On Whittle estimation for Lévy continuous-time moving average process

Yuma Uehara
Kansai University

(with Tetsuya Takabatake at Osaka University)

Time Series, Random Fields and beyond, 23-27, September, 2024

Outline

1. Model Setup
2. Main Results
3. Numerical Experiment
4. Summary

Model Setup

Lévy process

- A stochastic process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$ a.s. is called a Lévy process if the following conditions are satisfied:
  1. Independent and stationary increments ($0 = t_0 < t_1 < \cdots < t_n$; $n \in \mathbb{N}$)
     - $L_{t_1} - L_{t_0},\ L_{t_2} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent.
     - $L_{t_j} - L_{t_{j-1}} \ \sim \ L_{t_j - t_{j-1}}$
  2. Continuity in probability: $L_s \xrightarrow{p} L_t$ as $s \to t$.
- L is a continuous-time analogue of random walk:

$$L_t = \sum_{i=1}^{n} \left(L_{\frac{i}{n}t} - L_{\frac{i-1}{n}t} \right),$$

and there exists a one-to-one correspondence between infinitely divisible distributions.

Lévy-Itô decomposition

- A Lévy process L can be decomposed as:

$$L_t = bt + \sigma B_t + J_t,$$

where $b \in \mathbb{R}, \sigma \geq 0$, and $B = (B_t)_{t\geq 0}$ is the standard Brownian motion.

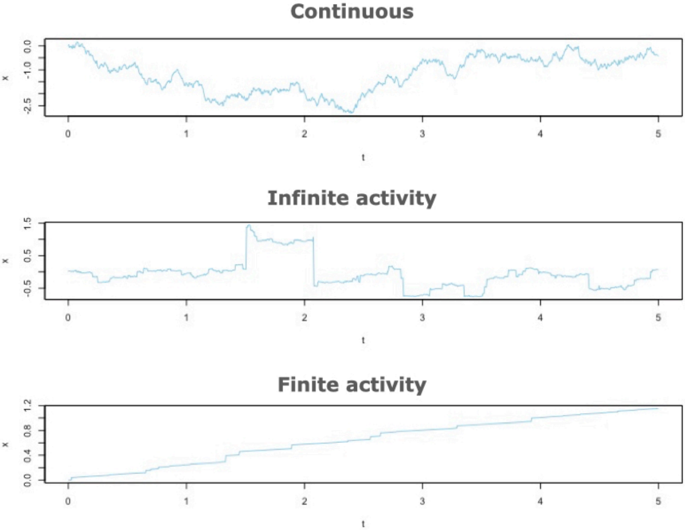
- Moreover, B and $J = (J_t)_{t\geq 0}$ are independent and J is a pure-jump process with

$$\log E[e^{iuJ_t}] = t \int (e^{iuz} - 1 - iuz\mathbb{1}_{|z|\leq 1})^{\exists}\nu(dz),$$

and $\nu(\cdot)$ is a so-called Lévy measure ($\int (1 \wedge |z|^2)\nu(dz) < \infty$).

- Non-Gaussian distribution corresponds to J and we suppose that $\sigma = 0$ in this talk.

Sample path of Lévy process



Continuous-time Moving Average Process

- One-dimensional continuous-time moving average (CMA) process $X^\theta = \{X^\theta_t\}_{t\in\mathbb{R}}$ of the form

$$X^\theta_t := \int_{\mathbb{R}} k_\theta(t-u)\,\mathrm{d}L_u, \quad t \in \mathbb{R}, \tag{1}$$

  where $\{L_t\}_{t\in\mathbb{R}}$ is a pure-jump Lévy process satisfying that

$$\mathbb{E}[L_1] = 0, \mathbb{E}[L_1^2] = 1, \mathbb{E}[L_1^4] < \infty.$$

- Assume the function $x \mapsto k_\theta(x)$ belongs to $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$.
- Notice that $X^\theta = \{X^\theta_t\}_{t\in\mathbb{R}}$ is a centered stationary process with covariance function

$$\mathrm{Cov}[X^\theta_s, X^\theta_t] = \int_{\mathbb{R}} k_\theta(u-|t-s|)k_\theta(u)\,\mathrm{d}u.$$

Objective

$$X_t^\theta := \int_{\mathbb{R}} k_\theta(t-u)\,\mathrm{d}L_u, \quad t \in \mathbb{R},$$

- We consider the situation where the process X_t^θ has long memory, that is, the spectral density function $s_\theta(z)$ of X_t^θ behaves as
$$s_\theta(z) \sim c_\theta |z|^{-\alpha_\theta} \text{ as } |z| \to 0,$$
with some positive constant c_θ and $\alpha_\theta \in (0,1)$.
- Observe $\{X_{j\Delta}\}_{j=1,2,\cdots,n}$ with a fixed positive constant Δ.

Objective

- Estimate the p-dimensional parameter θ from the observations.

Examples

1. Fractional Ornstein-Uhlenbeck Kernel:

$$k_\theta(x) = \sigma c_H \left(x^{H-\frac{1}{2}} - \kappa \int_0^x e^{-\kappa(x-s)} s^{H-\frac{1}{2}} \, \mathrm{d}s \right) \mathbb{1}_{(0,\infty)}(x)$$

   where $\theta = (H, \kappa, \sigma)$ with $\sigma > 0$, $H \in (1/2, 1)$ and $\kappa > 0$.

2. Power Law Kernel:

$$k_\theta(x) = \sigma x^\beta (1 + x)^{\eta - \beta - 1} \mathbb{1}_{(0,\infty)}(x)$$

   where $\theta = (\beta, \eta, \sigma)$ with $\sigma > 0$, $\beta \in (-1/4, 1/2)$ and $\eta \in (0, 1/2)$.

Example: SPD for fOU Process

- The spectral density of the fOU process, *e.g.* see Cheridito-Kawaguchi-Maejima (2003-EJP), is given by

$$s_\theta(z) = \sigma^2 c_H \frac{|z|^{1-2H}}{z^2 + \kappa^2}$$

where $c_H = (2\pi)^{-1}\Gamma(2H+1)\sin(\pi H)$ and $\theta = (H, \kappa, \sigma)$ with $H \in (1/2, 1)$, $\kappa > 0$ and $\sigma > 0$.

- Then we can show that

$$s_\theta(z) \sim \frac{\sigma^2 c_H}{\kappa^2}|z|^{1-2H} \quad \text{as } |z| \to 0$$

which implies

$$\alpha_\theta = 2H - 1 \quad \text{and} \quad c_\theta = \sigma^2 c_H/\kappa^2.$$

Continuous-time Spectral Density Function

- Recall that the Itô isometry yields

$$\gamma_\theta(t) := \mathrm{Cov}[X_t^\theta, X_0^\theta] = \int_{\mathbb{R}} k_\theta(u - |t|) k_\theta(u) \, \mathrm{d}u. \qquad (2)$$

- Plancherel-Parseval's equality yields

$$\begin{aligned} \gamma_\theta(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(k_\theta(\cdot - |t|))(z) \cdot \overline{\mathcal{F} k_\theta(z)} \, \mathrm{d}z \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\sqrt{-1}|t|z} \left| \mathcal{F} k_\theta(z) \right|^2 \, \mathrm{d}z \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\sqrt{-1}tz} \left| \mathcal{F} k_\theta(z) \right|^2 \, \mathrm{d}z \end{aligned}$$

so that the spectral density function of X^θ is given by

$$s_\theta(z) = \frac{1}{2\pi} \left| \mathcal{F} k_\theta(z) \right|^2, \quad z \in \mathbb{R}. \qquad (3)$$

Discrete-time Spectral Density Function

- Set $Y_j := X_{j\Delta}$ for $j \in \mathbb{Z}$. Notice that $Y = \{Y_j\}_{j \in \mathbb{Z}}$ is a centered *discrete-time* stationary process.
- Using the aliasing formula of the spectral density function, the spectral density function of the discrete-time stationary process Y, denoted by $s_\theta^\Delta(\omega)$, is given by

  $$s_\theta^\Delta(\omega) := \frac{1}{\Delta} \sum_{\tau \in \mathbb{Z}} s_\theta\left(\frac{2\pi\tau + \omega}{\Delta}\right), \quad \omega \in [-\pi, \pi],$$

  where $s_\theta(z)$ is the spectral density function of X^θ.
- In particular, for the CMA process, we have

  $$s_\theta^\Delta(\omega) = \frac{1}{2\pi\Delta} \sum_{\tau \in \mathbb{Z}} \left| \mathcal{F} k_\theta\left(\frac{2\pi\tau + \omega}{\Delta}\right)\right|^2$$

  since $s_\theta(z) = (2\pi)^{-1} |\mathcal{F} k_\theta(z)|^2$.

Main Results

Estimation scheme: Whittle estimation

- Denote $\mathbb{L}_n^W$ by the Whittle likelihood function of the observations $\{Y_j\}_{j=1,2,\ldots,n}$, which is given by

$$\mathbb{L}_n^W(\theta) = \frac{1}{n}\sum_{j=1}^{n}\left[\log s_\theta^\Delta(\omega_j) + \frac{I_n(\omega_j)}{s_\theta^\Delta(\omega_j)}\right],$$

  where $\omega_j = \frac{j}{2\pi n}$ and $I_n(\omega) = (2\pi n)^{-1}\left|\sum_{t=1}^{n} Y_t e^{it\omega}\right|^2$.

- We define the Whittle estimator of θ by

$$\hat{\theta}_n \in \arg\min_{\theta\in\Theta} \mathbb{L}_n^W(\theta).$$

- We suppose that the parameter space Θ is a bounded convex domain.

Approximation by discretized process

- We approximate $Y = \{Y_j\}_{j\in\mathbb{Z}}$ by $\bar{Y}^n := \{\bar{Y}^n_j\}_{j\in\mathbb{Z}}$ defined as

$$\bar{Y}^n_t = \sum_{i\in\mathbb{Z}\setminus\{0\}} k_\theta(ih_n\Delta)(L_{t\Delta-(i-1)h_n\Delta} - L_{t\Delta-ih_n\Delta}) = \sum_{i\in\mathbb{Z}\setminus\{0\}} k_\theta(ih_n\Delta)\Delta_{t,i}L,$$

  where $\Delta_{t,i}L := L_{\{t-(i-1)h_n\}\Delta} - L_{(t-ih_n)\Delta}$.

- $\{\bar{Y}^n\}_{n\in\mathbb{N}}$ is an approximating sequence of Y in the sense of

$$\sup_{j\in\mathbb{Z}} \mathbb{E}[|\bar{Y}^n_j - Y_j|^2] = O(h_n^{2\beta+1}) \text{ as } n\to\infty, \tag{4}$$

  when $\partial_z k_{\theta_0}(z) \sim |z|^{\beta-1}$ around the origin.

- We can also derive the approximation of the spectral density function:

$$s_\theta^{\bar{Y}^n}(\omega) := \frac{1}{2\pi}\sum_{\tau\in\mathbb{Z}}\left(h_n\Delta\sum_{i_1\in\mathbb{Z}\setminus\{0\}} k_\theta((i_1h_n)\Delta)k_\theta((\tau+i_1h_n)\Delta)\right)e^{\sqrt{-1}\tau\omega} \to s_\theta^\Delta(\omega)$$

Assumption 1 (Continuous-Time SPD $s_\theta(z)$)

The spectral density $s_\theta(z) = s(z, \theta)$ of $X^\theta = \{X_t^\theta\}_{t \in \mathbb{R}}$ satisfying the following conditions:

(1) For each $\theta \in \Theta$, $z \mapsto s_\theta(z)$ is a non-negative integrable even function on $\mathbb{R}$. Moreover, it satisfies that $s(\cdot, \cdot) \in C^{1,3}((\mathbb{R} \backslash \{0\}) \times \Theta)$.

(2) There exists a continuous function $\alpha_0 : \Theta \to (0, 1)$ such that for some constants $c_1, c_2 > 0$ and for any $\iota > 0$ and some constant $c_{3,\iota} > 0$, the following conditions hold for every $(z, \theta) \in ([-\Delta\pi, \Delta\pi] \backslash \{0\}) \times \Theta$:

  (a) $c_1 |z|^{-\alpha_0(\theta)} \leq s_\theta(z) \leq c_2 |z|^{-\alpha_0(\theta)}$.

  (b) For any $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1\}$, it holds $\left| \partial_z^k \partial_\theta^j s_\theta(z) \right| \leq c_{3,\iota} |z|^{-\alpha_0(\theta) - k - \iota}$.

(3) For any $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1\}$, it holds

$$\sum_{\tau \in \mathbb{Z} \backslash \{0\}} \sup_{(\omega, \theta) \in [-\pi, \pi] \times \Theta} \left| \partial_z^k \partial_\theta^j s_\theta \left(\frac{2\pi\tau + \omega}{\Delta} \right) \right| < \infty.$$

Lemma (Properties of Discrete-time SPD $s_\theta^\Delta(\omega)$)

(1) For each $\theta \in \Theta$ and $\omega \in [-\pi, \pi] \setminus \{0\}$, $s_\theta^\Delta(\omega)$ is finite, and $s_\theta^\Delta(\cdot) \in C^{1,3}\left(([-\pi, \pi]\setminus\{0\}) \times \Theta\right)$.

(2) For some constants $c_1, c_2 > 0$ and for any $\iota > 0$ and some constant $c_{3,\iota} > 0$, the following conditions hold for every $(\omega, \theta) \in ([-\pi, \pi]\setminus\{0\}) \times \Theta$:

- (a) $c_1|\omega|^{-\alpha(\theta)} \leq s_\theta^\Delta(\omega) \leq c_2|\omega|^{-\alpha(\theta)}$.
- (b) For any $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1\}$, it holds $\left|\partial_\omega^k \partial_\theta^j s_\theta^\Delta(\omega)\right| \leq c_{3,\iota}|\omega|^{-\alpha(\theta)-k-\iota}$.

Derivation of Lemma

- Recall that $s_\theta^\Delta(\omega)$ is given by

$$s_\theta^\Delta(\omega) = \frac{1}{\Delta} s_\theta\left(\frac{\omega}{\Delta}\right) + \frac{1}{\Delta} \sum_{\tau \in \mathbb{Z} \setminus \{0\}} s_\theta\left(\frac{2\pi\tau + \omega}{\Delta}\right).$$

- Under Assumption 1, it follows that

$$s_\theta^\Delta(\omega) \lesssim \frac{1}{\Delta}\left[\left|\frac{\omega}{\Delta}\right|^{-\alpha_0(\theta)} + 1\right] \lesssim |\omega|^{-\alpha_0(\theta)}.$$

  On the other hand, using the positiveness of $s_\theta(z)$, we can directly show the lower estimate

$$s_\theta^\Delta(\omega) \gtrsim \frac{1}{\Delta}\left|\frac{\omega}{\Delta}\right|^{-\alpha_0(\theta)} \gtrsim |\omega|^{-\alpha_0(\theta)} > 0.$$

Asymptotic behavior of the Whittle estimator

In addition to Assumption 1, we assume the following conditions.

Assumption 2

- If θ_1 and θ_2 are distinct elements of Θ, the set $\{\omega \in [-\pi, \pi] \setminus \{0\} : s_{\theta_1}^{\Delta}(\omega) \neq s_{\theta_2}^{\Delta}(\omega)\}$ has a positive Lebesgue measure.

Theorem 1: Asymptotic Normality of MLE $\widehat{\theta}_n$

Under Assumptions 1 and 2, a sequence of Whittle estimator $\{\widehat{\theta}_n\}_{n\in\mathbb{N}}$ is consistent and asymptotically normal:

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to \mathcal{N}_p\left(\mathbf{0}_p, {}^{\exists}\Sigma\right),$$

where the asymptotic variance Σ is supposed to be positive definite.

Numerical Experiment

Numerical experiment

- We consider the following fractional Lévy process:

$$X_t^\theta = c_H \int_{-\infty}^{t} \sigma\left[\left\{(t-u)\mathbb{1}_{t-u>0}\right\}^d - \left\{-u\mathbb{1}_{-u>0}\right\}^d\right] dL_u, \quad t \geq 0,$$

  where $d = H - \frac{1}{2}$ and c_H is the normalizing constant such that $E[(X_t^\theta - X_{t-1}^\theta)^2] = \sigma^2$.

- Although our method cannot be applied for the above process directly, by taking the difference $Y_t^\theta = X_t^\theta - X_{t-1}^\theta$, a similar approach will be valid (we expect).

- We suppose that the driving Lévy process is a compound Poisson process whose intensity is 1 and the jump distribution is the standard normal distribution.

- Since the difference of jump times of compound Poisson processes obeys the exponential distribution, we can simulate the path of

$$X_t^{\theta,(L)} = c_H \int_{-L}^{t} \sigma \left[\left\{ (t-u)\mathbb{1}_{t-u>0} \right\}^d - \left\{ -u\mathbb{1}_{-u>0} \right\}^d \right] dL_u$$

  for some positive constant $L > 0$ and we set $L = 5000$ for the approximation of X_t^{θ}.
- We independently simulate 1000 paths for $L = 5000$, and calculate the Whittle estimator of H and σ based on the discrete observations $\{Y_t^{\theta}\}_{t=0}^{n}$.
- We consider two cases: $(H, \sigma) = (0.9, 1)$ and $(H, \sigma) = (0.75, 1)$.

Result: case 1

n	$\hat{H}_n$	$\hat{\sigma}_n$
250	0.8879627	1.216178
	(0.04525944)	(1.157441)
500	0.8961821	1.113735
	(0.03300364)	(0.6215016)
1000	0.8970059	1.025469
	(0.02215364)	(0.2720051)

Table: The performance of the Whittle estimator with the true value $(H, \sigma) = (0.9, 1)$; the mean of each estimator is given with the standard deviation in parenthesis.

Result: case 2

n	$\hat{H}_n$	$\hat{\sigma}_n$
250	0.7370236	0.9938576
	(0.04389529)	(0.1782975)
500	0.7420348	0.992157
	(0.03252598)	(0.1216881)
1000	0.7459363	0.9951483
	(0.02210757)	(0.08555319)

Table: The performance of the Whittle estimator with the true value $(H, \sigma) = (0.75, 1)$; the mean of each estimator is given with the standard deviation in parenthesis.

Summary

Summary:

- For general Lévy continuous-time moving average process with long memory, we proved the consistency and asymptotic normality of the Whittle estimator.
- We used the approximation process based on discretization and derived the convergence of the spectral density, the decomposition of periodogram.