b_n=n^{1/\alpha}$, cf. Theorems 2.1.3,2.1.4, Corollary 2.1.1 and Remark 2.1.5. Then the tail balance condition (2.4.2) holds with $p=\frac{1+\beta}{2},q=\frac{1-\beta}{2}$. By Remark 2.4.1 c), putting $b_n=n^{1/\alpha}$ into it yields that $L(x)$ in (2.4.3) has the property $\lim_{x\to+\infty}L(x)=c_\alpha\lambda$. It follows from (2.4.2) and (2.4.3) of Theorem 2.4.1 that for $x\to+\infty$

$$x^\alpha\mathbb{P}(X>x)\sim x^\alpha p\mathbb{P}(|X|>x)\sim p,\quad x^\alpha x^{-\alpha}\lim_{x\to+\infty}L(x)=pc_\alpha\lambda=c_\alpha\frac{1+\beta}{2}\lambda,$$

$x^\alpha\mathbb{P}(X<-x)\sim qc_\alpha\lambda=c_\alpha\frac{1-\beta}{2}\lambda, x\to+\infty$ is shown analogously. □

Proof of Theorem 2.4.1 F belongs to the domain of attraction of a distribution function G if, by Definition 2.4.1, $\exists$ i.i.d. r.v.'s $\{X_n\}_{n\in\mathbb{N}}, X_n\sim F$, $\{a_n\}_{n\in\mathbb{N}},\{b_n\}_{n\in\mathbb{N}}\subset\mathbb{R}: b_n>0\ \forall n$, s.t. $S_n=\frac{1}{b_n}\sum_{i=1}^n X_i-a_n=\sum_{i=1}^n\frac{X_i-a_nb_n\frac{1}{n}}{b_n}\xrightarrow{d}X\sim G, n\to\infty$. Denote $c_n=a_n\frac{1}{n}, n\in\mathbb{N}$. In terms of characteristic functions, $\varphi_{S_n}(s)\underset{n\to\infty}{\to}\varphi_X(s)\ \forall s\in\mathbb{R}$, where

$$\begin{aligned}\varphi_{S_n}(s)=\mathbb{E}\exp\left(is\sum_{k=1}^n\frac{X_k-c_nb_n}{b_n}\right)&=\prod_{k=1}^n\mathbb{E}\exp\left(is\frac{X_k-c_nb_n}{b_n}\right)\\ &\underset{X_i-i.i.d.}{=}\left(e^{-isc_n}\varphi_{X_1}(s/b_n)\right)^n.\end{aligned}$$

Put $\varphi_n(s)=\varphi_{X_1}(s/b_n), F_n(x)=F(b_nx)$. Then the statement of Theorem 2.4.1 is equivalent to

$$\left(e^{-isc_n}\varphi_n(s)\right)^n\underset{n\to\infty}{\to}\varphi_X(s),\tag{2.4.4}$$

where X is stable.

Lemma 2.4.1
Under assumptions of Theorem 2.4.1, relation (2.4.4) is equivalent to

$$n(\varphi_n(s) - 1 - ic_n s) \to \eta(s), n \to \infty \tag{2.4.5}$$

where $\eta(s)$ is a continuous function of the form $\eta(s) = isa - bs^2 + \int_{\{x \neq 0\}} (e^{isx} - 1 - is \sin x) dH(x)$ (cf. (2.1.6)) with $H(\cdot)$ from Theorem 2.1.2 and $\varphi_X(s) = e^{\eta(s)}, s \in \mathbb{R}$.

Proof 1) Show this equivalence in the symmetric case, i.e., if $X_1 \stackrel{d}{=} -X_1$. Then it is clear that we may assume $c_n = 0, \forall n \in \mathbb{N}$. Show that

$$\varphi_n^n(s) \underset{n\to\infty}{\to} e^{\eta(s)} \quad \Leftrightarrow \tag{2.4.6}$$

$$n(\varphi_n(s) - 1) \underset{n\to\infty}{\to} \eta(s), \tag{2.4.7}$$

and η is continuous. First, if a characteristic function $\varphi(s) \neq 0 \forall s : |s| < s_0$, then $\exists!$ representation $\varphi(s) = r(s)e^{i\theta(s)}$, where $\theta(\cdot)$ is continuous and $\theta(0) = 0$. Hence, $\log \varphi(s) = \log r(s) + i\theta(s)$ is well-defined, continuous, and $\log \varphi(0) = \log r(0) + i\theta(0) = \log 1 + i0 = 0$.

Let us show (2.4.7) $\Rightarrow$ (2.4.6). It follows from (2.4.7) that $\varphi_n(s) \underset{n\to\infty}{\to} 1$ and by continuity theorem for characteristic functions, this convergence is uniform in any finite interval $s \in (-s_0, s_0)$. Then, $\log \varphi_n(s)$ is well-defined for large n (since $\varphi_n(s) \neq 0$ there). Since

$$\log z = z - 1 + o((z-1)^2) \text{ for } |z - 1| < 1, \tag{2.4.8}$$

it follows $\log \varphi_n^n(s) = n \log \varphi_n(s) = n\left(\varphi_n(s) - 1 + o((\varphi_n(s) - 1)^2)\right) \underset{n\to\infty}{\sim} n(\varphi_n(s) - 1) \underset{n\to\infty}{\to} \eta(s)$ by (2.4.7). Then, $\varphi_n^n(s) \underset{n\to\infty}{\to} e^{\eta(s)}, \forall s \in \mathbb{R}$ and (2.4.6) holds.

Let us show (2.4.6) $\Rightarrow$ (2.4.7). Since $\eta(0) = 0$, then $e^{\eta}(s) \neq 0$ $\forall s \in (-s_0, s_0)$ for some $s_0 > 0$. Since the convergence of characteristic functions is uniform by continuity theorem, $\varphi_n(s) \neq 0$ for all n large enough and for $s \in (-s_0, s_0)$. Taking logarithms in (2.4.6), we get $n \log \varphi_n(s) \underset{n\to\infty}{\to} \eta(s)$. Using Taylor expansion (2.4.8), we get $n(\varphi_n(s) - 1) \underset{n\to\infty}{\to} \eta(s)$, and (2.4.7) holds.

2) Show this equivalence in the general case $c_n \neq 0$. More specifically, show that it holds if $\varphi_n(s) \underset{n\to\infty}{\to} 1$ $\forall s \in \mathbb{R}$, and $n\beta_n^2 \underset{n\to\infty}{\to} 0$, where $\beta_n = \int_{\mathbb{R}} \sin\left(\frac{x}{b_n}\right) F(dx)$. Then

$$n(\beta_n - c_n) \underset{n\to\infty}{\to} a, \tag{2.4.9}$$

and (2.4.5) writes equivalently as

$$n(\varphi_n(s) - 1 - i\beta_n s) \underset{n\to\infty}{\to} \eta(s). \tag{2.4.10}$$

Without loss of generality set $a = 0$.

Notice that the proof of 1) does not essentially depend on the symmetry of X_1, i.e., equivalence (2.4.6) $\Leftrightarrow$ (2.4.7) holds for any characteristic functions $\{\varphi_n\}$ s.t. $\varphi_n(s) \underset{n\to\infty}{\to} 1$ $\forall s \in \mathbb{R}$. Applying this equivalence to $\{\varphi_n(s)e^{-isc_n}\}_{n\in\mathbb{N}}$ leads to $n(\varphi_n(s)e^{-ic_n s} - 1) \underset{n\to\infty}{\to} \eta(s) = -bs^2 + \int_{\{x\neq 0\}} (e^{isx} - 1 - is\sin x) dH(x)$. Since we assumed that $\varphi_n(s) \underset{n\to\infty}{\to} 1$ it follows $c_n \underset{n\to\infty}{\to} 0$, while $b_n \to \infty$. Consider $\operatorname{Im}(n(\varphi_n(s) - e^{ic_n s})) \underset{n\to\infty}{\to} \operatorname{Im}(e^{ic_n s}\eta(s))$ for $s = 1$. Since $\eta(1) \in \mathbb{R}$ and $c_n \to 0$, we get $n(\operatorname{Im}\varphi_n(1) - \sin c_n) \underset{n\to\infty}{\sim} \eta(1) \sin c_n$, $\sin c_n \sim c_n$ as $c_n \to 0$, where

$\operatorname{Im}\varphi_n(1) = \operatorname{Im}\left(\int_{\mathbb{R}} e^{is/b_n} dF(x)\right)\Big|_{s=1} = \int_{\mathbb{R}} \sin(x/b_n) dF(x) = \beta_n \Rightarrow n(\beta_n - c_n) \underset{n\to\infty}{\to} 0$. Hence, relation $n(\varphi_n(s)e^{-ic_n s} - 1) \underset{n\to\infty}{\to} \eta(s)$ one can write as $n(\varphi_n(s)e^{-i\beta_n s} - 1) \underset{n\to\infty}{\to} \eta(s)$. But

$$n(\varphi_n(s)e^{-i\beta_n s} - 1) = n(\varphi_n(s) - 1 - i\beta_n s)e^{-i\beta_n s} + \underbrace{n((1 + i\beta_n s)e^{-i\beta_n s} - 1)}_{\to 0, n\to\infty},$$

since $n((1+i\beta_n s)e^{-i\beta_n s} - 1) = n((1+i\beta_n s)(1 - i\beta_n s + o(b_n)) - 1) = n(1 + \beta_n^2 s^2 + o(b_n) - 1) = n\beta_n^2 s^2 + o(nb_n) \underset{n\to\infty}{\to} 0$ by our assumption. We conclude that (2.4.4) $\Rightarrow$ (2.4.5) holds.

Conversely, if (2.4.9) and (2.4.8) hold then reading the above reasoning in reverse order we go back to (2.4.4).

Now we have to show that $\varphi_n(s) \underset{n\to\infty}{\to} 1$, $n\beta_n^2 \underset{n\to\infty}{\to} 0$. The first statement is trivial since $\varphi_n(s) = \varphi(s/b_n) \to \varphi(0) = 1$, as $b_n \to \infty$. Let us show $n\beta_n^2 = n\left(\int_{\mathbb{R}} \sin(x/b_n)F(dx)\right)^2 \underset{n\to\infty}{\to} 0$. By Corollary 2.1.1 $b_n \sim n^{1/\alpha}h(n), n \to \infty$, where $h(\cdot)$ is slowly varying at $+\infty$. It follows from (2.4.3) that $\mathbb{E}|X_1|^p < \infty$ $\forall p \in (0, \alpha)$. Then for $p \in (0, 1]$ it holds $|\beta_n| \le 2\int_0^\infty \left|\frac{x}{b_n}\right|^p dF(x) = O(|b_n|^{-p}) = O(n^{-p/\alpha}h^{-p}(n))$ and $n\beta_n^2 = O(n^{1-2p/\alpha}) \underset{n\to\infty}{\to} 0$ if β is chosen s.t. $p \in (\alpha/2, 1]$. □

Now prove the following.

Lemma 2.4.2
Conditions of Theorem 2.4.1 are necessary and sufficient for relation (2.4.5) to hold with some special sequences of constants $\{b_n\}, \{c_n\}$.

If this lemma is proven, then the proof of Theorem 2.4.1 is complete, since by Lemma 2.4.1 relation (2.4.5) and (2.4.4) are equivalent, and thus F belongs to the domain of attraction of some $\alpha-$stable law.

Proof of Lemma 2.4.2. Let relation (2.4.5) holds with some $b_n > 0$ and a_n. This means, equivalently, that $S_n \underset{n\to\infty}{\overset{d}{\to}} X \sim G$. Since the case $X \sim N(0,1)$ is covered by the CLT, let us exclude it as well as the trivial case $X \equiv const$. By Theorem 2.1.2-2.1.3 with $k_n = n$ $X_{n_j} = X_j/b_n$, $a_n = A_n(y) - a - \int_{|u|<y} u dH(u) + \int_{|u|\ge y} \frac{1}{u} dH(u)$, $X_1 \sim F$,

$$A_n(y) = n\mathbb{E}\left(\frac{X_1}{b_n}\mathbb{I}(|X_1|/b_n < y)\right) = \frac{n}{b_n}\mathbb{E}(X_1\mathbb{I}(|X_1| < b_n y))\frac{n}{b_n}\int_{-yb_n}^{yb_n} x dF(x),$$

$\pm y$ being continuity points of H, it follows that $\begin{cases} n(F(xb_n) - 1) \underset{n\to\infty}{\to} H(x), & x > 0, \\ nF(xb_n) \underset{n\to\infty}{\to} H(x), & x < 0, \end{cases}$ and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{b_n^2}\left(\int_{-\varepsilon b_n}^{\varepsilon b_n} x^2 dF(x) - \left(\int_{-\varepsilon b_n}^{\varepsilon b_n} x^2 dF(x)\right)^2\right) = b. \tag{2.4.11}$$

1) Show that $b_n \to +\infty$, $\frac{b_{n+1}}{b_n} \underset{n\to\infty}{\to} 1$, if $X \not\equiv const$ a.s. By Remark 2.1.3 2), it holds property (2.1.5), i.e., $\lim_{n\to\infty}\mathbb{P}(|X_1| < c) = 1$ then the central limit theorem can be applied to $\{X_n\}$ with

$$\frac{\sum_{i=1}^n X_i - n\mathbb{E}X_1}{\sqrt{n}\sqrt{\operatorname{Var}X_1}} \underset{n\to\infty}{\overset{d}{\to}} N(0,1)$$

and it is not difficult to show (see Exercise 2.3.2 below) that $b_n = const\sqrt{n} \to \infty$ in this case. If $\nexists\ c > 0 : \mathbb{P}(|X_1| < c) = 1$ then $b_n \neq O(1), n \to \infty$ since that would contradict $\lim_{n\to\infty} \mathbb{P}(|X_1| > b_n\varepsilon) = 0 \Rightarrow \exists\{n_k\}, n_k \to \infty$ as $k \to \infty : b_{n_k} \to +\infty$. W.l.o.g. identify sequences $\{n\}$ and $\{n_k\}$. Alternatively, one can agree that $\{S_n\}$ is stochastically bounded (which is the case if $S_n \underset{n\to\infty}{\overset{d}{\to}} X$) iff $b_n \to +\infty$.

Exercise 2.4.1
Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of c.d.f. s.t. $F_n(\alpha_n \cdot + \beta_n) \overset{d}{\to} U(\cdot), n \to \infty$, $F_n(\gamma_n \cdot + \delta_n) \overset{d}{\to} V(\cdot)$, $n \to \infty$ for some sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ s.t. $\alpha_n\gamma_n > 0$, where U and V are c.d.f.'s which are not concentrated at one point. Then

$$\frac{\gamma_n}{\alpha_n} \underset{n\to\infty}{\to} a \neq 0, \quad \frac{\delta_n - \beta_n}{\alpha_n} \underset{n\to\infty}{\to} b$$

and $V(x) = U(ax + b)$, $\forall x \in \mathbb{R}$.

Now show that $\frac{b_{n+1}}{b_n} \to 1, n \to \infty$. Since $S_n \underset{n\to\infty}{\overset{d}{\to}} X \not\equiv const$, it holds $S_{n+1} \underset{n\to\infty}{\overset{d}{\to}} X$, $\frac{X_{n+1}}{b_{n+1}} = S_{n+1} - S_n \underset{n\to\infty}{\overset{d}{\to}} 0 \Rightarrow \frac{X_{n+1}}{b_{n+1}} \overset{P}{\to} 0, n \to \infty$. Thus, $\frac{1}{b_{n+1}} S_n - a_{n+1} \underset{n\to\infty}{\overset{d}{\to}} X$ and $\frac{1}{b_n} S_n - a_n \underset{n\to\infty}{\overset{d}{\to}} X$, which means by Exercise 2.4.1, that $\frac{b_{n+1}}{b_n} \underset{n\to\infty}{\to} 1$.

2) Prove the following.

Proposition 2.4.1. *Let $\beta_n \underset{n\to\infty}{\to} +\infty$, $\frac{\alpha_{n+1}}{\alpha_n} \underset{n\to\infty}{\to} 1$. Let U be a monotone function s.t.*

$$\lim_{n\to\infty} \alpha_n U(\beta_n x) = \psi(x) \tag{2.4.12}$$

exists on a dense subset of $\mathbb{R}_+$, where $\psi(x) \in (0, +\infty)$ on some interval I, then U is regularly varying at $+\infty$, $\psi(x) = cx^\rho, \rho \in \mathbb{R}$.

Proof W.l.o.g. set $\psi(1) = 1$, and assume that U is non-decreasing and (2.4.12) holds for $x = 1$ (otherwise, a scaling in x can be applied). Set $n = \min\{k \in \mathbb{N}_0 : \beta_{k+1} > t\}$. Then it holds $\beta_n \leq t < \beta_{n+1}$, and

$$\psi(x) \underset{n\to\infty}{\sim} \frac{\lambda_n U(\beta_n x)}{\lambda_{n+1} U(\beta_{n+1})} \underset{n\to\infty}{\sim} \frac{U(\beta_n x)}{U(\beta_{n+1})} \leq \frac{U(tx)}{U(t)}$$
$$\leq \frac{U(\beta_{n+1} x)}{U(\beta_n)} \underset{n\to\infty}{\sim} \frac{\lambda_{n+1} U(\beta_{n+1} x)}{\lambda_n U(\beta_n)} \underset{n\to\infty}{\sim} \frac{\psi(x)}{\psi(1)} = \psi(x)$$

for all x, for which (2.4.12) holds. The application of Lemma 2.1.1 finishes the proof. □

3) Apply Proposition 2.4.1 to $\begin{cases} n(F(xb_n) - 1) \to H(x), & x > 0 \\ nF(-xb_n) \to H(-x), & x < 0 \end{cases}$ as $n \to \infty$ with $\alpha_n = n, \beta_n = b_n \Rightarrow 1 - F(x) = \mathbb{P}(X_1 > x), F(-x) = \mathbb{P}(X_1 < -x)$ are regularly varying at $+\infty$, and $H(x) = c_1 x^{\rho_1}, H(-x) = c_2 x^{\rho_2}$,

$$\mathbb{P}(X_1 > x) \sim x^{\rho_1} L_1(x), \quad \mathbb{P}(X_1 < -x) \sim x^{\rho_2} L_2(x), x \to +\infty, \tag{2.4.13}$$

where L_1, L_2 are slowly varying at $+\infty$.

Since (2.4.11) holds, $\lim_{n\to\infty} \frac{n}{b_n^2} \left(\mu(\varepsilon b_n) - \left(\int_{-\varepsilon b_n}^{\varepsilon b_n} x dF(x) \right)^2 \right)$ is a bounded function of ε in the neighborhood of zero, hence by Proposition 2.4.1 with $\alpha_n = \frac{n}{b_n^2}, \beta_n = b_n$, $\mu(x) - \left(\int_{-x}^{x} y dF(y) \right)^2$

is regularly varying at $+\infty$. By Theorem 2.1.4, $\rho_1 = \rho_2 = -\alpha$, $c_1 < 0, c_2 > 0$, and evidently, $\mathbb{P}(|X_1| > x) = 1 - F(x) + F(-x) \underset{x\to+\infty}{\sim} x^{-\alpha} \underbrace{(L_1(x) + L_2(x))}_{L(x)}$, so (2.4.3) holds.

Exercise 2.4.2
Show that then $\mu(x) \underset{x\to+\infty}{\sim} x^{2-\alpha} L_3(x)$ is equivalent to (2.4.3). Show that tail balance condition (2.4.2) follows from (2.4.13) with $\rho_1 = \rho_2 = -\alpha$.

So we have proven that (2.4.5) $\Rightarrow$ (2.4.2),(2.4.3) (or, equivalently, (2.4.1),(2.4.2)). Now let us prove the inverse statement.

4) Let (2.4.1) hold. Since L_1 is slowly varying, one can find a sequence $\{b_n\}, b_n \to \infty, n \to \infty$ s.t. $\frac{n}{b_n^\alpha} L(b_n) \underset{n\to\infty}{\to} c > 0$ – some constant. (Compare Remark 2.4.1, c).) Then $\frac{n}{b_n^2}\mu(b_n x) \underset{n\to\infty}{\sim} \frac{n}{b_n^2}(b_n x)^{2-\alpha} L(b_n x) = \frac{n}{b_n^\alpha} L(b_n x) x^{-\alpha} \underset{n\to\infty}{\sim} c x^{-\alpha}, x > 0$ and hence

$$\begin{cases} n(F(xb_n) - 1) \underset{n\to\infty}{\to} c_1 x^{-\alpha}, \\ nF(-xb_n) \underset{n\to\infty}{\to} c_2 x^{-\alpha}. \end{cases} \tag{2.4.14}$$

Exercise 2.4.3
1) Show the last relation. Then 1) of Theorem 2.1.3 holds.
2) Prove that 2) of Theorem 2.1.3 holds as well, as a consequence of $\frac{n}{b_n^2}\mu(b_n x) \underset{n\to\infty}{\sim} c x^{-\alpha}$ and (2.4.14).

Then, by Theorem 2.1.3 $S_n \underset{n\to\infty}{\overset{d}{\to}} X$, and (2.4.5) holds. Lemma 2.4.2 is proven. □

The proof of Theorem 2.4.1 is thus complete. Part a) and the second half of part c) of Remark 2.4.1 will remain unproven. □

2.5 Further properties of stable laws

Proposition 2.5.1. *Let $X \sim S_\alpha(\lambda, \beta, \gamma)$ with $\alpha \in (1, 2]$. Then $\mathbb{E}X = \lambda\gamma$.*

In addition to a proof a using the law of large numbers, (see Exercise 4.1.14) let us give an alternative proof here.

Proof By Corollary 2.3.3, $\mathbb{E}|X| < \infty$ if $\alpha \in (1, 2)$. For $\alpha = 2$ X is Gaussian and hence $\mathbb{E}|X| < \infty$ is trivial. Let $\mu = \lambda\gamma$. By Remark 2.3.1 5), $X - \mu$ is strictly stable, i.e., $X_1 - \mu + X_2 - \mu \overset{d}{=} c_2(X - \mu)$ by Definition 2.1.3, where $X_1 \overset{d}{=} X_2 \overset{d}{=} X$, all independent r.v.'s. Taking expectations on both sides yields $2\mathbb{E}(X - \mu) = c_2\mathbb{E}(X - \mu)$. Since $c_n = n^{1/\alpha}$ by Remark 2.1.5, $c_2 = 2^{1/\alpha}$, and hence $\mathbb{E}(X - \mu) = 0 \Rightarrow \mathbb{E}X = \mu$. □

Now we go on to show series representation of stable random variables. Some preparatory definitions are in order.

Definition 2.5.1
Let X and Y be two random variables defined possibly on different probability space. One says that X is *a representation* of Y if $X \overset{d}{=} Y$.

Definition 2.5.2
Let $\{T_i\}_{i\in\mathbb{N}}$ be the sequence of i.i.d. $Exp(\lambda)$-distributed random variables with $\lambda > 0$. Set $\tau_n = \sum_{i=1}^n T_i\ \forall n \in \mathbb{N}$, $\tau_0 = 0$, and $N(t) = \max\{n \in \mathbb{N}_0 : \tau_n \le t\}, t \ge 0$. The random process $N = \{N(t), t \ge 0\}$ is called *Poisson* with intensity λ. Time instants τ_i are called *arrival times*, T_i are *interarrival times.*

Exercise 2.5.1
Prove the following properties of a Poisson process N :

1. $N(t) \sim Poisson(\lambda t), t > 0$, and, in particular, $\mathbb{E}N(t) = \mathrm{Var}N(t) = \lambda t$.
2. Let $N_i = \{N_i(t), t \ge 0\}$ be two independent Poisson processes with intensities $\lambda_i, i = 1, 2$. Then $\{N_1(t) + N_2(t), t \ge 0\}$ is a Poisson process with intensity $\lambda_1 + \lambda_2$ (which is called the *superposition* $N_1 + N_2$ of N_1 and N_2.)
3. $\tau_n \sim \Gamma(n, \lambda), n \in \mathbb{N}$, where $\Gamma(n, \lambda)$ is a Gamma distribution with parameters n and λ, $\mathbb{E}\tau_n = n/\lambda$.

Clearly, all T_n are dependent random variables.

Proposition 2.5.2. *Let $N = \{N(t), t \ge 0\}$ be a Poisson process with intensity one ($\lambda = 1$) and arrival times $\{\tau_n\}_{n\in\mathbb{N}}$. Let $\{R_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables, independent of $\{\tau_n\}_{n\in\mathbb{N}}$. Then $X = \sum_{n=1}^\infty \tau_n^{-1/\alpha} R_n$ is a strictly $\alpha-$stable random variable provided that $\alpha \in (0, 2]$ and this series converges a.s.*

Proof Let $X_i = \sum_{n=1}^\infty (\tau_n^{(i)})^{-1/\alpha} R_n^{(i)}, i = 1, 2, 3$ be three independent copies of X, where $\{R_n\} \stackrel{d}{=} \{R_n^{(i)}\}, i = 1, 2, 3$, $\{\tau_n\} \stackrel{d}{=} \{\tau_n^{(i)}\}, i = 1, 2, 3$, and all three sequences are independent. By Definition 2.1.4 and Remark 2.1.5, it suffices to show that $\begin{cases} X_1 + X_2 \stackrel{d}{=} 2^{1/\alpha} X, \\ X_1 + X_2 + X_3 \stackrel{d}{=} 3^{1/\alpha} X, \end{cases}$ $2^{1/\alpha} X = \sum_{n=1}^\infty (\tau_n/2)^{-1/\alpha} R_n$, where $\{\tau_n/2\}_{n\in\mathbb{N}}$ forms a Poisson process $2N$ with intensity $\lambda = 2$, since $T_n/2 = (\tau_n - \tau_{n-1})/2\ \forall n$, and $\mathbb{P}(T_n/2 \ge x) = \mathbb{P}(T_n \ge 2x) = \exp(-2x), x \ge 0$. It is clear that $X_1 + X_2 = \sum_{n=1}^\infty (\tau'_n)^{-1/\alpha} R'_n$, where $\{\tau'_n\}$ are arrival times of the superposition $N_1 + N_2$ (being a Poisson process of intensity 2, cf. Exercise 2.5.1), and $R'_n = \begin{cases} R_k^{(1)}, & \text{if } \tau'_n = \tau_k^{(1)} \text{ for some } k \in \mathbb{N} \\ R_k^{(2)}, & \text{if } \tau'_n = \tau_m^{(2)} \text{ for some } m \in \mathbb{N}. \end{cases}$ Since $\{R_n\}_{n\in\mathbb{N}} \stackrel{d}{=} \{R'_n\}_{n\in\mathbb{N}}$, and $N_1 + N_2 \stackrel{d}{=} 2N$, we have $X_1 + X_2 \stackrel{d}{=} X$, so we are done. For $X_1 + X_2 + X_3$, the proof is analogous. □

In order to get a series representation of a $S\alpha S$ random variable X, we'll have to ensure the a.s. convergence of this series. For that, we impose restrictions on $\alpha \in (0, 2)$ and on $\{R_n\}$: we assume $R_n = \varepsilon_n W_n$, where $\varepsilon_n = \mathrm{sign}(R_n) = \begin{cases} +1, & \text{if } R_n > 0, \\ -1, & \text{if } R_n \le 0, \end{cases}$ $W_n = |R_n|, \mathbb{E}W_n^\alpha < \infty$.

Theorem 2.5.1 (LePage representation):
Let $\{\varepsilon_n\}, \{W_n\}, \{\tau_n\}$ be independent sequences of random variables, where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ are i.i.d. Rademacher random variables, $\varepsilon_n = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2 \end{cases}$, $\{W_n\}_{n\in\mathbb{N}}$ are i.i.d. random variables with $\mathbb{E}|W_n|^\alpha < \infty, \alpha \in (0, 2)$, and $\{\tau_n\}_{n\in\mathbb{N}}$ is the sequence of arrival times of a unit rate Poisson process N ($\lambda = 1$).

Then $X \overset{a.s.}{:=} \sum_{n=1}^{\infty} \varepsilon_n \tau_n^{-1/\alpha} W_n \sim S_\alpha(\sigma, 0, 0)$, where this series converges a.s., $\sigma = \frac{\mathbb{E}|W_1|^\alpha}{c_\alpha}$, and c_α is a constant introduced in Proposition 2.3.5.

Remark 2.5.1
1) Proposition 2.5.2 yields the fact that $X \sim S\alpha S$, but it does not give insights into the value of σ.

2) Since the distribution of X depends only on $\mathbb{E}|W_1|^\alpha$, it does not matter, which $\{W_n\}$ we choose. A usual choice can be $W_n \sim U[0,1]$, or $W_n \sim N(0,1)$. Hence, W_n do not need to be non-negative, as in the comment before Theorem 2.5.1.

3) The LePage representation is not used to simulate stable variables, since the convergence of the series is rather slow. Indeed, methods in Chapter 3 are widely used.

4) Skewed stable variables have another series representation which will be given (without proof) in Theorem 2.5.3 below.

5) It follows directly from Theorem 2.5.1 that for any $S\alpha S$ random variable $X \sim S_\alpha(\lambda, 0, 0)$, it has the LePage representation $X \overset{d}{=} \left(\frac{c_\alpha \lambda}{\mathbb{E}|W_1|^\alpha}\right)^{1/\alpha} \sum_{n=1}^{\infty} \varepsilon_n \tau_n^{-1/\alpha} W_n$, where sequences $\{\varepsilon_n\}, \{W_n\}, \{\tau_n\}$ are chosen as above. In particular, choosing the law of W_1 s.t. $\mathbb{E}|W_1|^\alpha = \lambda$ reduces the representation to $X \overset{d}{=} c_\alpha^{1/\alpha} \sum_{n=1}^{\infty} \varepsilon_n \tau_n^{-1/\alpha} W_n$. Since $\tau_n \uparrow$ a.s. as $n \to \infty$, the terms $\varepsilon_n \tau_n^{-1/\alpha} W_n \downarrow$ stochastically, and one can show that the very first term $\varepsilon_1 \tau_1^{-1/\alpha} W_1$ dominates the whole tail behaviour of X. In more details, by Proposition 2.3.5, it holds $\mathbb{P}(X > x) \underset{x\to\infty}{\sim} \frac{1}{2} c_\alpha \lambda x^{-\alpha}$, and it is not difficult to see that

a) $\mathbb{P}(c_\alpha^{1/\alpha} \varepsilon_1 \tau_1^{-1/\alpha} W_1 > x) \sim \frac{1}{2} c_\alpha \lambda x^{-\alpha}$ as $x \to +\infty$,

b) $\mathbb{P}(\sum_{n=0}^{\infty} \varepsilon_n \tau_n^{-1/\alpha} W_n > x) = o(x^{-\alpha})$ as $x \to +\infty$.

Exercise 2.5.2
Prove the statement a) of Remark 2.5.1.

Proof of Theorem 2.5.1 1) Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. $U[0,1]-$distributed random variables, independent of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{W_n\}_{n\in\mathbb{N}}$. Then $\{Y_n\}_{n\in\mathbb{N}}$ given by $Y_n = \varepsilon_n U_n^{-1/\alpha} W_n, n \in \mathbb{N}$ is a sequence of symmetric i.i.d. random variables. Let us show that the law of Y_1 lies in the domain of attraction of a $S\alpha S$ random variable. For that, compare its tail probability

$$\begin{aligned}\mathbb{P}(|Y_1| > x) &= \mathbb{P}(U_1^{-1/\alpha}|W_1| > x) = \mathbb{P}(U_1 < x^{-\alpha}|W_1|^\alpha) \\ &= \int_0^\infty \mathbb{P}(U_1 < x^{-\alpha}\omega^\alpha) dF_{|W|}(\omega) = \int_0^x x^{-\alpha}\omega^\alpha dF_{|W|}(\omega) + \int_x^\infty dF_{|W|}(\omega) \\ &= x^{-\alpha}\int_0^x \omega^\alpha dF_{|W|}(\omega) + \mathbb{P}(|W_1| > x),\end{aligned}$$

where $F_{|W|}(x) = \mathbb{P}(|W_1| \le x)$. So,

$$\lim_{x\to+\infty} x^\alpha \mathbb{P}(|Y_1| > x) = \underbrace{\int_0^\infty \omega^\alpha dF_{|W|}(\omega)}_{\mathbb{E}|W_1|^\alpha} + \underbrace{\lim_{x\to+\infty} x^\alpha \mathbb{P}(|W_1| > x)}_{=0, \text{ since } \mathbb{E}|W_1|^\alpha < \infty} = \mathbb{E}|W_1|^\alpha.$$

Hence, condition (2.4.3) of Theorem 2.4.1 is satisfied. Due to symmetry of Y_1, tail balance condition (2.4.2) is obviously true with $p = q = 1/2$. Then, by Theorem 2.4.1 and Corollary 2.1.1, it holds $\frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} Y_k \underset{n\to\infty}{\overset{d}{\to}} X \sim S_\alpha(\sigma, 0, 0)$, where the parameters (λ, β, γ) of the limiting stable law come from the proof of Theorem 2.1.1 with $c_1 = c_2 = \frac{\mathbb{E}|W_1|^\alpha}{2}$ (due to the symmetry of Y_1 and X).

2) Rewrite $\frac{1}{n^{1/\alpha}}\sum_{k=1}^n Y_k$ to show that its limiting random variable X coincides with $\sum_{k=1}^\infty \varepsilon_k \tau_k^{-1/\alpha} W_k$.

Exercise 2.5.3

Let N be the Poisson process with intensity $\lambda > 0$ built upon arrival times $\{\tau_n\}_{n\in\mathbb{N}}$. Show that

a) under the condition $\{\tau_{n+1} = t\}$ it holds $(\tau_{1/t}, \dots, \tau_{n/t}) \stackrel{d}{=} (u_{(1)}, \dots, u_{(n)})$, where $u_{(k)}, k = 1, \dots, n$ are order statistics of a sample $(u_1, \dots, u_n)$ with $u_k \sim U(0,1)$ being i.i.d. random variables.

b) $\left(\frac{\tau_1}{\tau_{n+1}}, \dots, \frac{\tau_n}{\tau_{n+1}}\right) \stackrel{d}{=} (u_{(1)}, \dots, u_{(n)})$.

Reorder the terms Y_k in the sum $\sum_{k=1}^n Y_k$ in order of ascending u_k, so to have $\sum_{k=1}^n \varepsilon_k u_{(k)}^{-1/\alpha} W_k$. Since W_k and ε_k are i.i.d., this does not change the distribution of the whole sum. Then

$$\frac{1}{n^{1/\alpha}}\sum_{k=1}^n Y_k \stackrel{d}{=} \frac{1}{n^{1/\alpha}}\sum_{k=1}^n \varepsilon_k u_{(k)}^{-1/\alpha} W_k \stackrel{d}{=} \frac{1}{n^{1/\alpha}}\sum_{k=1}^n \varepsilon_k \left(\frac{\tau_k}{\tau_{n+1}}\right)^{-1/\alpha} W_k$$

by Exercise 2.5.3 b). Then, by part 1), $\underbrace{\left(\frac{\tau_{n+1}}{n}\right)^{1/\alpha}\sum_{k=1}^n \varepsilon_k \tau_k^{-1/\alpha} W_k}_{=:S_n} \xrightarrow[n\to\infty]{d} X$ with X as above.

3) Show that $S_n \xrightarrow{d} \sum_{k=1}^\infty \varepsilon_k \tau_k^{-1/\alpha} W_k$, then we are done, since then $S_\alpha(\sigma, 0, 0) \sim X \stackrel{d}{=} \sum_{k=1}^\infty \varepsilon_k \tau_k^{-1/\alpha} W_k$. By the strong law of large numbers, it holds $\frac{\tau_{n+1}}{n} = \frac{\tau_{n+1}}{n+1}\frac{n+1}{n} = \frac{\sum_{i=1}^{n+1} T_i}{n+1}\frac{n+1}{n} \xrightarrow{a.s.} \mathbb{E}T_1 = 1$, as $n \to \infty$, since the Poisson process N has the unit rate, and $T_1 \sim Exp(1)$. Then $\mathbb{P}(A) = 1$, where $A = \{\lim_{n\to\infty}\frac{\tau_n}{n} = 1\} \cap \tau_1 > 0$. Let us show that $\forall \omega \in A$ $\sum_{k=1}^\infty \varepsilon_k(\omega)(\tau_k(\omega))^{-1/\alpha} W_k(\omega) < \infty$. Apply the following three-series theorem by Kolmogorov (without proof).

Theorem 2.5.2 (Three-series theorem by Kolmogorov):

Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables. Then $\sum_{n=1}^\infty Y_n < \infty$ a.s. iff $\forall s > 0$

a) $\sum_{n=1}^\infty \mathbb{P}(|Y_n| > s) < \infty$

b) $\sum_{n=1}^\infty \mathbb{E}(Y_n \mathbb{I}(|Y_n| \le s)) < \infty$

c) $\sum_{n=1}^\infty \mathrm{Var}(Y_n \mathbb{I}(|Y_n| \le s)) < \infty$

See the proof in [1, Theorem IX.9.2.]

Let us check conditions a)-c) above. $\forall s > 0$

a) $\sum_{n=1}^\infty \mathbb{P}(|\varepsilon_n \tau_n^{-1/\alpha} W_n| > s) = \sum_{n=1}^\infty \mathbb{P}(|W_n|^\alpha > s^\alpha \tau_n) \le \sum_{n=1}^\infty \mathbb{P}(|W_1|^\alpha > s^\alpha c_1 n) < \infty$, since $\exists c_1, c_2 > 0 : c_1 n \le \tau_n(\omega) \le c_2 n \ \forall n > N(\omega)$ (due to $\frac{\tau_n(\omega)}{n} \xrightarrow[n\to\infty]{d} 1 \ \forall \omega \in A$) and $\mathbb{E}|W_1|^\alpha < \infty$ by assumptions.

b) It holds $\mathbb{E}\left[\varepsilon_n \tau_n^{-1/\alpha} W_n \mathbb{I}(|\varepsilon_n \tau_n^{-1/\alpha} W_n| \le s)\right] = \underbrace{\mathbb{E}\varepsilon_n}_{=0} \mathbb{E}\left[\tau_n^{-1/\alpha} W_n \mathbb{I}(|\tau_n^{-1/\alpha} W_n| \le s)\right] = 0$ by independence of ε_n from τ_n and W_n, and by symmetry of $\{\varepsilon_n\}$. Then $\sum_{n=1}^\infty \mathbb{E}\left[\varepsilon_n \tau_n^{-1/\alpha} W_n \mathbb{I}(|\varepsilon_n \tau_n^{-1/\alpha} W_n| \le s))\right] = 0 < \infty$.

c)

$$\sum_{n=1}^{\infty}\mathrm{Var}\left[\varepsilon_n\tau_n^{-1/\alpha}W_n\mathbb{I}(|\varepsilon_n\tau_n^{-1/\alpha}W_n|\le s)\right]\overset{\text{by b)}}{=}\sum_{n=1}^{\infty}\mathbb{E}\left[\tau_n^{-2/\alpha}W_n^2\mathbb{I}(|\tau_n^{-1/\alpha}W_n|\le s)\right]$$
$$\le\sum_{n=1}^{\infty}c_1^{-2/\alpha}n^{-2/\alpha}\mathbb{E}\left[W_1^2\mathbb{I}(|W_1|\le s(c_2n)^{1/\alpha})\right]=c_1^{-2/\alpha}\sum_{n=1}^{\infty}n^{-2/\alpha}\int_0^{s(c_2n)^{1/\alpha}}w^2dF_{|W|}(w)$$
$$\le c_3\int_0^{\infty}x^{-2/\alpha}\int_0^{s(c_2x)^{1/\alpha}}w^2dF_{|W|}(w)dx\overset{Fubini}{=}c_3\int_0^{\infty}w^2dF_{|W|}(w)\int_{s^{-\alpha}c_2^{-1}w^{\alpha}}^{\infty}x^{-2/\alpha}dx$$
$$=c_4\int_0^{\infty}w^{\alpha}dF_{|W|}(w)=c_4\mathbb{E}|W_1|^{\alpha}<\infty,$$

where $c_3, c_4 > 0$.

Hence, by Theorem 2.5.2 $S_n\xrightarrow[n\to\infty]{d}\sum_{k=1}^{\infty}\varepsilon_k\tau_k^{-1/\alpha}W_k<\infty$ a.s. and $X\overset{d}{=}\sum_{k=1}^{\infty}\varepsilon_k\tau_k^{-1/\alpha}W_k\sim S_\alpha(\sigma,0,0)$. □

Theorem 2.5.3 (LePage representation for skewed stable variables):
Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables and let $N=\{N(t),t\ge 0\}$ be a unit rate Poisson process with arrival times $\{\tau_n\}_{n\in\mathbb{N}}$, independent of $\{W_n\}_{n\in\mathbb{N}}$. Assume $\mathbb{E}|W_1|^{\alpha}<\infty,\alpha\in(0,2),\alpha\ne 1$, and $\mathbb{E}|W_1\log(|W_1|)|<\infty,\alpha=1$. Then $X:=\sum_{n=1}^{\infty}(\tau_n^{-1/\alpha}W_n-\kappa_n^{(\alpha)})\sim S_\alpha(\lambda,\beta,0)$, where this convergence is a.s., $\lambda=\frac{\mathbb{E}|W_1|^{\alpha}}{c_\alpha}$ with c_α being a constant introducing in Proposition 2.3.5, $\beta=\frac{\mathbb{E}(|W_1|^{\alpha}\mathrm{sign}W_1)}{\mathbb{E}|W_1|^{\alpha}}$, and

$$\kappa_n^{(\alpha)}=\begin{cases}0, & 0<\alpha<1,\\ \mathbb{E}\left(W_1\int_{|W_1|/n}^{|W_1|/(n-1)}\frac{\sin x}{x^2}dx\right), & \alpha=1\\ \frac{\alpha}{\alpha-1}\left(n^{\frac{\alpha-1}{\alpha}}-(n-1)^{\frac{\alpha-1}{\alpha}}\right)\mathbb{E}W_1, & \alpha>1.\end{cases}$$

If $\alpha=1$, then

$$X:=\sum_{n=1}^{\infty}\left(T_n^{-1}W_n-\mathbb{E}\left(W_1\right)\int_{1/n}^{1/(n-1)}\frac{\sin x}{x^2}dx\right)\sim S_1(\lambda,\beta,\gamma),\qquad(2.5.1)$$

with λ and β as above, and $\gamma=-\frac{1}{\lambda}\mathbb{E}(W_1\log|W_1|)$.

Proof see [3, §1.5.] □

Some remarks are in order.

Remark 2.5.2
1) the statement of Theorem 2.5.3 can be easily converted into a representation: a random variable $X\sim S_\alpha(\lambda,\beta,\gamma),0<\alpha<2$, has a representation $X\overset{d}{=}\lambda\gamma+\sum_{n=1}^{\infty}(\tau_n^{-1/\alpha}W_n-\kappa_n^{(\alpha)})$, where the i.i.d. random variables $\{W_n\}_{n\in\mathbb{N}}$ satisfy $\mathbb{E}|W_1|^{\alpha}=c_\alpha\lambda,\mathbb{E}(|W_1|^2\mathrm{sign}W_1)=c_\alpha\beta\lambda$. Apart from this restrictions on $\{W_n\}_{n\in\mathbb{N}}$, the choice of their distribution is deliberate.

2) Theorem 2.5.1 is a special case of Theorem 2.5.3 if we replace W_n by ε_nW_n, where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ are independent of $\{W_n\}_{n\in\mathbb{N}}$ i.i.d. random variables.

3) The LePage representation of a stable subordinator $X \sim S_\alpha(\lambda, 1, 0), \lambda > 0, \alpha \in (0, 1)$, follows easily from Theorem 2.5.3. Indeed, set $W_n = 1, \forall n$. Then, $\sum_{n=1}^{\infty} \tau_n^{-1/\alpha} \sim S_\alpha(c_\alpha^{-1}, 1, 0)$, so $X \stackrel{d}{=} \lambda^{1/\alpha} c_\alpha^{1/\alpha} \sum_{n=1}^{\infty} \tau_n^{-1/\alpha}$.

4) For $\alpha \geq 1$, the series $\sum_{n=1}^{\infty} \tau_n^{-1/\alpha} W_n$ diverges in general, if W_n are not symmetric. Hence, the correction $\kappa_n^{(\alpha)}$ is needed, which is of order of the $\mathbb{E}(W_n \tau_n^{-1/\alpha})$. Indeed, for $\lambda > 1$ $\mathbb{E}(\tau_n^{-1/\alpha} W_n) = \mathbb{E}\tau_n^{-1/\alpha} \mathbb{E} W_n \sim n^{-1/\alpha} \mathbb{E} W_1 \sim \kappa_n^{(\alpha)}$. Analogously, for $\alpha = 1$ $\mathbb{E}(\tau_n^{-1} W_n) \sim n^{-1} \mathbb{E} W_1 \sim \int_{1/n}^{1/(n-1)} \frac{\sin x}{x^2} dx \cdot \mathbb{E} W_1$ as in (2.5.1).

The following result yields the integral form of the cumulative distribution function of a $S\alpha S$ law.

Theorem 2.5.4
1) Let $X \sim S_\alpha(1, 0, 0)$ be a $S\alpha S$ random variable, $\alpha \neq 1, \alpha \in (0, 2]$. Then

$$\frac{1}{\pi} \int_0^{\pi/2} \exp\left(-x^{\frac{\alpha}{\alpha-1}} \kappa_\alpha(t)\right) dt = \begin{cases} \mathbb{P}(0 \leq X \leq x), & \alpha \in (0, 1), \\ \mathbb{P}(X > x), & \alpha \in (1, 2] \end{cases}$$

for $x > 0$, where

$$\kappa_\alpha(t) = \left(\frac{\sin(\alpha t)}{\cos t}\right)^{\frac{\alpha}{1-\alpha}} \frac{\cos((1-\alpha)t)}{\cos t}, t \in \left(0, \frac{\pi}{2}\right].$$

2) Let $X \sim S_\alpha(1, 1, 0), \alpha \in (0, 1]$. Then

$$\mathbb{P}(X \leq x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp\left(-x^{\frac{\alpha}{\alpha-1}} \bar{\kappa}_\alpha(t)\right) dt, x > 0,$$

where

$$\bar{\kappa}_\alpha(t) = \left(\frac{\sin(\alpha(\pi/2 + t))}{\sin(\pi/2 + t)}\right)^{\frac{\alpha}{1-\alpha}} \frac{\sin((1-\alpha)(\pi/2 + t))}{\sin(\pi/2 + t)}, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

See [6, Remark 1 p.78.] and [5, formula (4.5.2)].

3 Simulation of stable variables

In general, the simulation of stable laws can be demanding. However, in some particular cases, it is quite easy.

Proposition 3.0.1 (Lévy distribution). *Let $X \sim S_{1/2}(\lambda, 1, \gamma)$. Then X can be simulated by representation $X \stackrel{d}{=} \lambda^2 Y^{-2} + \lambda\gamma$, where $Y \sim N(0,1)$.*

Proof It follows from Exercise 1.0.6,1) and Theorem 2.3.1, 3),4). □

Proposition 3.0.2 (Cauchy distribution). *Let $X \sim S_1(\lambda, 0, \gamma)$. Then X can be simulated by representations*
1) $X \stackrel{d}{=} \lambda \frac{Y_1}{Y_2} + \lambda\gamma$, where Y_1 and Y_2 are i.i.d. $N(0,1)$ random variables,
2) $X \stackrel{d}{=} \lambda \mathrm{tg}(\pi(U - 1/2)) + \lambda\gamma$, where $U \sim Uniform[0,1]$.

Proof 1) Use Exercise 4.1.29 and the scaling properties of stable laws given in Theorem 2.3.1, 3),4).

2) By Example 1.0.2 it holds $\mathrm{tg}Y \stackrel{d}{=} Z, Y \sim U[-\pi/2, \pi/2] \stackrel{d}{=} \pi(U - 1/2), Z \sim Cauchy(0,1) \sim S_1(1,0,0)$. Then use again Theorem 2.3.1, 3),4) to get $X \stackrel{d}{=} \lambda Z + \lambda\gamma$. □

Now we reduced the simulation of Lévy and Cauchy laws to the simulation of $U[0,1]$ and $N(0,1)$ random variables. A realisation of a $U[0,1]$ is given by generators of pseudorandom numbers built into any programming language. The simulation of $N(0,1)$ is more involved, and we give it in the following Proposition 3.0.3 below. From this, it can be easily seen that the method of Proposition 3.0.2, 2) is much more efficient and fast than that of Proposition 3.0.2, 1).

Proposition 3.0.3. *1) Let R and θ be independent random variables, $R^2 \sim Exp(1/2)$, $\theta \sim U[0, 2\pi]$. Then $X_1 = R\cos\theta$ and $X_2 = R\sin\theta$ are independent $N(0,1)$-distributed random variables.*

2) A random variable $X \sim N(\mu, \sigma^2)$ can be simulated by $X \stackrel{d}{=} \mu + \sigma\sqrt{-2\log U}\cos(2\pi V)$, where $U, V \sim U[0,1]$ are independent.

Proof 1) For any $x, y \in \mathbb{R}$ consider

$$
\begin{aligned}
\mathbb{P}(X_1 \le x, X_2 \le y) &= \mathbb{P}(\sqrt{R^2}\cos\theta) \le x, \sqrt{R^2}\sin\theta) \le y) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \mathbb{I}(\sqrt{t}\cos\varphi \le x, \sqrt{t}\sin\varphi \le y)\frac{1}{2}e^{-t/2}dt d\varphi = \left|t = r^2\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \mathbb{I}(r\cos\varphi \le x, r\sin\varphi \le y) r e^{-r^2/2} dr d\varphi = \begin{vmatrix} x_1 = r\cos\varphi, \\ x_2 = r\sin\varphi \end{vmatrix} \\
&= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty} \mathbb{I}(x_1 \le x, x_2 \le y) e^{-\frac{x_1^2 + x_2^2}{2}} dx_1 dx_2 \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{x} e^{-\frac{x_1^2}{2}} dx_1 \frac{1}{\sqrt{2\pi}}\int_0^{y} e^{-\frac{x_2^2}{2}} dx_2 = \mathbb{P}(X_1 \le x)\mathbb{P}(X_2 \le y).
\end{aligned}
$$

Hence, $X_1, X_2 \sim N(0,1)$ are independent.

2) If $X \sim N(\mu, \sigma^2)$ then $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$. By 1), $Y \stackrel{d}{=} R\cos\Theta$, where $R^2 \sim Exp(1/2)$, $\Theta \stackrel{d}{=} 2\pi V$, $V \sim U[0,1]$. Simulate R^2 by the inversion method, i.e. show that $R \stackrel{d}{=} -2\log U$, where $U \sim U[0,1]$, independent of V. Indeed, $\mathbb{P}(-2\log U \le x) = \mathbb{P}(\log U \ge -x/2) = \mathbb{P}(U \ge e^{-x/2}) = 1 - e^{-x/2}, x \ge 0$. Hence $-2\log U \sim Exp(1/2)$, then, it holds $\frac{X-\mu}{\sigma} \stackrel{d}{=} \sqrt{-2\log U}\cos(2\pi V)$, and we are done. □

Remark 3.0.1 (Inverse function method):
From the proof of Proposition 3.0.3, 2) it follows that for $X \sim Exp(\lambda)$ it holds $X \stackrel{d}{=} -\frac{1}{\lambda}\log U, U \sim U[0,1], \lambda > 0$. This is the particular case of the so-called inverse function simulation method: for any random variable X with c.d.f. $F_X(x) = \mathbb{P}(X \le x)$ s.t. F_X is increasing on (a,b) $-\infty \le a < b \le +\infty$, $\lim_{x\to a+} F_X(x) = 0, \lim_{x\to b-} F_X(x) = 1$: it holds $X \stackrel{d}{=} F_X^{-1}(U)$, where $U \sim U[0,1]$, and F_X^{-1} is the quantile function of X. Indeed, we may write $\mathbb{P}(F_X^{-1}(U) \le x) = \mathbb{P}(U \le F_X(x)) = F_X(x), x \in (a,b)$, since $\mathbb{P}(U \le y) = y, \forall y \in [0,1]$.

Theorem 3.0.1 (Simulation of $S_\alpha(1,0,0)$):
Let $X \sim S_\alpha(1,0,0), \alpha \in (0,2]$. Then X can be simulated by representation

$$X \stackrel{d}{=} \frac{\sin(\alpha\pi(U-1/2))}{(\cos(\pi(U-1/2)))^{1/\alpha}} \left(\frac{\cos((1-\alpha)\pi(U-1/2))}{-\log V}\right)^{\frac{1-\alpha}{\alpha}}, \tag{3.0.1}$$

where $U, V \sim U[0,1]$ are independent random variables.

Proof Denote $T = \pi(U-1/2), W = -\log V$. By Remark 3.0.1 it is clear that $T \sim U[-\pi/2, \pi/2]$, $W \sim Exp(1)$. So (3.0.1) reduces to

$$X \stackrel{d}{=} \frac{\sin(\alpha T)}{(\cos T)^{1/\alpha}} \left(\frac{\cos((1-\alpha)T)}{W}\right)^{\frac{1-\alpha}{\alpha}}. \tag{3.0.2}$$

1) $\alpha = 1$: Then (3.0.2) reduces to $X \stackrel{d}{=} \mathrm{tg}T$, which was proven in Proposition 3.0.2,2).

2) $\alpha \in (0,1)$: Under the condition $T > 0$, relation (3.0.2) rewrites as $X \stackrel{d}{=} Y = \left(\frac{K_\alpha(T)}{W}\right)^{\frac{1-\alpha}{\alpha}}$, where $K_\alpha(T) = \left(\frac{\sin(\alpha T)}{(\cos T)}\right)^{1/\alpha} \frac{\cos((1-\alpha)T)}{W}$ as in Theorem 2.5.1.

Then

$$\begin{aligned}
\mathbb{P}(0 \le Y \le x) &= \mathbb{P}(0 \le Y \le x, T > 0) = |Y \ge 0 \Leftrightarrow T > 0| \\
&= \mathbb{P}\left(0 \le \left(\frac{K_\alpha(T)}{W}\right)^{\frac{1-\alpha}{\alpha}} \le x, T > 0\right) = \mathbb{P}(W \ge K_\alpha(T)x^{-\frac{\alpha}{1-\alpha}}, T > 0) \\
&= \frac{1}{\pi}\int_0^{\pi/2} \mathbb{P}(W \ge K_\alpha(t)x^{-\frac{\alpha}{1-\alpha}})dt \stackrel{W\sim Exp(1)}{=} \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-K_\alpha(t)x^{\frac{\alpha}{1-\alpha}}\right)dt.
\end{aligned}$$

Hence, $Y \sim S_\alpha(1,0,0)$ by Theorem 2.5.4 $\Rightarrow X \stackrel{d}{=} Y \sim S_\alpha(1,0,0)$.

3) $\alpha \in (1,2]$ is proven analogously as in 2) considering $1-\alpha < 0$ and $\mathbb{P}(Y \ge x) = \mathbb{P}(Y \ge x, T > 0)$. □

Remark 3.0.2
In the Gaussian case $\alpha = 2$, the formula (3.0.2) reduces to $X \stackrel{d}{=} \sqrt{W}\frac{\sin(2T)}{\cos T} = \sqrt{W}\frac{2\sin T\cos T}{\cos T} = \sqrt{2}\sqrt{2W}\sin T$, where $\begin{cases} W \sim Exp(1) \\ T \sim U[-\pi/2, \pi/2] \end{cases}$, so $2W \sim Exp(1/2)$. Hence, $X \sim N(0,2)$ is generated by the algorithm 2) of Proposition 3.0.3, so formula (3.0.1) contains Proposition 2.4.7,2) as a spacial case.

Now let us turn to the general case of simulating a random variable $X \sim S_\alpha(\lambda, \beta, \gamma)$. We show first that, to this end, it sufficient to know how to simulate $X \sim S_\alpha(1,1,0)$.

Lemma 3.0.1
Let $X \sim S_\alpha(\lambda, \beta, \gamma), \alpha \in (0,2)$. Then

$$X \stackrel{d}{=} \begin{cases} \lambda\gamma + \lambda^{1/\alpha} Y, & \alpha \neq 1, \\ \lambda\gamma + \frac{2}{\pi}\beta\lambda\log\lambda + \lambda Y, & \lambda = 1, \end{cases} \tag{3.0.3}$$

where $Y \sim S_\alpha(1, \beta, 0)$ can be simulated by

$$Y \stackrel{d}{=} \begin{cases} \left(\frac{1+\beta}{2}\right)^{1/\alpha} Y_1 - \left(\frac{1-\beta}{2}\right)^{1/\alpha} Y_2, & \alpha \neq 1, \\ \left(\frac{1+\beta}{2}\right) Y_1 - \left(\frac{1-\beta}{2}\right) Y_2 + \frac{\lambda}{\pi}\left((1+\beta)\log\left(\frac{1+\beta}{2}\right) - (1-\beta)\log\left(\frac{1-\beta}{2}\right)\right), & \alpha = 1, \end{cases} \tag{3.0.4}$$

with $Y_1, Y_2 \sim S_\alpha(1,1,0)$ being independent random variables.

Proof Relation (3.0.4) follows from the proof of Proposition 2.3.3 and Exercise 4.1.28. Relation (3.0.3) follows easily from Theorem 2.3.1,3)-4). □

Now let us simulate $X \sim S_\alpha(1,1,0)$. First, we do it for $\alpha \in (0,1)$.

Lemma 3.0.2
Let $X \sim S_\alpha(1,1,0), \alpha \in (0,1)$. Then X can be simulated by $X \stackrel{d}{=} \frac{\sin(\alpha\theta)}{\sin\theta}\left(\frac{\sin((1-\alpha)\theta)}{W\sin\theta}\right)^{\frac{1-\alpha}{\alpha}}$, where θ and W are independent random variables, $\theta \sim U[0,\pi], W \sim Exp(1)$. As before, θ and W can be simulated by $\theta \stackrel{d}{=} \pi U, U \sim U[0,1]$, where $W \stackrel{d}{=} -\log V, V \sim U[0,1]$, where U and V are independent.

Proof By Theorem 2.5.4, 2) we have true following representation formula for the c.d.f. $\mathbb{P}(X \leq x) = F_X(x)$:

$$F_X(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \exp\left(-x^{\frac{\alpha}{\alpha-1}}\bar{K}_\alpha(t)\right) dt, x > 0,$$

where

$$\bar{K}_\alpha(t) = \left(\frac{\sin(\alpha(\pi/2+t))}{\sin(\pi/2+t)}\right)^{\frac{\alpha}{1-\alpha}} \frac{\sin((1-\alpha)(\pi/2+t))}{\sin(\pi/2+t)}, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The rest of the proof is exactly as in Theorem 3.0.1, 2). □

Similar results can be proven for $\alpha \in [1,2)$:

Theorem 3.0.2
The random variable $X \sim S_\alpha(1,1,0), \alpha \in [1,2)$ can be simulated by

$$X \stackrel{d}{=} \begin{cases} \frac{2}{\pi}\left((\pi/2+T)\mathrm{tg}T - \log\left(\frac{\frac{\pi}{2}W\cos T}{\frac{\pi}{2}+T}\right)\right), & \alpha = 1, \\ \left(1+\mathrm{tg}^2\left(\frac{\pi}{2}\alpha\right)\right)^{\frac{1}{2\alpha}} \frac{\sin(\alpha(T+\pi/2))}{(\cos T)^{1/\alpha}} \left(\frac{\cos((1-\alpha)T-\alpha\pi/2)}{W}\right)^{\frac{1-\alpha}{\alpha}}, & \alpha \in (1,2), \end{cases}$$

where $W \sim Exp(1)$ and $T \sim U[-\pi/2, \pi/2]$ are independent random variables.

Without proof.

4 Additional exercises

4.1

Exercise 4.1.1
Let X_1, X_2 be two i.i.d. r.v.'s with probability density φ. Find a probability density of aX_1+bX_2, where $a, b \in \mathbb{R}$.

Exercise 4.1.2
Let X be a symmetric stable random variable and X_1, X_2 be its two independent copies. Prove that X is a strictly stable r.v., i.e., for any positive numbers A and B, there is a positive number C such that

$$AX_1 + BX_2 \overset{d}{=} CX.$$

Exercise 4.1.3

1. Prove that $\varphi = \{e^{-|x|}, x \in \mathbb{R}\}$ is a characteristic function. (Check Pólya's criterion for characteristic functions.[1])
2. Let X be a real r.v. with characteristic function φ. Is X a stable random variable? (Verify definition.)

Exercise 4.1.4
Let real r.v. X be Lévy distributed (see Exercise Sheet 1, Ex. 1-4). Find the characteristic function of X. Give parameters $(\alpha, \sigma, \beta, \mu)$ for the stable random variable X.

Hint: You may use the following formulas. [2]

$$\int_0^\infty \frac{e^{-1/(2x)}}{x^{3/2}} \cos(yx)dx = \sqrt{2\pi}e^{-\sqrt{|y|}} \cos(\sqrt{|y|}), y \in \mathbb{R},$$
$$\int_0^\infty \frac{e^{-1/(2x)}}{x^{3/2}} \sin(yx)dx = \sqrt{2\pi}e^{-\sqrt{|y|}} \sin(\sqrt{|y|})\text{sign}y, y \in \mathbb{R}.$$

Exercise 4.1.5
Let Y be a Cauchy distributed r.v. Find the characteristic function of Y. Give parameters $(\alpha, \sigma, \beta, \mu)$ for the stable random variable Y.
Hint: Use Cauchy's residue theorem.

Exercise 4.1.6
Let $X \sim S_1(\sigma, \beta, \mu)$ and $a > 0$. Is aX stable? If so, define new $(\alpha_2, \sigma_2, \beta_2, \mu_2)$ of aX.

Exercise 4.1.7
Let $X \sim N(0, \sigma^2)$ and A be a positive $\alpha-$stable r.v. Is the new r.v. AX stable, strictly stable? If so, find its stability index α_2.

---

[1] **Pólya's theorem.** If φ is a real-valued, even, continuous function which satisfies the conditions $\varphi(0) = 1$, φ is convex for $t > 0$, $\lim_{t\to\infty} \varphi(t) = 0$, then φ is the characteristic function of an absolutely continuous symmetric distribution.

[2] Oberhettinger, F. (1973). Fourier transforms of distributions and their inverses: a collection of tables. Academic press, p.25

Exercise 4.1.8
Let L be a positive slowly varying function, i.e., $\forall x > 0$

$$\lim_{t\to+\infty} \frac{L(tx)}{L(t)} = 1. \tag{4.1.1}$$

1. Prove that $x^{-\varepsilon} \leq L(x) \leq x^{\varepsilon}$ for any fixed $\varepsilon > 0$ and all x sufficiently large.

2. Prove that limit (4.1.1) is uniform in finite intervals $0 < a < x < b$.

Hint: Use a representation theorem:[3]
A function Z varies slowly iff it is of the form $Z(x) = a(x) \exp\left(\int_1^x \frac{\varepsilon(y)}{y} dy\right)$, where $\varepsilon(x) \to 0$ and $a(x) \to c < \infty$ as $x \to \infty$.

Definition 4.1.1 (Infinitely divisible distributions):
A distribution function F is called infinitely divisible if for all $n \geq 1$, there is a distribution function F_n such that

$$Z \stackrel{d}{=} X_{n,1} + \cdots + X_{n,n},$$

where $Z \sim F$ and $X_{n,k}, 1 \leq k \leq n$ are i.i.d. r.v.'s with the distribution function F_n.

Exercise 4.1.9
For the following distribution functions check whether they are infinitely divisible.

1. (1 point) Gaussian distribution.

2. (1 point) Poisson distribution.

3. (1 point) Gamma distribution.

Exercise 4.1.10
Find parameters (a, b, H) in the canonic Lévy-Khintchin representation of a characteristic function for

1. (1 point) Gaussian distribution.

2. (1 point) Poisson distribution.

3. (1 point) Lévy distribution.

Exercise 4.1.11
What is wrong with the following argument? If $X_1, \ldots, X_n \sim Gamma(\alpha, \beta)$ are independent, then $X_1 + \cdots + X_n \sim Gamma(n\alpha, \beta)$, so gamma distributions must be stable distributions.

Exercise 4.1.12
Let $X_i, i \in \mathbb{N}$ be i.i.d. r.v.'s with a density symmetric about 0 and continuous and positive at 0. Prove

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \xrightarrow{d} X, n \to \infty,$$

where X is a Cauchy distributed random variable.

Hint: At first, apply Khintchin's theorem (T.2.2 in the lecture notes). Then find parameters a, b and a spectral function H from Gnedenko's theorem (T.2.3 in the lecture notes).

[3]Feller, W. (1973). An Introduction to Probability Theory and its Applications. Vol 2, p.282

Exercise 4.1.13
Show that the sum of two independent stable random variables with different α-s is not stable.

Exercise 4.1.14
Let $X \sim S_\alpha(\lambda, \beta, \gamma)$. Using the weak law of large numbers prove that when $\alpha \in (1, 2]$, the shift parameter $\mu = \lambda\gamma$ equals $\mathbb{E}X$.

Exercise 4.1.15
Let X be a standard Lévy distributed random variable. Compute its Laplace transform

$$\mathbb{E}\exp(-\gamma X), \gamma > 0.$$

Exercise 4.1.16
Let $X \sim S_{\alpha'}(\lambda', 1, 0)$, and $A \sim S_{\alpha/\alpha'}(\lambda_A, 1, 0)$, $0 < \alpha < \alpha' < 1$ be independent. The value of λ_A is chosen s.t. the Laplace transform of A is given by $\mathbb{E}\exp(-\gamma A) = \exp(-\gamma^{\alpha/\alpha'}), \gamma > 0$. Show that $Z = A^{1/\alpha'}X$ has a $S_\alpha(\lambda, 1, 0)$ distribution for some $\lambda > 0$.

Exercise 4.1.17
Let $X \sim S_\alpha(\lambda, 1, 0), \alpha < 1$ and the Laplace transform of X be given by $\mathbb{E}\exp(-\gamma X) = \exp(-c_\alpha \gamma^\alpha), \gamma > 0$,where $c_\alpha = \lambda^\alpha / \cos(\pi\alpha/2)$.

1. Show that
$$\lim_{x\to\infty} x^\alpha \mathbb{P}\{X > x\} = C_\alpha,$$
where C_α is a positive constant.
Hint: Use the Tauberian theorem.[4]

2. (2 points) Prove that
$$\mathbb{E}|X|^p < \infty, \text{ for any } 0 < p < \alpha,$$
$$\mathbb{E}|X|^p = \infty, \text{ for any } p \geq \alpha.$$

Exercise 4.1.18
Let X_1, X_2 be two independent α-stable random variables with parameters (λ, β, γ). Prove that $X_1 - X_2$ is a stable random variable and find its parameters $(\alpha_1, \lambda_1, \beta_1, \gamma_1)$.

Exercise 4.1.19
Let $X_1, \ldots, X_n$ be i.i.d $S_\alpha(\lambda, \beta, \gamma)$ distributed random variables and $S_n = X_1 + \cdots + X_n$. Prove that the limiting distribution of

1. $n^{-1/\alpha} S_n, n \to \infty$, if $\alpha \in (0, 1)$;
2. $n^{-1}(S_n - 2\pi^{-1}\lambda\beta n \log n) - \lambda\gamma, n \to \infty$, if $\alpha = 1$;
3. $n^{-1/\alpha}(S_n - n\lambda\gamma), n \to \infty$, if $\alpha \in (1, 2]$;

[4](Feller 1971 Theorem XIII.5.4.) If L is slowly varying at infinity and $\rho \in \mathbb{R}_+$, the following relations are equivalent

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^\rho L(t), t \to \infty, \qquad \int_0^\infty e^{-\tau x} dU(x) \sim \frac{1}{\tau^\rho} L\left(\frac{1}{\tau}\right), \tau \to 0.$$

is $S_\alpha(\lambda, \beta, 0)$.

Exercise 4.1.20
Let $X_1, X_2 \ldots,$ be a sequence of i.i.d. random variables and let $p > 0$. Applying the Borel-Cantelli lemmas, show that

1. $\mathbb{E}|X_1|^p < \infty$ if and only if $\lim_{n\to\infty} n^{-1/p} X_n = 0$ a.s.,
2. $\mathbb{E}|X_1|^p = \infty$ if and only if $\limsup_{n\to\infty} n^{-1/p} X_n = \infty$ a.s.

Exercise 4.1.21
Let ξ be a non-negative random variable with the Laplace transform $\mathbb{E}\exp(-\lambda\xi) = \exp(-\lambda^\alpha), \lambda \geq 0$. Prove that

$$\mathbb{E}\xi^{\alpha s} = \frac{\Gamma(1-s)}{\Gamma(1-\alpha s)}, s \in (0,1).$$

Exercise 4.1.22
Denote by

$$\tilde{f}(s) := \int_0^\infty e^{-sx} f(x) dx,$$

the Laplace transform of a real function f defined for all $s > 0$, whenever $\tilde{f}$ is finite. For the following functions find the Laplace transforms (in terms of $\tilde{f}$):
1. For $a \in \mathbb{R}$ $f_1(x) := f(x-a), x \in \mathbb{R}_+$, and $f(x) = 0, x < 0$.
2. For $b > 0$ $f_2(x) := f(bx), x \in \mathbb{R}_+$.
3. $f_3(x) := f'(x), x \in \mathbb{R}_+$.
4. $f_4(x) := \int_0^x f(u) du, x \in \mathbb{R}_+$.

Exercise 4.1.23
Let $\tilde{f}, \tilde{g}$ be Laplace transforms of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$.

1. Find the Laplace transform of the convolution $f * g$.
2. Prove the final value theorem: $\lim_{s\to 0} s\tilde{f}(s) = \lim_{t\to\infty} f(t)$.

Exercise 4.1.24
Let $\{X_n\}_{n\geq 0}$ be i.i.d. r.v.'s with a density symmetric about 0 and continuous and positive at 0. Applying the Theorem 2.8 from the lecture notes, prove that cumulative distribution function $F(x) := \mathbb{P}(X_1^{-1} \leq x), x \in \mathbb{R}$ belongs to the domain of attraction of a stable law G. Find its parameters $(\alpha, \lambda, \beta, \gamma)$ and sequences a_n, b_n s.t. $\frac{1}{b_n}\sum_{i=1}^n X_i^{-1} - a_n \xrightarrow{d} Y \sim G$ as $n \to \infty$.

Exercise 4.1.25
Let $\{X_n\}_{n\geq 0}$ be i.i.d. r.v.'s with for $x > 1$

$$\mathbb{P}(X_1 > x) = \theta x^{-\delta}, \quad \mathbb{P}(X_1 < -x) = (1-\theta)x^{-\delta},$$

where $0 < \delta < 2$. Applying the Theorem 2.8 from the lecture notes, prove that c.d.f. $F(x) := \mathbb{P}(X_1 \leq x), x \in \mathbb{R}$ belongs to the domain of attraction of a stable law G. Find its parameters $(\alpha, \lambda, \beta, \gamma)$ and sequences a_n, b_n s.t. $\frac{1}{b_n}\sum_{i=1}^n X_i - a_n \xrightarrow{d} Y \sim G$ as $n \to \infty$.

Exercise 4.1.26
Let X be a random variable with probability density function $f(x)$. Assume that $f(0) \neq 0$ and that $f(x)$ is continuous at $x = 0$. Prove that

1. if $0 < r \le \frac{1}{2}$, then $|X|^{-r}$ belongs to the domain of attraction of a Gaussian law,
2. if $r > 1/2$ then $|X|^{-r}$ belongs to the domain of attraction of a stable law with stability index $1/r$.

Exercise 4.1.27
Find a distribution F which has infinite second moment and yet it is in the domain of attraction of the Gaussian law.

Exercise 4.1.28
Prove the following statement which is used in the proof of Proposition 2.3.3.

Let $X \sim S_\alpha(\lambda, \beta, 0)$ with $\alpha \in (0, 2)$. Then there exist two i.i.d. r.v.'s Y_1 and Y_2 with common distribution $S_\alpha(\lambda, 1, 0)$ s.t.

$$X \stackrel{d}{=} \begin{cases} \left(\frac{1+\beta}{2}\right)^{1/\alpha} Y_1 - \left(\frac{1-\beta}{2}\right)^{1/\alpha} Y_2, & \text{if } \alpha \neq 1, \\ \left(\frac{1+\beta}{2}\right) Y_1 - \left(\frac{1-\beta}{2}\right) Y_2 + \frac{\lambda}{\pi}\left((1+\beta)\log\frac{1+\beta}{2} - (1-\beta)\log\frac{1-\beta}{2}\right), & \text{if } \alpha = 1. \end{cases}$$

Exercise 4.1.29
Prove that for $\alpha \in (0, 1)$ and fixed λ, the family of distributions $S_\alpha(\lambda, \beta, 0)$ is stochastically ordered in β, i.e., if $X_\beta \sim S_\alpha(\lambda, \beta, 0)$ and $\beta_1 \le \beta_2$ then $\mathbb{P}(X_{\beta_1} \ge x) \le \mathbb{P}(X_{\beta_2} \ge x)$ for $x \in \mathbb{R}$.

Exercise 4.1.30
Prove the following theorem.

Theorem 4.1.1
A distribution function F is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$ if and only if there are constants $C_+, C_- \ge 0, C_+ + C_- > 0$, such that

1. $$\lim_{y\to+\infty} \frac{F(-y)}{1 - F(y)} = \begin{cases} C_-/C_+, & \text{if } C_+ > 0, \\ +\infty, & \text{if } C_+ = 0, \end{cases}$$
2. and for every $a > 0$
$$\begin{cases} \lim_{y\to+\infty} \frac{1-F(ay)}{1-F(y)} = a^{-\alpha}, & \text{if } C_+ > 0, \\ \lim_{y\to+\infty} \frac{F(-ay)}{F(-y)} = a^{-\alpha}, & \text{if } C_- > 0. \end{cases}$$

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