Approximations and asymptotics of continuous-time locally stationary processes

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Abstract

We introduce a general theory on stationary approximations for locally stationary continuous-time processes. Based on the stationary approximation, we use θ-weak dependence to establish laws of large numbers and central limit type results under different observation schemes. Hereditary properties for a large class of finite and infinite memory transformations show the flexibility of the developed theory. Sufficient conditions for the existence of stationary approximations for time-varying Lévy-driven state space models are derived and compared to existing results. We conclude with comprehensive results on the asymptotic behavior of the first and second order localized sample moments of time-varying Lévy-driven state space models.

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1 Introduction

In the last years, extensive research on continuous-time processes led to a variety of flexible yet analytically tractable models [10, 12, 26, 29, 34]. Not least because irregularly spaced time series and high frequency data can be handled, continuous-time processes have been successfully applied in areas as diverse as finance, physics, and engineering [6, 7, 11, 28]. Despite this success, many of the established models as the well-known Lévy-driven state space model (or equivalently CARMA model), are not capable of modeling non-stationary behavior in observed data.

In the discrete-time setting, this problem has, for instance, been treated with a rich theory on locally stationary processes. Starting in [15], a consistent framework for locally stationary time-varying models (see [16] for a comprehensive overview) has been developed by Dahlhaus and others. Recently, this approach has been generalized in [5] and [17], where the authors introduce a remarkably versatile approach for local stationarity, including non-linear models of finite and infinite memory, which are beyond the classical linear time series models.

However, in the continuous-time setting, things look different. Except for results on time-varying Gaussian-driven diffusion models as in [27], the only definition on local stationarity in continuous-time including non-Gaussian models was recently given in [8]. There, the authors follow an approach that can be considered as the continuous-time analog of the original approach from

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Dahlhaus [15]. The authors define a locally stationary process as a sequence of moving average processes $Y_N(t)$ with a time-varying kernel function $g_N(t, t - \cdot)$ driven by a Lévy process, i.e.

$$Y_N(t) = \int_{\mathbb{R}} g_N(t, t - s) L(ds), \tag{1}$$

where the kernel function $g_N(Nt, \cdot)$ converges in L^2 to a continuous square integrable limiting kernel function $g(t, \cdot)$ as N tends to infinity.

Noticeable examples of (1) come from the class of time-varying Lévy-driven state space models, for which the time-varying kernel function, as well as the limiting kernel function, are of exponential type (see Section 5.3 for a detailed discussion). In [8] it is shown that, under conditions on the limiting kernel g_N and the Lévy process L, time-varying Lévy-driven state space models are locally stationary with respect to the given definition. Unfortunately, there are no results on statistical inference available based on this concept of local stationarity.

Overall it seems that, up to now, there is no general theory on non-stationary continuous-time models available that encompasses non-Gaussianity and provides asymptotic results.

In the present work, we address this problem by investigating stationary approximations in the spirit of [17] for sequences of non-parametric non-stationary continuous-time processes. Based on such approximations, we show hereditary properties under transformations and establish asymptotic results for localized sample moments of such processes. These asymptotic results form the foundation for many M-estimation procedures, to which powerful inference methodologies as the maximum likelihood estimator belong [35].

All asymptotic results will be obtained by assuming that the stationary approximation is θ-weakly dependent as introduced in [18]. θ-weak dependence is inherited under the same transformations as mentioned above such that our results also apply for a large class of transformations, including the variance and covariance operator.

Moreover, we extend the results from [8] and consider sequences of time-varying Lévy-driven state space models $Y_N(t)$ in the form of (1) and give sufficient conditions for the existence of a stationary approximation $\tilde{Y}_u(t)$ for a fixed approximation point u. This approximation can be expressed in terms of the limiting kernel function g from [8] as

$$\tilde{Y}_u(t) = \int_{\mathbb{R}} g(u, t - s) L(ds). \tag{2}$$

The paper is structured as follows. In Section 2 we start by introducing locally stationary approximations and discuss the different types of observations which will be considered in this work. Section 2.2 is dedicated to different dependence measures, which will be needed to establish asymptotic results. L^1-mixingale type dependence measures, θ-weak dependence, and important results on their relationship will be reviewed. Then, hereditary properties of locally stationary approximations and hereditary properties of θ-weak dependence under both finite and infinite transformations are discussed in Section 2.3 and 2.4. Based on the locally stationary approximations from Section 2.1 we give several asymptotic results in Section 3. Global and local laws of large numbers are proved in Section 3.1 under minimal moment assumptions for different observation schemes. In Section 3.2 we establish central limit type results for similar observation schemes. The proofs are mainly based on asymptotic theory for stationary triangular arrays [19]. Sufficient conditions from [19] are connected to the θ-weak dependence coefficients of the stationary approximation. In Section 4.1 we review elementary properties of Lévy processes, including stochastic integration with respect to them. A brief summary of the main results from

[8] form Section 4.2 where we additionally show that sequences of processes in the form (1) that possess a locally stationary approximation are locally stationary in the sense of [8]. In Section 5 we give, starting from time-varying Lévy-driven Ornstein-Uhlenbeck processes, a comprehensive introduction on general time-varying Lévy-driven state space models, time-varying Lévy-driven CARMA models, and comment on their relationship. Sufficient conditions for the existence of locally stationary approximations for these processes are given in Section 5.3. Finally, we apply the developed asymptotic theory from Section 3 and provide asymptotic results for the first and second order localized sample moments of time-varying Lévy-driven state space models under different observations. Results from [14] are used that give sufficient conditions for processes in the form (2) to be θ-weakly dependent if the kernel is of exponential decay.

1.1 Notation

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, non-negative integers by $\mathbb{N}_0$, positive real numbers by $\mathbb{R}^+$, non-negative real numbers by $\mathbb{R}_0^+$, the set of $m \times n$ matrices over a ring R by $M_{m \times n}(R)$ and $\mathbf{1}_n$ stands for the $n \times n$ identity matrix. For square matrices $A, B \in M_{n \times n}(R)$, $[A, B] = AB - BA$ denotes the commutator of A and B. Norms of matrices and vectors are denoted by $\|\cdot\|$. If the norm is not further specified, we take the Euclidean norm or its induced operator norm, respectively. For a bounded function h, $\|h\|_\infty$ denotes the uniform norm of h. In the following Lipschitz continuous is understood to mean globally Lipschitz. For $u, n \in \mathbb{N}$, let $\mathcal{G}_u^*$ be the class of bounded functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$ and $\mathcal{G}_u$ be the class of bounded, Lipschitz continuous functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$ with respect to the distance $\sum_{i=1}^u \|x_i - y_i\|$, where $x, y \in (\mathbb{R}^n)^u$. For $G \in \mathcal{G}_u$ we define

$$Lip(G) = \sup_{x \neq y} \frac{|G(x) - G(y)|}{\|x_1 - y_1\| + \ldots + \|x_u - y_u\|}.$$

We shortly write the transpose of a matrix $A \in M_{m \times n}(\mathbb{R})$ as A'. For two sets U and V, $|U|$ stands for the cardinality of a set U and $U \triangle V$ for the symmetric difference of the two sets. The Borel σ-algebras are denoted by $\mathcal{B}(\cdot)$ and λ stands for the Lebesgue measure, at least in the context of measures. For a normed vector space W we denote by $\ell^\infty(W)$ the space of all bounded sequences in W. In the following, we will assume all stochastic processes and random variables to be defined on a common complete probability space $(\Omega, \mathcal{F}, P)$ equipped with an appropriate filtration if necessary. Finally, we simply write L^p to denote the space $L^p(\Omega, \mathcal{F}, P)$ and $L^p(\mathbb{R})$ to denote the space $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with corresponding norms $\|\cdot\|_{L^p}$.

2 Locally stationary approximations in continuous-time and transformations

2.1 Locally stationary approximations in continuous-time

In this paper, we follow the intuitive idea of local stationarity and assume that a sequence of non-stationary processes $Y_N(t)$ can be locally approximated by a stationary process $\tilde{Y}_u(Nt)$, whenever t is close to u and N increases. We consider this approximation to hold in an L^p-sense, i.e. $\|Y_N(t) - \tilde{Y}_u(Nt)\|_{L^p}$ is small for t close to u. Considering a rescaled time domain Nt for the stationary process $\tilde{Y}_u(Nt)$, enables us to establish asymptotic results (see the introduction of [16] for a comprehensive discussion on the effects of rescaling).

Understanding this approximation in an L^p-sense has several advantages. On the one hand, we

show that many interesting asymptotic results carry over from the stationary approximation to the non-stationary process while, on the other hand, many continuous-time models are very well understood in an L^p-sense such that we present locally stationary approximations for a wide range of non-stationary models. This idea is motivated by recent work in [17], where the authors used similar locally stationary approximations in a discrete-time setting to obtain various analytical and statistical results.

Mathematically, we express the idea of a locally stationary approximation as follows.

Definition 2.1. *Let $Y_N = \{Y_N(t), t \in \mathbb{R}\}_{N \in \mathbb{N}}$ be a sequence of real-valued stochastic processes and $\tilde{Y} = \{\tilde{Y}_u(t), t \in \mathbb{R}\}_{u \in \mathbb{R}^+}$ a family of real-valued stationary processes. We assume that the process $\tilde{Y}_u$ is ergodic for all $u \in \mathbb{R}^+$ and $\sup_{u \in \mathbb{R}^+} \|\tilde{Y}_u(0)\|_{L^p} < \infty$ for some $p \geq 1$. If there exists a constant $C > 0$, such that uniformly in $t \in \mathbb{R}$ and $u, v \in \mathbb{R}^+$*

$$\left\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\right\|_{L^p} \leq C \left|u - v\right| \ \ and \ \ \left\|Y_N(t) - \tilde{Y}_t(Nt)\right\|_{L^p} \leq C \frac{1}{N}, \tag{LS}$$

then we call $\tilde{Y}_u$ a locally stationary approximation *of the sequence Y_N for p.*

Examples of sequences Y_N with locally stationary approximation $\tilde{Y}_u$ are given in Section 5. In particular, we derive sufficient conditions for the existence of locally stationary approximations of time-varying Lévy-driven state space models.

Throughout this paper, we assume that either continuous-time observations of the process Y_N are available or that the process has been sampled in the following way.

Assumption 2.2. *For fixed $N \in \mathbb{N}$ and $u \in \mathbb{R}^+$ we assume Y_N to be equidistantly observed at times $\tau_i^N = u + i\delta_N$ with grid size $\delta_N = |\tau_i^N - \tau_{i-1}^N|$ such that $\delta_N \downarrow 0$ for $N \to \infty$. For a sequence $b_N \downarrow 0$ we consider the observation window $[u - b_N, u + b_N]$ and set $m_N = \lfloor \frac{b_N}{\delta_N} \rfloor$. Thus, the number of observations is given by $2m_N + 1 = |\{i \in \mathbb{Z} : \tau_i^N \in [u - b_N, u + b_N]\}|$. We require $\frac{b_N}{\delta_N} \to \infty$ as $N \to \infty$ and either*

(O1) $N\delta_N = \delta > 0$ for all $N \in \mathbb{N}$ or

(O2) $N\delta_N \to \infty$ as $N \to \infty$.

Note that these conditions on N, b_N and δ_N immediately imply $Nb_N \to \infty$ as $N \to \infty$. The two conditions (O1) and (O2) describe how the locally stationary approximation $\tilde{Y}_u$ is observed, when the non-stationary process Y_N is sampled. In the case when (O1) holds, the stationary process $\tilde{Y}_u$ is observed equidistantly. Under (O2) the distance of two observations of the process $\tilde{Y}_u$ tends to ∞ as $N \to \infty$.

2.2 L^1-mixingale type dependence measures and θ-weak dependence

Considering a set of observations $\mathcal{T}_N = \{\tau_i^N, i = -m_N, \ldots, m_N\}$ as described above we note that this set contracts (with rate b_N) towards u as $N \to \infty$.

In fact, this contraction allows us to replace the observations $\{Y_N(\tau)\}_{\tau \in \mathcal{T}_N}$ in L^p by the corresponding observations of the triangular array $\{\tilde{Y}_u(N\tau)\}_{\tau \in \mathcal{T}_N}$ at rate $\frac{1}{N} + b_N$ (see e.g. the proof of Theorem 3.6). As usual, to establish asymptotic results for triangular arrays, it is necessary to impose additional conditions on the dependence structure as N increases.

By imposing conditions on the dependence structure of only the locally stationary approximation and avoiding assumptions on Y_N, we additionally allow for some model misspecification in

4

Y_N like an additional noise component which is decreasing in N and not captured in Y_N.
All upcoming asymptotic results will be based on the following L^1-mixingale type measure of dependence.

Definition 2.3. *For a σ-algebra $\mathscr{M}$ and a real-valued integrable random variable X we define*

(a) $\gamma(\mathscr{M}, X) = \|E[X|\mathscr{M}] - E[X]\|_1$ *and*

(b) $\theta(\mathscr{M}, X) = \sup_{g \in \mathscr{L}_1} \|E[g(X)|\mathscr{M}] - E[g(X)]\|_1$,

where $\mathscr{L}_1 = \{g : \mathbb{R} \to \mathbb{R}, \text{ bounded and Lipschitz with } Lip(g) \leq 1\}$. For a stationary and integrable stochastic process $(X_k)_{k \in \mathbb{Z}}$ we define the following dependence coefficients

$$\gamma_h = \gamma(\mathscr{M}_0, X_h) \text{ and}$$
$$\theta_h = \theta(\mathscr{M}_0, X_h),$$

where $h \in \mathbb{N}$ and $\mathscr{M}_0 = \sigma(\{X_k, k \leq 0\})$.

In particular, these measures will be readily accessible for many models through their connection to the following notion of weak dependence, called θ-weak dependence.

Definition 2.4. *Let $X = \{X(t)\}_{t \in \mathbb{R}}$ be a real-valued stochastic process. Then, X is called θ-weakly dependent if*

$$\theta(h) = \sup_{v \in \mathbb{N}} \theta_v(h) \underset{h \to \infty}{\longrightarrow} 0,$$

where

$$\theta_v(h) = \sup \left\{ \frac{|Cov(F(X(i_1), \ldots, X(i_v)), G(X(j)))|}{\|F\|_\infty Lip(G)}, F \in \mathcal{G}_v^*, G \in \mathcal{G}_1, i_1 \leq \ldots \leq i_v \leq i_v + h \leq j \right\}.$$

We call $(\theta(h))_{h \in \mathbb{R}_0^+}$ the θ-coefficients.

The following lemma gives a collection of auxiliary statements that help to connect the L^1-mixingale-type measure of dependence of a process with respect to its natural filtration to its θ-weak dependence coefficient.

Lemma 2.5. *For a σ-algebra $\mathscr{M}$, a real-valued integrable random variable Y and a real-valued stochastic process $X = \{X(t)\}_{t \in \mathbb{R}}$ it holds*

(a) $\gamma(\mathscr{M}, Y) \leq \theta(\mathscr{M}, Y)$ *such that* $\gamma_h \leq \theta_h, \quad h \in \mathbb{N}$,

(b) $\theta_h = \theta(h), \quad h \in \mathbb{N}$,

where $\theta(h)$ is the θ-coefficient of X.
Now, additionally assume that X is centered and θ-weakly dependent with θ-coefficient $\theta(h)$, $\mathscr{M}_0 = \sigma(\{X_k, k \in \mathbb{Z}, k \leq 0\})$ and $E[|X_0|^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Then,

(c) $\sum_{k=1}^\infty \|X_0 E[X_k|\mathscr{M}_0]\|_{L^1} \leq D \left(\sum_{h=1}^\infty (h+1)^{\frac{1}{\varepsilon}} \theta(h) \right)^{\frac{\varepsilon}{1+\varepsilon}}$ *for some constant $D > 0$.*

Proof. The first part of this Lemma is clear. The second part follows from [14, Lemma 5.1] for $m = 1$. The third part follows from the proof of [18, Lemma 2 and Corollary 1]. $\square$

Later, Lemma 2.5 will pave the way to show asymptotic results for sequences of processes that possess a locally stationary approximation that is θ-weakly dependent.

2.3 Hereditary properties under finite memory transformations

In this and the next subsection we investigate hereditary properties of locally stationary approximations and θ-weak dependence under transformations.

Consider a sequence of stochastic processes Y_N with locally stationary approximation $\tilde{Y}_u$ for some $p \geq 1$. For $k \in \mathbb{N}_0$ define the finite memory vectors

$$Z_N^k(t) = \left(Y_N(t), \ldots, Y_N\left(t - \tfrac{k}{N}\right)\right) \text{ and } \tilde{Z}_u^k(t) = (\tilde{Y}_u(t), \ldots, \tilde{Y}_u(t-k)).$$

Definition 2.6 ([17, Definition 2.4]). *A measurable function $g : \mathbb{R}^{k+1} \to \mathbb{R}$ is said to be in the class $\mathcal{L}_{k+1}(M, C)$ for $M, C \geq 0$, if*

$$\sup_{x \neq y} \frac{|g(x) - g(y)|}{\|x - y\|_1 \left(1 + \|x\|_1^M + \|y\|_1^M\right)} \leq C.$$

Proposition 2.7. *Let Y_N be a sequence of stochastic processes with locally stationary approximation $\tilde{Y}_u$ for some $\tilde{p} = p(M + 1)$, where $p \geq 1$, $M \geq 0$. Then, for $g \in \mathcal{L}_{k+1}(M, C)$, it holds:*

(a) *$g(\tilde{Z}_u^k)$ is a locally stationary approximation of the sequence $g(Z_N^k)$ for p.*

(b) *If $\tilde{Y}_u$ is θ-weakly dependent with θ-coefficients $\theta_{\tilde{Y}_u}(h)$, then $\tilde{Z}_u^k$ is θ-weakly dependent with θ-coefficients $\theta_{\tilde{Z}_u^k}(h) \leq (k+1)\theta_{\tilde{Y}_u}(h - (k+1))$ for $h \geq (k+1)$.*

(c) *If $\tilde{Y}_u$ is θ-weakly dependent with θ-coefficients $\theta_{\tilde{Y}_u}(h)$, $E[|\tilde{Y}_u(t)|^{(1+M+\varepsilon)}] < \infty$ for some $\varepsilon > 0$ and additionally $|g(x)| \leq \tilde{C} \|x\|_1^{M+1}$ for a constant $\tilde{C} > 0$, then $g(\tilde{Z}_u^k)$ is θ-weakly dependent with θ-coefficients*

$$\theta_{g(\tilde{Z}_u^k)}(h) = \mathcal{O}\left(\theta_{\tilde{Y}_u}(h)^{\frac{\varepsilon}{M+\varepsilon}}\right).$$

Proof. Part (a) follows immediately from Hoelder's inequality. For part (b) we refer to [20, Lemma 1]. Finally, part (c) follows from [4, Lemma 6]. $\qquad\square$

Remark 2.8. *Proposition 2.7 implies that all results on Y_N, that we give in the next sections, are also valid for transformations $g(Y_N^k)$ with $g \in \mathcal{L}_{k+1}(M, C)$ under additional moment conditions. Important examples of functions in $\mathcal{L}_{k+1}(M, C)$ are polynomials $h : \mathbb{R}^{k+1} \to \mathbb{R}$ with degree at most $(M + 1)$, satisfying $h(0) = 0$. Note that this also includes the covariance operator at lag k, i.e. $g(x_0, x_1, \ldots, x_k) = x_0 x_k$, such that $g \in \mathcal{L}_{k+1}(1, 1)$.*

Remark 2.9. *In fact, the condition $|g(x)| \leq C \|x\|_1^{M+1}$ from Proposition 2.7 (c) can be replaced by the equivalent condition $g(0) = 0$ (see [13, Proposition 3.4]).*

2.4 Hereditary properties under infinite memory transformations

Although the class $\mathcal{L}_{k+1}$ from above includes important transformations as the covariance operator, it does not cover transformations of vectors that are of infinite memory, i.e.

$$Z_N(t) = \left(Y_N(t), Y_N\left(t - \tfrac{1}{N}\right), \ldots\right) \text{ and } \tilde{Z}_u(t) = (\tilde{Y}_u(t), \tilde{Y}_u(t-1), \ldots).$$

Controlling such transformations is for instance important to prove asymptotic properties of certain estimators that are defined in terms of the full history of a process. The following class of functions will preserve locally stationary approximations of such infinite memory vectors and is a modification of equation (9) in [5].

Definition 2.10. *A measurable function $g : \mathbb{R}^\infty \to \mathbb{R}$ belongs to the class $\mathcal{L}_\infty^{p,q}(\alpha)$ for $p, q \geq 1$, if there exists a sequence $\alpha = (\alpha_k(g))_{k \in \mathbb{N}_0} \subset \mathbb{R}_0^+$ satisfying $\sum_{k=0}^\infty \alpha_k(g) < \infty$ and a function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for all sequences $X = (X_k)_{k \in \mathbb{N}_0} \in \ell^\infty(L^q)$ and $Y = (Y_k)_{k \in \mathbb{N}_0} \in \ell^\infty(L^q)$ it holds*

$$\|g(X) - g(Y)\|_{L^p} \leq f\left(\sup_{k \in \mathbb{N}_0} \{\|X_k\|_{L^q} \vee \|Y_k\|_{L^q}\} \right) \sum_{k=0}^\infty \alpha_k(g) \|X_k - Y_k\|_{L^q}.$$

The following result follows immediately from the Definition of the class $\mathcal{L}_\infty^{p,q}$ and Hoelder's inequality.

Proposition 2.11. *Let Y_N be a sequence of stochastic processes with locally stationary approximation $\tilde{Y}_u$ for some $q \geq 1$ and $g \in \mathcal{L}_\infty^{p,q}(\alpha)$ for some $p \geq 1$ such that $\sum_{k=0}^\infty k\alpha_k(g) < \infty$. Then, $g(\tilde{Z}_u(t))$ is a locally stationary approximation of $g(Z_N(t))$ for p.*

As we will see in the next remark, the class $\mathcal{L}_{k+1}$ can be embedded in the class $\mathcal{L}_\infty^{1,(M+1)}$.

Remark 2.12. *Asumme that $g \in \mathcal{L}_{k+1}(M, C)$. Then, $\tilde{g} = (g, 0, \ldots) \in \mathcal{L}_\infty^{1,(M+1)}(\alpha)$, where*

$$\alpha_n(\tilde{g}) = \begin{cases} 1, n \leq k+1, \\ 0, n > k+1 \end{cases} \quad \text{and } f(x) = C(1 + 2(k+1)x^M).$$

3   Asymptotic results

In this section, we discuss various asymptotic results including law of large numbers and central limit type results for sequences of stochastic processes that possess a locally stationary approximation. These asymptotic results provide the foundation for many statistical inference methodologies.

3.1   Law of large numbers

We give a series of law of large numbers for different observation schemes.

3.1.1   Law of large numbers under continuous observations

In the following, we give a global and a local law of large numbers. In order to obtain localized estimators, we introduce localizing kernels.

Definition 3.1. *Let $K : \mathbb{R} \to \mathbb{R}$ be a bounded function. If K is of bounded variation, has compact support $[-1, 1]$ and satisfies $\int_\mathbb{R} K(x)dx = 1$, then we call K a localizing kernel.*

From now on, if not otherwise stated, K always denotes a localizing kernel.

Theorem 3.2. *Consider a sequence of stochastic processes Y_N with locally stationary approximation $\tilde{Y}_u$ for some $p \geq 1$. Then:*

(a) For all $t \geq 0$,

$$\frac{1}{t} \int_0^t Y_N(\nu)d\nu \xrightarrow[N \to \infty]{L^p} \int_0^t E[\tilde{Y}_u(0)]du.$$

7

(b) Let K be a localizing kernel that is additionally differentiable and $b_N \downarrow 0$ with $Nb_N \to \infty$. Then,

$$\frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) Y_N(\tau) d\tau \xrightarrow[N\to\infty]{L^p} E[\tilde{Y}_u(0)].$$

The following lemma is an ergodic type result for weighted integrals of ergodic processes.

Lemma 3.3. Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a stationary ergodic process, $\|X(0)\|_{L^p} < \infty$ for some $p \geq 1$ and K a differentiable localizing kernel. Then, for all $u \in \mathbb{R}^+$

$$\frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) X(N\tau) d\tau \xrightarrow[N\to\infty]{L^p} E[X(0)],$$

where $b_N \to 0$ and $Nb_N \to \infty$ for $N \to \infty$.

Proof. First, note that $\frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) d\tau = \int_{-1}^{1} K(x) dx = 1$. Therefore, without loss of generality, we can assume that Y is centered, i.e. $E[Y(0)] = 0$. Let us first assume that $u = 0$. Now, using the integration by parts formula

$$\tilde{\mu}_N(0) = \frac{1}{b_N} \int_{-b_N}^{b_N} K\left(\frac{\tau}{b_N}\right) X(N\tau) d\tau$$

$$= \frac{1}{b_N} \left(K(1) \int_{-b_N}^{b_N} X(N\tau) d\tau - K(-1) \int_{-b_N}^{-b_N} X(N\tau) d\tau - \frac{1}{b_N} \int_{-b_N}^{b_N} K'\left(\frac{\tau}{b_N}\right) \int_{-b_N}^{\tau} X(Nu) du \, d\tau \right).$$

Since K is of bounded variation, we get

$$\|\tilde{\mu}_N(0)\|_{L^p} \leq C_1 \sup_{\tau\in[-b_N,b_N]} \left\| \frac{1}{b_N} \int_{-b_N}^{\tau} X(Nu) du \right\|_{L^p} = C_1 \sup_{\tau\in[0,2Nb_N]} \left\| \frac{1}{Nb_N} \int_{0}^{\tau} X(u) du \right\|_{L^p}$$

for a constant $C_1 > 0$. To lighten notation, we introduce $F(\tau) = \frac{1}{\tau} \int_0^\tau X(u) du$. From the continuous-time version of the Birkhoff-Khinchin theorem [25, Theorem 9.8] we get $F(\tau) \xrightarrow{L^p} E[X(0)] = 0$ as $\tau \to \infty$, such that

(a) $\|F(\tau)\|_{L^p} \leq C_2$ for a constant $C_2 > 0$ for all $\tau \in \mathbb{R}^+$ and

(b) for all $\varepsilon > 0$ there exists $c > 0$ such that $\|F(\tau)\|_{L^p} \leq \varepsilon$ for all $\tau > c$.

Now, let $\frac{\varepsilon}{2C_1} > 0$. Consider $c > 0$ such that $\|F(\tau)\|_{L^p} \leq \frac{\varepsilon}{2C_1}$ for all $\tau > c$. Then, for N sufficiently large

$$\begin{aligned}
\|\tilde{\mu}_N(0)\|_{L^p} &\leq 2C_1 \sup_{\tau\in[0,2Nb_N]} \left\| \frac{1}{2Nb_N} \int_0^\tau X(u) du \right\|_{L^p} \\
&\leq 2C_1 \sup_{\tau\in[0,c]} \left\| \frac{\tau}{2Nb_N} F(\tau) \right\|_{L^p} + 2C_1 \sup_{\tau\in[c,2Nb_N]} \left\| \frac{\tau}{2Nb_N} F(\tau) \right\|_{L^p} \\
&\leq \frac{cC_1}{Nb_N} \sup_{\tau\in[0,c]} \|F(\tau)\|_{L^p} + 2C_1 \sup_{\tau\in[c,2Nb_N]} \|F(\tau)\|_{L^p} \leq \frac{cC_1C_2}{Nb_N} + \varepsilon \xrightarrow[N\to\infty]{} \varepsilon,
\end{aligned}$$ (3)

which concludes the proof for $u = 0$. Since Y is stationary, we obtain for $u \in \mathbb{R}^+$

$$\|\tilde{\mu}_N(u)\|_{L^p} = \left\| \frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) X(N\tau) d\tau \right\|_{L^p} = \left\| \frac{1}{b_N} \int_{-b_N}^{b_N} K\left(\frac{\tau-u}{b_N}\right) X(N\tau + Nu) d\tau \right\|_{L^p}$$

8

$$= \left\| \frac{1}{b_N} \int_{-b_N}^{b_N} K\left(\frac{\tau}{b_N}\right) X(N\tau) d\tau \right\|_{L^p} \xrightarrow[N\to\infty]{} 0$$

by (3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Theorem 3.2.

(a) For $J, N \in \mathbb{N}$ and $j = 1, \ldots, 2^J$ we define $I_{J,j,N,t} = \{s : \frac{s}{N} \in (\frac{t(j-1)}{2^J}, \frac{tj}{2^J}]\}$, such that $\bigcup_{j=1}^{2^J} I_{J,j,N,t} = (0, tN]$. Notice that $\lambda(I_{J,j,N,t}) = \frac{Nt}{2^J}$. By (LS) we have

$$\left\| \frac{1}{t} \int_0^t (Y_N(\nu) - \tilde{Y}_\nu(N\nu)) d\nu \right\|_{L^p} \leq \sup_{s\in[0,t]} \left\| Y_N(s) - \tilde{Y}_s(Ns) \right\|_{L^p} \xrightarrow[N\to\infty]{} 0. \qquad (4)$$

Let us fix $J \in \mathbb{N}$. Then,

$$\left\| \frac{1}{t} \int_0^t \tilde{Y}_\nu(N\nu) d\nu - \frac{1}{2^J} \sum_{j=1}^{2^J} \frac{1}{\lambda(I_{J,j,N,t})} \int_{I_{J,j,N,t}} \tilde{Y}_{\frac{jt}{2^J}}(u) du \right\|_{L^p}$$

$$= \left\| \frac{1}{Nt} \int_0^{Nt} \tilde{Y}_{\frac{\nu}{N}}(\nu) d\nu - \frac{1}{2^J} \sum_{j=1}^{2^J} \frac{1}{\lambda(I_{J,j,N,t})} \int_{I_{J,j,N,t}} \tilde{Y}_{\frac{jt}{2^J}}(u) du \right\|_{L^p} \qquad (5)$$

$$= \left\| \frac{1}{2^J} \sum_{j=1}^{2^J} \frac{1}{\lambda(I_{J,j,N,t})} \int_{I_{J,j,N,t}} \left(\tilde{Y}_{\frac{\nu}{N}}(\nu) - \tilde{Y}_{\frac{jt}{2^J}}(\nu) \right) d\nu \right\|_{L^p}$$

$$\leq \sup_{s\in\mathbb{R}_0^+} \sup_{|u-v|\leq\frac{t}{2^J}} \left\| \tilde{Y}_u(s) - \tilde{Y}_v(s) \right\|_{L^p} \xrightarrow[J\to\infty]{} 0$$

by (LS). For fixed $J \in \mathbb{N}$ we use the continuous-time version of the Birkhoff-Khinchin theorem [25, Theorem 9.8] to obtain

$$E(J,N) := \frac{1}{2^J} \sum_{j=1}^{2^J} \frac{1}{\lambda(I_{J,j,N,t})} \int_{I_{J,j,N,t}} \tilde{Y}_{\frac{jt}{2^J}}(u) du \xrightarrow[N\to\infty]{L^p} \frac{1}{2^J} \sum_{j=1}^{2^J} E\left[\tilde{Y}_{\frac{jt}{2^J}}(0) \right] =: E(J). \qquad (6)$$

Furthermore, by the continuity of $u \mapsto E[\tilde{Y}_u(0)]$ (follows again from (LS)) we have

$$E(J) = \frac{1}{2^J} \sum_{j=1}^{2^J} E\left[\tilde{Y}_{\frac{jt}{2^J}}(0) \right] \xrightarrow[J\to\infty]{} \int_0^t E[\tilde{Y}_u(0)] du =: E. \qquad (7)$$

Finally, combining (4)-(7), we obtain for all $J \in \mathbb{N}$

$$\limsup_{N\to\infty} \left\| \frac{1}{t} \int_0^t Y_N(\nu) d\nu - \int_0^t E[\tilde{Y}_u(0)] du \right\|_{L^p}$$

$$\leq \limsup_{N\to\infty} \sup_{s\in[0,t]} \left\| Y_N(s) - \tilde{Y}_s(Ns) \right\|_{L^p} + \sup_{s\in\mathbb{R}_0^+} \sup_{|u-v|\leq\frac{t}{2^J}} \left\| \tilde{Y}_u(s) - \tilde{Y}_v(s) \right\|_{L^p}$$

$$+ \limsup_{N\to\infty} \| E(J,N) - E(J) \|_{L^p} + |E(J) - E|$$

$$\leq \sup_{|u-v|\leq\frac{t}{2^J}} C|u - v| + |E(J) - E|$$

by (LS), which converges to 0 for $J \to \infty$.

(b) It holds

$$\left\| \frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) Y_N(\tau)d\tau - E[\tilde{Y}_u(0)]\right\|_{L^p}$$

$$\leq \left\| \frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) \left(Y_N(\tau) - \tilde{Y}_\tau(N\tau)\right) d\tau\right\|_{L^p}$$

$$+ \left\| \frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) \left(\tilde{Y}_\tau(N\tau) - \tilde{Y}_u(N\tau)\right) d\tau\right\|_{L^p}$$

$$+ \left\| \frac{1}{b_N} \int_{u-b_N}^{u+b_N} K\left(\frac{\tau-u}{b_N}\right) \tilde{Y}_u(N\tau)d\tau - E[\tilde{Y}_u(0)]\right\|_{L^p} =: P_1 + P_2 + P_3.$$

Since P_3 converges to zero for $N \to \infty$ by Lemma 3.3 it remains to analyze P_1 and P_2. Considering P_1 we get

$$P_1 \leq 2\left\|K\right\|_\infty \sup_{\tau\in[u-b_N,u+b_N]} \left\|Y_N(\tau) - \tilde{Y}_\tau(N\tau)\right\|_{L^p} \xrightarrow[N\to\infty]{} 0$$

by (LS). For the second summand it holds

$$P_2 \leq 2\left\|K\right\|_\infty \sup_{\tau\in[u-b_N,u+b_N]} \sup_{|u-v|\leq 2b_N} \left\|\tilde{Y}_u(N\tau) - \tilde{Y}_v(N\tau)\right\|_{L^p} \xrightarrow[N\to\infty]{} 0,$$

again by (LS), since $b_N \to 0$. □

3.1.2 Law of large numbers under discrete observations

In this section we give a series of local law of large numbers under discrete observations as considered in Assumption 2.2. Depending on whether (O1) or (O2) holds, we obtain different results.

We start by assuming that (O1) holds and give a discrete-time version of Lemma 3.3.

Lemma 3.4. *Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a stationary ergodic process and $\|X(0)\|_{L^p} < \infty$ for some $p \geq 1$. Additionally, assume that (O1) holds. Then, for all $u \in \mathbb{R}^+$*

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) X(N\tau_i^N) \xrightarrow[N\to\infty]{L^p} E[X(0)].$$

Proof. We first note that, since (O1) holds, we have $N\tau_i^N = Nu + i\delta$ for some $\delta > 0$. In principle the proof is a discrete time version of the proof of Lemma 3.3. We give the most important steps. Since $\lim_{N\to\infty} \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N-u}{b_N}\right) = \int_{-1}^1 K(x)dx = 1$, we can assume without loss of generality that X is centered. For $u = 0$ and $S_k = \sum_{i=-m_N}^k X(i\delta)$ we obtain from the summation by parts formula

$$\hat{\mu}_N(0) := \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N}{b_N}\right) X(i\delta)$$

$$= \frac{\delta_N}{b_N} \left(S_{m_N} K\left(\frac{\tau_{m_N}^N}{b_N}\right) + \sum_{k=-m_N}^{m_N-1} S_k \left(K\left(\frac{\tau_k^N}{b_N}\right) - K\left(\frac{\tau_{k+1}^N}{b_N}\right)\right)\right).$$

Since K is of bounded variation, we obtain

$$\|\hat{\mu}_N(0)\|_{L^p} \leq C_1 \frac{\delta_N}{b_N} \sup_{k \in \{-m_N, \ldots, m_N\}} \|S_k\|_{L^p}$$

$$= C_1 \frac{(2m_N + 1)\delta_N}{b_N} \sup_{k \in \{0, \ldots, 2m_N\}} \left\| \frac{1}{2m_N + 1} \sum_{i=0}^{k} X(i\delta) \right\|_{L^p} \xrightarrow[N \to \infty]{} 0$$

for a constant $C_1 > 0$ by similar arguments as in (3). For general $u \in \mathbb{R}^+$, the result follows from the stationarity of X. $\qquad\square$

Theorem 3.5. *Consider a sequence of stochastic processes Y_N with locally stationary approximation $\tilde{Y}_u$ for some $p \geq 1$. For observations as given in Assumption 2.2 such that (O1) holds, i.e. $N\delta_N = \delta$ for some $\delta > 0$, we obtain for all $u \in \mathbb{R}^+$*

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N \to \infty]{L^p} E[\tilde{Y}_u(0)].$$

Proof. Consider the decomposition

$$\left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) - E[\tilde{Y}_u(0)] \right\|_{L^p}$$

$$\leq \left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) - \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_{\tau_i^N}(N\tau_i^N) \right\|_{L^p}$$

$$+ \left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_{\tau_i^N}(N\tau_i^N) - \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_u(N\tau_i^N) \right\|_{L^p}$$

$$+ \left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_u(N\tau_i^N) - E[\tilde{Y}_u(0)] \right\|_{L^p} =: P_1 + P_2 + P_3.$$

For P_1 and P_2 we obtain

$$P_1 + P_2 \leq \frac{C \|K\|_\infty \delta_N (2m_N + 1)}{b_N} \left(\frac{1}{N} + b_N\right) \xrightarrow[N \to \infty]{} 0$$

for some constant $C > 0$. Finally, due to Lemma 3.4, we also have $P_3 \to 0$, as $N \to \infty$. $\qquad\square$

If (O2) holds, we have to impose conditions on the dependence structure of the locally stationary approximation $\tilde{Y}_u$ to obtain similar results as in Theorem 3.5 for $p = 1$. Under additional moment conditions a rough inequality extends the result to $p = 2$.

Theorem 3.6. *Consider a sequence of stochastic processes Y_N with locally stationary approximation $\tilde{Y}_u$ for some $p \geq 1$. For $u \in \mathbb{R}^+$ fixed, assume that $\tilde{Y}_u$ is θ-weakly dependent. Then, for observations as considered in Assumption 2.2 such that (O2) holds, we obtain the following convergences:*

(a) It holds

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N \to \infty]{L^1} E[\tilde{Y}_u(0)].$$

(b) *Assume that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for some $p \geq 2$ and that $E[|\tilde{Y}_u(0)|^{2+\varepsilon}] < \infty$, for some $\varepsilon > 0$, then for all $u \in \mathbb{R}^+$*

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N\to\infty]{L^2} E[\tilde{Y}_u(0)].$$

Proof. (a) Consider the same decomposition as in the proof of Theorem 3.5 for $p = 1$. In the same way we have $P_1 \to 0$ and $P_2 \to 0$ as $N \to \infty$. To show that $P_3 \xrightarrow[N\to\infty]{} 0$, we apply [1, Theorem 2]. Again, since $\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \longrightarrow \int_{-1}^{1} K(x)dx = 1$ as $N \to \infty$, we can assume without loss of generality that $\tilde{Y}_u$ is centered. Consider the centered triangular array $X_{i,2m_N+1}$ and the corresponding filtration $\mathcal{M}_{i,2m_N+1}$, respectively, defined as

$$W_{i,2m_N+1} = \tilde{Y}_u\left(N(u + (i - m_N - 1)\delta_N)\right)$$

$$X_{i,2m_N+1} = \frac{(2m_N+1)\delta_N}{b_N} K\left(\frac{\tau_{i-m_N-1}^N - u}{b_N}\right) W_{i,2m_N+1} \text{ and}$$

$$\mathcal{M}_{i,2m_N+1} := \sigma(\{\tilde{Y}_u(N(u + (j - m_N + 1)\delta_N)), j \leq i, j \in \mathbb{Z}\}), \ i \in \mathbb{Z},$$

where the family $\{\mathcal{M}_{i,2m_N+1}\}_{i=1,\ldots,2m_N+1}$ is non-decreasing.
Let us note that the family $\{X_{i,2m_N+1}\}_{i=1,\ldots,2m_N+1,N\in\mathbb{N}}$ is uniformly integrable due to the stationarity of $\tilde{Y}_u$. Moreover, since $\tilde{Y}_u$ is θ-weakly dependent with θ-coefficients $\theta(h)$, also the sampled process $\{W_{i,2m_N+1}\}_{i\in\mathbb{Z}}$ is θ-weakly dependent with θ-coefficients $\theta^{(2m_N+1)}(h) = \theta(N\delta_N h)$. Now, because $N\delta_N \to \infty$ as $N \to \infty$ and $\theta(h)$ is non-increasing, we have

$$\max_{N\in\mathbb{N}} \theta^{(2m_N+1)}(h) = \theta(C_1 h) \tag{8}$$

for the constant $C_1 = \min_{N\in\mathbb{N}} N\delta_N > 0$. In view of Definition 2.3 and Lemma 2.5 we obtain from the stationarity of $\tilde{Y}_u$ for all $m \in \mathbb{N}_0$

$$\left\| E[X_{i,2m_N+1}|\mathcal{M}_{(i-m),2m_N+1}] \right\|_{L^1} = \gamma(\mathcal{M}_{(i-m),2m_N+1}, X_{i,2m_N+1})$$

$$\leq C_2\gamma(\mathcal{M}_{0,2m_N+1}, W_{m,2m_N+1}) \leq C_2\theta(\mathcal{M}_{0,2m_N+1}, W_{m,2m_N+1}) = C_2\theta^{2m_N+1}(m)$$

$$\leq C_2\theta(N\delta_N m) \leq C_2\theta(C_1 m) \xrightarrow[m\to\infty]{} 0$$

for some constant $C_2 > 0$, where we used (8). Overall, we have that the triangular array $X_{i,2m_N+1}$ is an L^1-mixingale with respect to the filtration $\mathcal{M}_{i,2m_N+1}$ in the sense of [1, Definition 2]. An application of [1, Theorem 2] gives $P_3 \xrightarrow[N\to\infty]{} 0$.

(b) Consider the same decomposition as in the proof of Theorem 3.5 for $p = 2$. In the same way we have $P_1 \to 0$ and $P_2 \to 0$ as $N \to \infty$. Again, without loss of generality we assume that $\tilde{Y}_u$ is centered. For P_3 we obtain

$$\left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_u(N\tau_i^N) \right\|^2 = \frac{\delta_N^2}{b_N^2}\left(\sum_{i=-m_N}^{m_N} Var\left(K\left(\frac{\tau_i^N - u}{b_N}\right)\tilde{Y}_u(N\tau_i^N)\right) \right.$$

$$+ \sum_{i\neq j} Cov\left(K\left(\frac{\tau_i^N - u}{b_N}\right)\tilde{Y}_u(N\tau_i^N), K\left(\frac{\tau_j^N - u}{b_N}\right)\tilde{Y}_u(N\tau_j^N)\right) \right)$$

$$\leq \frac{\|K\|_\infty^2 \delta_N^2 (2m_N+1)^2}{b_N^2}\left(\frac{E[\tilde{Y}_u(0)^2]}{2m_N + 1} + \sup_{i\neq j}\left|Cov\left(\tilde{Y}_u(0), \tilde{Y}_u(N\delta_N|j-i|)\right)\right|\right)$$

12

$$\leq \frac{\|K\|_\infty^2 \, \delta_N^2 (2m_N+1)^2}{b_N^2} \left(\frac{E[\tilde{Y}_u(0)^2]}{2m_N+1} + 9E[|\tilde{Y}_u(0)|^{2+\varepsilon}]^{\frac{1}{1+\varepsilon}} \theta(N\delta_N)^{\frac{\varepsilon}{1+\varepsilon}} \right),$$

which converges to zero as $N \to \infty$, since $\theta(h) \downarrow 0$ as $h \to \infty$ and $i \neq j$. The last inequality follows from [13, Remark 3.3]. For the convergence we use the results on the θ-coefficients of part (a).

$\square$

3.2 Central limit type results

The following theorem gives a central limit type result for sequences of non-stationary processes that possess a locally stationary approximation in the sense of Definition 2.1. We again consider the previous observation schemes and restrict ourselves to the rectangular kernel

$$K_{rect}(x) = \frac{1}{2}\mathbb{1}_{\{x \in [-1,1]\}} \tag{9}$$

as localization kernel. It is easy to see that K_{rect} is a localizing kernel as defined in Definition 3.1. Depending on whether (O1) or (O2) holds, we obtain different asymptotic variances.
In addition, we assume the θ-coefficients $\theta(h)$ of the locally stationary approximation to satisfy for a fixed $\varepsilon > 0$ the condition

$$DD(\varepsilon): \qquad \sum_{h=1}^{\infty} \theta(h) h^{\frac{1}{\varepsilon}} < \infty.$$

Sufficient conditions for $DD(\varepsilon)$ to hold are for instance $\theta(h) \in \mathcal{O}(h^{-\alpha})$ or $\theta(h) \in \mathcal{O}(h \ln(h)^{-\alpha})$ for some $\alpha > (1 + \frac{1}{\varepsilon})$.

Theorem 3.7. *Consider a sequence of stochastic processes Y_N with centered locally stationary approximation $\tilde{Y}_u$ for some $p \geq 2$ such that $\tilde{Y}_u(0) \in L^{2+\varepsilon}$ for some $\varepsilon > 0$. Additionally, assume that $\tilde{Y}_u$ is θ-weakly dependent with θ-coefficients $\theta(h)$ satisfying $DD(\varepsilon)$. Assume that we observe Y_N as described in Assumption 2.2, such that additionally $\sqrt{m_N} b_N \to 0$. Now, if either*

(a) (O1) holds, i.e. $N\delta_N = \delta > 0$ and $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2] + \sum_{k=1}^{\infty} E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)] > 0$ or

(b) (O2) holds and $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2] > 0$,

then $\sigma(u)^2 < \infty$ and

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N \to \infty]{d} \sigma(u)Z, \tag{10}$$

where Z is a standard normally distributed random variable and K_{rect} the rectangular kernel as defined in (9).

It is natural to ask for conditions on stronger concepts of convergence as the stated convergence in distribution in (10). The following corollary provides additional assumptions that strengthen this convergence to stable convergence. Recall that a sequence of integrable random variables $(Y_n)_{n \in \mathbb{N}}$ converges stably with limit Y, where Y is defined on an extension $(\Omega', \mathcal{F}', P')$, if $E[g(Y_n)Z] \longrightarrow E'[g(Y)Z]$ as $n \to \infty$ for all bounded, continuous functions g and any bounded $\mathcal{F}$-measurable random variable Z. Then, we write $Y_n \xrightarrow[n \to \infty]{st} Y$. Note that stable convergence immediately implies convergence in distribution.

Corollary 3.8. *Consider a sequence of stochastic processes Y_N with centered locally stationary approximation $\tilde{Y}_u$ for some $p \geq 2$ satisfying the assumptions of Theorem 3.7. If the family*

$$\{\mathcal{M}_{i,2m_N+1}\}_{N\in\mathbb{N}} = \sigma\left(\left\{\tilde{Y}_u(N(u+(j-m_N+1)\delta_N)), j \leq i, j \in \mathbb{Z}\right\}\right), \quad i \in \mathbb{Z}$$

is non-decreasing in N for all $i \in \mathbb{Z}$, then

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N\to\infty]{st} \sigma(u)Z.$$

Proof. It is sufficient to follow the proof of Theorem 3.7 and use [22, Theorem 3.7 (i)] as well as [19, Corollary 1]. $\qquad\square$

In the following lemma we give sufficient conditions for $\{\mathcal{M}_{i,2m_N+1}\}_{N\in\mathbb{N}}$ to be non-decreasing.

Lemma 3.9. *Consider a sequence of stochastic processes Y_N with centered locally stationary approximation $\tilde{Y}_u$ for some $p \geq 2$ satisfying the assumptions of Theorem 3.7 such that (O1) holds with $N\delta_N = \Delta > 0$. Additionally assume that u is a multiple of Δ and that the increments of Nb_N are bounded by u, i.e. $(N+1)b_{N+1} - Nb_N \leq u$. Then, the family $\{\mathcal{M}_{i,2m_N+1}\}_{N\in\mathbb{N}}$ is non-decreasing for all $i \in \mathbb{Z}$.*

Proof. We need to show that for any $N \in \mathbb{N}$, $i \in \mathbb{Z}$ and $j \leq i$ there exists $j_1 \in \mathbb{Z}$, $j_1 \leq i$ such that $Nu + \delta(j - \lfloor\frac{Nb_N}{\delta}\rfloor + 1) = (N+1)u + \delta(j_1 - \lfloor\frac{(N+1)b_{N+1}}{\delta}\rfloor + 1)$. We set $j_1 = j - \lfloor\frac{Nb_N}{\delta}\rfloor + \lfloor\frac{(N+1)b_{N+1}}{\delta}\rfloor - \frac{u}{\delta} \in \mathbb{Z}$. Then, $j_1 \leq i$, since

$$\left\lfloor\frac{(N+1)b_{N+1}}{\delta}\right\rfloor - \left\lfloor\frac{Nb_N}{\delta}\right\rfloor \leq \left\lfloor\frac{(N+1)b_{N+1} - Nb_N}{\delta}\right\rfloor \leq \left\lfloor\frac{u}{\delta}\right\rfloor = \frac{u}{\delta}.$$

$\qquad\square$

A particular choice for b_N and δ_N satisfying the assumptions of Lemma 3.9 is for instance $b_N = bN^{-\frac{2}{3}}$ and $\delta_N = \delta N^{-1}$, where $0 < b < u$ and $u = m\delta$ for some $m \in \mathbb{N}$, $\delta \in \mathbb{R}^+$.

Proof of Theorem 3.7. Since we obtain from (LS)

$$\left\|\sqrt{\frac{\delta_N}{b_N}}\sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right)\left(Y_N(\tau_i^N) - \tilde{Y}_{\tau_i^N}(N\tau_i^N) + \tilde{Y}_{\tau_i^N}(N\tau_i^N) - \tilde{Y}_u(N\tau_i^N)\right)\right\|_{L^1}$$

$$\leq \frac{C(2m_N+1)}{2}\sqrt{\frac{\delta_N}{b_N}}\left(\frac{1}{N} + b_N\right) \xrightarrow[N\to\infty]{} 0,$$

it is enough to show that

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) \tilde{Y}_u(N\tau_i^N) \xrightarrow[N\to\infty]{d} \sigma(u)Z,$$

where Z is a standard normally distributed random variable. We define for $i \in \mathbb{Z}$ and $N \in \mathbb{N}$

$$W_{i,2m_N+1} := \tilde{Y}_u(N\tau_{i-m_N-1}^N) = \tilde{Y}_u(N(u+(i-m_N-1)\delta_N)),$$

14

$$X_{i,2m_N+1} := \sqrt{2m_N+1}\sqrt{\frac{\delta_N}{b_N}}K_{rect}\left(\frac{\tau_{i-m_N-1}^N - u}{b_N}\right)W_{i,2m_N+1},$$

$$S_N(t) := \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} X_{i,2m_N+1}, \quad S_N := S_N(1) \quad \text{and}$$

$$\mathcal{M}_{i,2m_N+1} := \sigma\left(\left\{\tilde{Y}_u(N(u + (j - m_N + 1)\delta_N)), j \leq i, j \in \mathbb{Z}\right\}\right).$$

As a major first step we show that the conditional central limit theorem [19, Theorem 2] holds. From this conditional central limit theorem we then derive the convergence in (10).

Note that $\{\mathcal{M}_{i,2m_N+1}\}_{i\in\mathbb{N}}$ is non-decreasing and

$$P\left(\left|\frac{1}{\sqrt{2m_N+1}}X_{0,2m_N+1}\right| > \varepsilon\right) \leq \frac{\left\|\tilde{Y}_u(0)\right\|_{L^1}}{2\varepsilon}\sqrt{\frac{\delta_N}{b_n}} \xrightarrow[N\to\infty]{} 0$$

for all $\varepsilon > 0$, since $\tilde{Y}_u$ is stationary. We proceed by considering the conditions (a) and (b) separately.

(b) Let us assume that (O2) holds, i.e. $N\delta_N \to \infty$ as $N \to \infty$. Following [19, Corollary 3 and Remark 9], it is enough to show that for all $t \in (0,1]$

$$\lim_{N\to\infty} \sum_{k=1}^{2m_N+1} \|X_{0,2m_N+1}E\left[X_{k,2m_N+1}|\mathcal{M}_{0,2m_N+1}\right]\|_{L^1} = 0, \tag{11}$$

$$\lim_{N\to\infty} \frac{1}{\sqrt{2m_N+1}}\left\|E\left[S_N(t)\middle|\mathcal{M}_{0,2m_N+1}\right]\right\|_{L^1} = 0 \text{ and} \tag{12}$$

$$\frac{1}{t(2m_N+1)} \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} X_{i,2m_N+1}^2 \xrightarrow[N\to\infty]{L^1} \sigma(u)^2, \tag{13}$$

where $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2]$ is non-negative.

With the help of Lemma 2.5 we will now connect the conditions (11) and (12) to the θ-coefficients of $\tilde{Y}_u$. We start by proving (11). The sampled process $\{W_{i,2m_N+1}\}_{i\in\mathbb{N}}$ is θ-weakly dependent with θ-coefficients $\theta^{(2m_N+1)}(h) = \theta(N\delta_N h)$, where $\max_{N\in\mathbb{N}}\theta^{(2m_N+1)}(h) = \theta(C_1 h)$ for a constant $C_1 = \min_{N\in\mathbb{N}} N\delta_N > 0$. Then, since $\lim_{N\to\infty}(2m_N+1)\frac{\delta_N}{b_N} = 2$, (11) is bounded from above by

$$\lim_{N\to\infty} \frac{1}{2} \sum_{k=1}^{\infty} \|W_{0,2m_N+1}E\left[W_{k,2m_N+1}|\mathcal{M}_{0,2m_N+1}\right]\|_{L^1}.$$

For $U_k^{(2m_N+1)} = \|W_{0,2m_N+1}E\left[W_{k,2m_N+1}|\mathcal{M}_{0,2m_N+1}\right]\|_{L^1}$, we obtain due to part (c) of Lemma 2.5 and (8)

$$\sum_{k=1}^{\infty} U_k^{(2m_N+1)} \leq D\left(\sum_{h=1}^{\infty}(h+1)^{\frac{1}{\varepsilon}}\theta^{(2m_N+1)}(h)\right)^{\frac{\varepsilon}{1+\varepsilon}} \leq D\left(\sum_{h=1}^{\infty}(h+1)^{\frac{1}{\varepsilon}}\theta(C_1 h)\right)^{\frac{\varepsilon}{1+\varepsilon}}.$$

Note that $(x + y)^p \leq \max(1, 2^{p-1})(x^p + y^p)$ for all $x, y, p \in \mathbb{R}_0^+$. Hence, for some α such

that $0 < \alpha < \min(1, C_1)$ we obtain

$$D\left(\sum_{h=1}^{\infty}(h+1)^{\frac{1}{\varepsilon}}\theta(C_1 h)\right)^{\frac{\varepsilon}{1+\varepsilon}} \leq D\left(\sum_{h=1}^{\infty}(h+1)^{\frac{1}{\varepsilon}}\theta(\lfloor \alpha h\rfloor)\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

$$= D\left(\sum_{k=0}^{\infty}\theta(k)\sum_{\{h\in\mathbb{N}:\frac{k}{\alpha}\leq h<\frac{k+1}{\alpha}\}}(h+1)^{\frac{1}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \leq D\left(\left(1+\frac{1}{\alpha}\right)\sum_{k=0}^{\infty}\theta(k)(\frac{k}{\alpha}+\frac{1+\alpha}{\alpha})^{\frac{1}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\leq D\left(\left(\max(1,2^{\frac{1}{\varepsilon}-1})\right)\left(\frac{1+\alpha}{\alpha^{1+\frac{1}{\varepsilon}}}\right)\sum_{k=0}^{\infty}\theta(k)\left(2^{\frac{1}{\varepsilon}}+k^{\frac{1}{\varepsilon}}\right)\right)^{\frac{\varepsilon}{1+\varepsilon}},$$

$$\tag{14}$$

which is finite, since DD(ε) holds. By the dominated convergence theorem we obtain

$$\lim_{N\to\infty}D\left(\sum_{h=1}^{\infty}(h+1)^{\frac{1}{\varepsilon}}\theta(N\delta_N h)\right)^{\frac{\varepsilon}{1+\varepsilon}} = 0, \text{ such that (11) holds.}$$

To show (12), we first note that since $\tilde{Y}_u$ is centered, we have

$$\|E\left[W_{i,2m_N+1}|\mathcal{M}_{0,2m_N+1}\right]\|_{L^1} = \gamma(\mathcal{M}_{0,2m_N+1}, W_{i,2m_N+1})$$
$$\leq \theta(\mathcal{M}_{0,2m_N+1}, W_{i,2m_N+1}) = \theta^{(2m_N+1)}(i),$$

where we used part (a) and (b) of Lemma 2.5. Thus,

$$\lim_{N\to\infty}\frac{1}{\sqrt{2m_N+1}}\left\|E\left[S_N(t)\big|\mathcal{M}_{0,2m_N+1}\right]\right\|_{L^1} \leq \frac{1}{\sqrt{2m_N+1}}\sum_{i=1}^{2m_N+1}\|E\left[X_{i,2m_N+1}|\mathcal{M}_{0,2m_N+1}\right]\|_{L^1}$$

$$\leq \frac{1}{2}\sqrt{\frac{\delta_N}{b_N}}\sum_{i=1}^{2m_N+1}\theta^{(2m_N+1)}(i) = \frac{1}{2}\sqrt{\frac{\delta_N}{b_N}}\sum_{i=1}^{2m_N+1}\theta(N\delta_N i) \leq C_2\sqrt{\frac{\delta_N}{b_N}} \to 0$$

as $N\to\infty$ for some constant $C_2 > 0$, since DD(ε) holds, using similar arguments as in (8) and (14).

To show (13) for $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2]$, we define for $t\in(0,1]$

$$G_t(x) = \frac{2K_{rect}^2(x)}{t}\mathbb{1}_{\{x\in[-1,-1+2t]\}}, \ x\in\mathbb{R}. \tag{15}$$

It is easy to see that G_t is a localizing kernel for all $t\in(0,1]$. As the function $x\mapsto x^2 \in \mathcal{L}_1(1,1)$, Proposition 2.7 ensures that the stationary process $\tilde{Y}_u^2$ is θ-weakly dependent. Then, following the same steps as in the proof of Theorem 3.6 part (a), using the localizing kernel $G_t(x)$, we obtain

$$\left\|\frac{\delta_N}{b_N}\sum_{i=-m_N}^{m_N}\frac{1}{2}G_t\left(\frac{\tau_i^N - u}{b_N}\right)\tilde{Y}_u(N\tau_i^N)^2 - \frac{E[\tilde{Y}_u(0)^2]}{2}\right\|_{L^1}\xrightarrow[N\to\infty]{}0.$$

Therefore, to show (13) for $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2]$, it is enough to observe

$$
\frac{\delta_N}{b_N}\Bigg\|\frac{1}{t}\sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} K_{rect}\Big(\frac{\tau^N_{i-m_N-1}-u}{b_N}\Big)^2 \tilde{Y}_u(N\tau^N_{i-m_N-1})^2
$$
$$
-\sum_{i=-m_N}^{m_N}\frac{1}{2}G_t\Big(\frac{\tau^N_i-u}{b_N}\Big)\tilde{Y}_u(N\tau^N_i)^2\Bigg\|_{L^1}
$$
$$
\leq \frac{E[\tilde{Y}_u(0)^2]}{4t}\frac{\delta_N}{b_N}\sum_{i=1}^{2m_N+1}\Bigg|\mathbb{1}_{\{i\in[1,\lfloor t(2m_N+1)\rfloor]\}}-\mathbb{1}_{\left\{\frac{(i-m_N-1)\delta_N}{b_N}\in[-1,2t-1]\right\}}\Bigg| \tag{16}
$$
$$
= \frac{E[\tilde{Y}_u(0)^2]}{4t}\frac{\delta_N}{b_N}\sum_{i=1}^{2m_N+1}\Bigg|\mathbb{1}_{\{i\in[1,\lfloor t(2m_N+1)\rfloor]\}}-\mathbb{1}_{\left\{i\in\left[1,\frac{b_N}{\delta_N}(2t-1)+m_N+1\right]\right\}}\Bigg|
$$
$$
\leq \frac{E[\tilde{Y}_u(0)^2]}{4t}\frac{\delta_N}{b_N}\Big|\Big\{i\in\mathbb{Z},i\in[1,\lfloor t(2m_N+1)\rfloor]\triangle[1,\tfrac{b_N}{\delta_N}(2t-1)+m_N+1]\Big\}\Big|
$$
$$
\leq \frac{E[\tilde{Y}_u(0)^2]}{4t}\frac{\delta_N}{b_N}\Big(\Big|\lfloor t(2m_N+1)\rfloor-2t\tfrac{b_N}{\delta_N}+\tfrac{b_N}{\delta_N}-m_N-1\Big|+1\Big)\xrightarrow[N\to\infty]{}0,
$$

since $\Big(\Big|\lfloor t(2m_N+1)\rfloor-2t\tfrac{b_N}{\delta_N}+\tfrac{b_N}{\delta_N}-m_N-1\Big|+1\Big)$ is bounded by $|\lfloor t(2m_N+1)\rfloor-2t\tfrac{b_N}{\delta_N}-1|+|m_N-\tfrac{b_N}{\delta_N}|+1\leq 2t+2\leq 4$. Using [19, Corollary 3 and Remark 9], the conditional central limit theorem [19, Theorem 2] holds and for $t=1$ we have

$$
\lim_{N\to\infty}\Big\|E\Big[\varphi\Big(\tfrac{1}{\sqrt{2m_N+1}}S_N\Big)-\int_{\mathbb{R}}\varphi(x\sigma(u))g(x)dx\Big|\mathcal{M}_{k,2m_N+1}\Big]\Big\|_{L^1}=0 \tag{17}
$$

for all $k\in\mathbb{N}$ and $\varphi\in\mathcal{H}=\{\varphi:\mathbb{R}\to\mathbb{R},\text{continuous such that } x\mapsto|(1+x^2)^{-1}\varphi(x)| \text{ is bounded}\}$. In order to show the stated convergence in (10), it is sufficient to observe that for all continuous and bounded functions h and a standard normally distributed random variable Z it holds

$$
\Big|E\Big[h\Big(\tfrac{1}{\sqrt{2m_N+1}}S_N\Big)\Big]-E\Big[h\Big(\sigma(u)Z\Big)\Big]\Big|
$$
$$
\leq\Big\|E\Big[h\Big(\tfrac{1}{\sqrt{2m_N+1}}S_N\Big)-\int_{\mathbb{R}}h\Big(x\sigma(u)\Big)g(x)dx\Big|\mathcal{M}_{1,2m_N+1}\Big]\Big\|_{L^1}\xrightarrow[N\to\infty]{}0, \tag{18}
$$

since $h\in\mathcal{H}$ and (17) holds.

(a) Let us assume that (O1) holds, i.e. $N\delta_N=\delta>0$. In the following we use [19, Corollary 3], where we define

$$
\sigma(u)^2=\frac{1}{2}E[\tilde{Y}_u(0)^2]+\sum_{k=1}^{\infty}E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)]<\infty.
$$

Let us briefly mention that, due to the decay of $\theta(h)$ (i.e. since $DD(\varepsilon)$ holds) and [13, Remark 3.3], $\sigma(u)^2=\frac{E[\tilde{Y}_u(0)^2]}{2}+\sum_{k=1}^{\infty}E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)]$ is finite.
Overall, we obtain a similar convergence as given in (17), if for any $t\in(0,1]$ and $k\in\mathbb{N}_0$

$$
\lim_{K\to\infty}\limsup_{N\to\infty}\sup_{K\leq m\leq 2m_N+1}\Big\|X_{0,2m_N+1}\sum_{k=K}^{m}E[X_{k,2m_N+1}|\mathcal{M}_{0,2m_N+1}]\Big\|_{L^1}=0, \tag{19}
$$

17

$$\lim_{N\to\infty} \frac{1}{\sqrt{2m_N+1}} \left\| E\left[S_N(t) \big| \mathcal{M}_{0,2m_N+1} \right] \right\|_{L^1} = 0 \text{ and} \tag{20}$$

$$\frac{1}{t(2m_N+1)} \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} X_{i,2m_N+1} X_{i+k,2m_N+1} \xrightarrow[N\to\infty]{L^1} \lambda_k, \tag{21}$$

where $\lambda_k = E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)]$. We first note that (20) follows analogously to the proof of (12).

To show (19), we follow similar steps as in the proof of (11) and obtain for $K \in \mathbb{N}$

$$\limsup_{N\to\infty} \sup_{K\le m\le 2m_N+1} \left\| X_{0,2m_N+1} \sum_{k=K}^{m} E[X_{k,2m_N+1}|\mathcal{M}_{0,2m_N+1}] \right\|_{L^1}$$

$$\le \frac{1}{2} \limsup_{N\to\infty} \sum_{k=K}^{\infty} \left\| \tilde{Y}_u(0) E[\tilde{Y}_u(k\delta)|\sigma(\{\tilde{Y}_u(j\delta): j\le 0, j\in\mathbb{Z}\})] \right\|_{L^1} =: V(K),$$

since $\tilde{Y}_u$ is stationary. Now, due to the third part of Lemma 2.5, we obtain

$$\limsup_{N\to\infty} \sum_{k=1}^{\infty} \left\| \tilde{Y}_u(0) E[\tilde{Y}_u(k\delta)|\sigma(\{\tilde{Y}_u(j\delta): j\le 0, j\in\mathbb{Z}\}] \right\|_{L^1}$$

$$\le D\left(\sum_{h=1}^{\infty} (h+1)^{\frac{1}{\varepsilon}} \theta(\delta h) \right)^{\frac{\varepsilon}{1+\varepsilon}} < \infty,$$

since DD(ε) holds, using similar arguments as in (14). By the dominated convergence theorem we obtain $V(K) \to 0$ as $K \to \infty$. Finally, it is left to show (21). First, we note that

$$\left\| \frac{1}{t} \frac{\delta_N}{b_N} \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} K_{rect}\left(\frac{\tau^N_{i-m_N-1}-u}{b_N} \right)^2 \tilde{Y}_u(N\tau^N_{i-m_N-1}) \tilde{Y}_u(N\tau^N_{i+k-m_N-1}) \right.$$

$$\left. - \frac{1}{t(2m_N+1)} \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} X_{i,2m_N+1} X_{i+k,2m_N+1} \right\|_{L^1}$$

$$\le \frac{1}{2t} \frac{\delta_N}{b_N} E[|\tilde{Y}_u(0)\tilde{Y}_u(k\delta)|] \sum_{i=-m_N}^{m_N} \left| K_{rect}\left(\frac{\tau^N_i-u}{b_N} \right) - K_{rect}\left(\frac{\tau^N_{i+k}-u}{b_N} \right) \right|$$

$$\le \frac{1}{2t} \frac{\delta_N}{b_N} E[|\tilde{Y}_u(0)\tilde{Y}_u(k\delta)|] k B_K \xrightarrow[N\to\infty]{} 0$$

for some constant $B_K \ge 0$, since K is of bounded variation. Similar arguments as in (16) give

$$\left\| \frac{1}{t} \frac{\delta_N}{b_N} \sum_{i=1}^{\lfloor t(2m_N+1)\rfloor} K_{rect}\left(\frac{\tau^N_{i-m_N-1}-u}{b_N} \right)^2 \tilde{Y}_u(N\tau^N_{i-m_N-1}) \tilde{Y}_u(N\tau^N_{i+k-m_N-1}) \right.$$

$$\left. - \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} \frac{1}{2} G_t\left(\frac{\tau^N_i-u}{b_N} \right) \tilde{Y}_u(N\tau^N_i) \tilde{Y}_u(N\tau^N_{i+k}) \right\|_{L^1} \xrightarrow[N\to\infty]{} 0,$$

where $G_t(x)$ is defined as in (15). Thus, (21) holds, since

$$\left\| \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} \frac{1}{2} G_t\left(\frac{\tau^N_i-u}{b_N} \right) \tilde{Y}_u(N\tau^N_i) \tilde{Y}_u(N\tau^N_{i+k}) - \frac{E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)]}{2} \right\|_{L^1} \xrightarrow[N\to\infty]{} 0,$$

18

by Lemma 3.4. We conclude by similar steps as in (18).

$\square$

Remark 3.10. *We restrict ourselves to the rectangular kernel (9) as for general kernels it seems only possible to establish the conditions (13) and (21) for $t = 1$, which is not sufficient to apply [19, Theorem 2].*

4 Locally stationary processes in continuous-time

In the sequel we first review the definition of locally stationary processes in continuous-time from [8]. Then, we relate it to processes that possess a locally stationary approximation as introduced in Definition 2.1.

4.1 Preliminaries

We summarize elementary properties of Lévy processes and stochastic integration with respect to them. For further insight we refer to [2] and [31].

Definition 4.1. *A real-valued stochastic process $L = \{L(t), t \in \mathbb{R}_0^+\}$ is called Lévy process if*

(a) $L(0) = 0$ almost surely,

(b) for any $n \in \mathbb{N}$ and $t_0 < t_1 < t_2 < \cdots < t_n$, the random variables $(L(t_0), L(t_1) - L(t_0), \ldots, L(t_n) - L(t_{n-1}))$ are independent,

(c) for all $s, t \geq 0$, the distribution of $L(s + t) - L(s)$ does not depend on s and

(d) L is stochastically continuous.

Without loss of generality we additionally consider L to be càdlàg, i.e. right continuous with finite left limits.

Let $L = \{L(t), t \in \mathbb{R}_0^+\}$ be a real-valued Lévy process. Then, $L(1)$ is an infinitely divisible real-valued random variable with characteristic triplet (γ, Σ, ν), where $\gamma \in \mathbb{R}$, $\Sigma > 0$ and ν is a Lévy measure on $\mathbb{R}$, i.e. $\nu(0) = 0$ and $\int_{\mathbb{R}} \left(1 \wedge |x|^2\right) \nu(dx) < \infty$. The characteristic function of $L(t)$ is given by

$$\varphi_{L(t)}(z) = E[e^{izL(t)}] = e^{t\Psi_L(z)},$$
$$\Psi_L(z) = \left(i\gamma z - \frac{\Sigma z^2}{2} + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbb{1}_Z(x)\right) \nu(dx)\right), \tag{22}$$

where $z \in \mathbb{R}$ and $Z = \{x \in \mathbb{R}, |x| \leq 1\}$. If ν has finite second moment, i.e.

$$\int_{|x|>1} |x|^2 \, \nu(dx) < \infty \left(\iff \int_{\mathbb{R}} |x|^2 \, \nu(dx) < \infty \right), \tag{23}$$

then $L(t) \in L^2$ for all $t \geq 0$ and we have $E[L(t)] = t(\gamma + \int_{|x|>1} x\nu(dx)) < \infty$ and $Var(L(t)) = t(\Sigma + \int_{\mathbb{R}} x^2\nu(dx)) < \infty$. In the remainder we work with two-sided Lévy process, i.e. $L(t) =$

19

$L_1(t)\mathbb{1}_{\{t\geq 0\}} - L_2(-t)\mathbb{1}_{\{t<0\}}$, where L_1 and L_2 are independent copies of a one-sided Lévy process. Consider

$$X(t) = \int_{\mathbb{R}} f(t,s)L(ds), \tag{24}$$

where $t \in \mathbb{R}$ and $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{B}(\mathbb{R}\times\mathbb{R}) - \mathcal{B}(\mathbb{R})$ measurable. Necessary and sufficient conditions for the stochastic integral (24) to exist are given in [32, Theorem 3.3], namely if

- $\Sigma \int_{\mathbb{R}} f(t,s)^2 ds < \infty$,

- $\int_{\mathbb{R}} \int_{\mathbb{R}} \left((f(t,s)x)^2 \wedge 1\right)\nu(dx)ds < \infty$ and

- $\int_{\mathbb{R}} \left| f(t,s)(\gamma + \int_{\mathbb{R}} x(\mathbb{1}_{[0,1]}(|f(t,s)x|) - \mathbb{1}_{[0,1]}(|x|)\nu(dx))) \right| ds < \infty$

are satisfied, then (24) is well-defined. If L satisfies (23) and $f(t,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the above conditions are satisfied and $X(t) = \int_{\mathbb{R}} f(t,s)L(ds)$ exists in L^2. If $X = \{X(t), t \in \mathbb{R}\}$ with $X(t)$ as in (24) is well-defined, then $X(t)$ is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$, where

- $\gamma_{int} = \int_{\mathbb{R}} f(t,s)\gamma ds + \int_{\mathbb{R}} \int_{\mathbb{R}} f(t,s)x \left(\mathbb{1}_{[0,1]}(|f(t,s)x|) - \mathbb{1}_{[0,1]}(|x|)\nu(dx)\right) ds$,

- $\Sigma_{int} = \Sigma \int_{\mathbb{R}} f(t,s)^2 ds$ and

- $\nu_{int}(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(f(t,s)x)\nu(dx)ds$, where $B \in \mathcal{B}(\mathbb{R})$.

4.2 Locally stationary processes in continuous-time

The following definition was first given in [8, Definition 3.1].

Definition 4.2. *Let $Y_N = \{Y_N(t), t \in \mathbb{R}\}_{N\in\mathbb{N}}$ be a sequence of stochastic processes. Then, Y_N is called locally stationary if there exists a representation*

$$Y_N(t) = \int_{\mathbb{R}} g_N(Nt, Nt - s)L(ds), \tag{25}$$

where

(a) L is a two sided real-valued Lévy process such that (23) holds,

(b) $g_N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $g_N(Nt, \cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$,

(c) there exists a limiting local kernel function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(t, \cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, $t \mapsto g(t, \cdot)$ is continuous for all $t \in \mathbb{R}$ and

$$g_N(Nt, \cdot) \xrightarrow[N\to\infty]{L^2} g(t, \cdot)$$

for all $t \in \mathbb{R}$.

Remark 4.3. *(a) Definition 4.2 can also be reformulated in the frequency domain. To this end one defines locally stationary processes via a time-varying spectral representation, which can be locally approximated by the spectral representation of a stationary process (see [8, Definition 3.2]). In the continuous-time framework, the definition of local stationarity in the time and frequency domain are equivalent (see [8, Proposition 3.3]).*

(b) From [8, Proposition 3.5] it follows that the sequence (25) converges pointwise for each $t \in \mathbb{R}$ in distribution for $N \to \infty$ to the stationary process $\int_{\mathbb{R}} g(t, -s)L(ds)$.

Proposition 4.4. *Consider a sequence of stochastic processes Y_N in the form of (25) such that (23) holds and $E[L(1)] = 0$. Let $\tilde{Y}_u = \{\tilde{Y}_u(t), t \in \mathbb{R}\}_{u \in \mathbb{R}^+}$ be the family of stationary processes $\tilde{Y}_u(t) = \int_{\mathbb{R}} g(u, t - s)L(ds)$, such that $g_N(Nt, \cdot), g(t, \cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Assume that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for some $p \geq 1$. Then, Y_N is locally stationary.*

Proof. It holds

$$\|g_N(Nt, \cdot) - g(t, \cdot)\|_{L^2} = \|g_N(Nt, Nt - \cdot) - g(t, Nt - \cdot)\|_{L^2} = \frac{1}{\Sigma} \left\| Y_N(t) - \tilde{Y}_t(Nt) \right\|_{L^2} \leq \frac{1}{\Sigma} \frac{C}{N} \overset{N \to \infty}{\longrightarrow} 0.$$

$\square$

Proposition 4.5. *Let $Y_N = \{Y_N(t), t \in \mathbb{R}\}_{N \in \mathbb{N}}$ be a locally stationary process. Define $\tilde{Y}_u(t) = \int_{\mathbb{R}} g(u, t - s)L(ds)$, where g is the corresponding limiting local kernel function of Y_N. If it holds that $\sup_{u \in \mathbb{R}} \|g(u, \cdot)\|_{L^2} < \infty$,*

$$\|g(u, \cdot) - g(v, \cdot)\|_{L^2} \leq C |u - v| \quad and \quad \|g(Nt, \cdot) - g(t, \cdot)\|_{L^2} \leq C \frac{1}{N},$$

then $\tilde{Y}_u$ is a locally stationary approximation of Y_N for $p = 2$.

Proof. The ergodicity of $\tilde{Y}_u$ follows for instance from [21, Theorem 3.5]. Then it is sufficient to observe that

$$\left\| \tilde{Y}_u(t) - \tilde{Y}_v(t) \right\|_{L^2} = \Sigma \|g(u, \cdot) - g(v, \cdot)\|_{L^2} \leq C\Sigma |u - v| \quad and$$

$$\left\| Y_N(t) - \tilde{Y}_t(Nt) \right\|_{L^2} = \Sigma \|g_N(Nt, Nt - \cdot) - g(t, Nt - \cdot)\|_{L^2} \leq \frac{C\Sigma}{N}.$$

$\square$

5 Time-varying Lévy-driven state space models and CARMA(p,q) processes

In this section we first introduce time-varying Lévy-driven autoregressive moving average (tv-CARMA) processes. Then we review conditions such that sequences of tvCARMA processes are locally stationary and discuss conditions for the existence of a locally stationary approximation. Since the class of time-varying Lévy-driven CARMA processes is embedded in the class of Lévy-driven linear state space models, we state our results in terms of this class of processes. Throughout this section, $L = \{L(t), t \in \mathbb{R}\}$ denotes a two-sided Lévy process with values in $\mathbb{R}$, where we assume that the characteristic triplet (γ, Σ, ν) satisfies (23).

5.1 Lévy-driven Ornstein-Uhlenbeck (CAR(1)) processes

The simplest Lévy-driven CARMA(p,q) process is obtained for $p = 1$ and $q = 0$, i.e. the Lévy-driven CAR(1) or Lévy-driven Ornstein-Uhlenbeck process. It is given as the stationary solution to the differential equation $dY(t) = -aY(t)dt + L(dt)$ for a constant $a > 0$. The

solution can be expressed as $Y(t) = \int_{-\infty}^{t} e^{-a(t-s)} L(ds)$. If one allows a to be time-varying, we arrive at the so called time-varying Lévy-driven CAR(1) process, which is given by $Y(t) = \int_{-\infty}^{t} e^{\int_s^t a(\tau)d\tau} L(ds)$. To investigate processes of the above form in the context of local stationarity, we consider $Y_N = \{Y_N(t), t \in \mathbb{R}\}_{N \in \mathbb{N}}$ to be a sequence of rescaled time-varying CAR(1) processes defined by

$$Y_N(t) = \int_{-\infty}^{\infty} g_N(Nt, Nt - s) L(ds), \text{ with kernel function}$$

$$g_N(Nt, Nt - s) = \mathbb{1}_{\{Nt-s \geq 0\}} e^{-\int_s^{Nt} a\left(\frac{\tau}{N}\right)d\tau} = \mathbb{1}_{\{Nt-s \geq 0\}} e^{-\int_{-(Nt-s)}^{0} a\left(\frac{\tau+Nt}{N}\right)d\tau}, \tag{26}$$

where $a : \mathbb{R} \to \mathbb{R}_0^+$ is continuous such that $s \mapsto e^{-\int_{-s}^{0} a\left(\frac{\tau+Nt}{N}\right)d\tau} \in L^1(\mathbb{R}^+)$ for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$, which ensures the existence of (26), since additionally (23) holds. In the next proposition we revise sufficient conditions such that the sequence Y_N is locally stationary in the sense of Definition 4.2. Then we give additional conditions for the existence of a locally stationary approximation for $p = 2$ and $p = 4$, where the locally stationary approximation $\tilde{Y}_u$ is given by

$$\tilde{Y}_u(t) = \int_{\mathbb{R}} g(u, t - s) L(ds), \text{ with kernel function}$$

$$g(u, t - s) = \mathbb{1}_{\{t-s \geq 0\}} e^{-a(u)(t-s)}, \tag{27}$$

where we assume that $s \mapsto \mathbb{1}_{\{s \geq 0\}} e^{-a(u)s} \in L^1(\mathbb{R}^+)$ for all $u \in \mathbb{R}^+$ and $N \in \mathbb{N}$.

Proposition 5.1 ([8, Proposition 4.1]). *Let Y_N be a sequence of time-varying Lévy-driven CAR(1) processes as given in (26). Then, Y_N is locally stationary if*

(a) *the coefficient function a is continuous and*

(b) *for every $T \in \mathbb{R}^+$ there exists $\varepsilon_T > 0$ such that $a(s) > \varepsilon_T$ for all $s \leq T$.*

The corresponding limiting local kernel function is given by $g(u, t)$ as defined in (27).

Lemma 5.2. *For fixed $u \in \mathbb{R}$ let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R} \times \mathbb{R}) - \mathcal{B}(\mathbb{R})$ measurable function such that $f(u, \cdot) \in L^1(\mathbb{R}) \cap L^4(\mathbb{R})$ and let L be a two sided Lévy-process such that $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$. Then, $\int_{-\infty}^{t} f(u, t - s) L(ds) \in L^4$ and*

$$E\left[\left(\int_{-\infty}^{t} f(u, t - s) L(ds)\right)^4\right] = \int_{\mathbb{R}} f(u, s)^4 ds \int_{\mathbb{R}} x^4 \nu(dx) + 3\Sigma_L^2 \left(\int_{\mathbb{R}} f(u, s)^2 ds\right)^2$$

$$+ \mu_L^4 \left(\int_{\mathbb{R}} f(u, s) ds\right)^4 + 6\mu_L^2 \Sigma_L \left(\int_{\mathbb{R}} f(u, s) ds\right)^2 \int_{\mathbb{R}} f(u, s)^2 ds$$

$$+ 4\mu_L \left(\int_{\mathbb{R}} f(u, s)^3 ds \int_{\mathbb{R}} x^3 \nu(dx) \int_{\mathbb{R}} f(u, s) ds\right),$$

where $\Sigma_L = \Sigma + \int_{\mathbb{R}} x^2 \nu(dx)$ and $\mu_L = \gamma + \int_{|x|>1} x\nu(dx)$.

Proof. The finiteness of the fourth moments follows e.g. from [31]. The characteristic function of $\int_{-\infty}^{t} f(u, t - s) L(ds)$ is known from (22). Usual calculations then give the above representation, where we used the results from Section 4.1. □

Proposition 5.3. *Let Y_N be a sequence of time-varying Lévy-driven CAR(1) processes as given in (26). Then, $\tilde{Y}_u$ as given in (27) is a locally stationary approximation of Y_N for $p = 2$, if*

(a) *the coefficient function a is Lipschitz with constant L and*

(b) $\inf_{s\in\mathbb{R}} a(s) > 0$.

If additionally

(c) $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$,

then $\tilde{Y}_u$ is also a locally stationary approximation of Y_N for $p = 4$.

Proof. It is clear that $\tilde{Y}_u$ is stationary for all $u \in \mathbb{R}^+$. From [21, Theorem 3.5] it also follows that $\tilde{Y}_u$ is ergodic. Let $\Sigma_L = \Sigma + \int_{\mathbb{R}} x^2 \nu(dx)$ and $\mu_L = \gamma + \int_{|x|>1} x\nu(dx)$. Now, for $\varepsilon = \inf_{s\in\mathbb{R}} a(s) > 0$ it holds

$$\left\|\tilde{Y}_u(t)\right\|_{L^2}^2 = \Sigma_L \int_{\mathbb{R}} g(u, t-s)^2 \, ds + \mu_L^2 \left(\int_{\mathbb{R}} g(u, t-s) \, ds\right)^2 = \frac{\Sigma_L}{2a(u)} + \frac{\mu_L^2}{a(u)^2} \leq \frac{\Sigma_L}{2\varepsilon} + \frac{\mu_L^2}{\varepsilon^2} < \infty.$$

For $u, v \in \mathbb{R}$ we obtain

$$\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\|_{L^2}^2 = \Sigma_L \|g(u, t-\cdot) - g(v, t-\cdot)\|_{L^2}^2 + \mu_L^2 \left(\int_{\mathbb{R}} g(u, t-\cdot) - g(v, t-\cdot) \, ds\right)^2$$

$$= \Sigma_L \int_0^\infty \left(e^{-a(u)s} - e^{-a(v)s}\right)^2 ds + \mu_L^2 \left(\int_0^\infty e^{-a(u)s} - e^{-a(v)s} ds\right)^2 =: P_1 + P_2.$$

Then,

$$P_1 = \Sigma_L \int_0^\infty e^{-s\varepsilon} \left(e^{-(a(u)s - \frac{\varepsilon}{2}s)} - e^{-(a(v)s - \frac{\varepsilon}{2}s)}\right)^2 ds \leq \Sigma_L \int_0^\infty e^{-s\varepsilon} s^2 (a(u) - a(v))^2 ds \leq C_1^2 (u-v)^2$$

for a constant $C_1 > 0$, since $a(\cdot)$ is Lipschitz and $x \mapsto e^{-x}$ is Lipschitz on $\mathbb{R}_0^+$ with constant 1. For P_2 it holds analogously

$$P_2 \leq \mu_L^2 \left(\int_0^\infty e^{-a(u)s} - e^{-a(v)s} ds\right)^2 \leq C_2^2 (u-v)^2$$

for a constant $C_2 > 0$. Therefore,

$$\left\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\right\|_{L^2} \leq \sqrt{C_1^2 + C_2^2}(u-v).$$

Second,

$$\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^2}^2 = \Sigma_L \|g_N(Nt, Nt - \cdot) - g(t, Nt - \cdot)\|_{L^2}^2$$

$$+ \mu_L^2 \left(\int_{\mathbb{R}} g_N(Nt, Nt - u) - g(t, Nt - u) \, du\right)^2$$

$$= \Sigma_L \int_0^\infty \left(e^{-\int_{-s}^0 a(\frac{\tau+Nt}{N}) d\tau} - e^{-a(t)s}\right)^2 ds + \mu_L^2 \left(\int_{\mathbb{R}} e^{-\int_{-s}^0 a(\frac{\tau+Nt}{N}) d\tau} - e^{-a(t)s} ds\right)^2$$

$$=: P_3 + P_4.$$

Then,

$$P_3 = \Sigma_L \int_0^\infty e^{-s\varepsilon} \left(e^{-\int_{-s}^0 (a(\frac{\tau}{N}+t) - \frac{\varepsilon}{2}) d\tau} - e^{-(a(t) - \frac{\varepsilon}{2})s}\right)^2 ds$$

$$\leq \Sigma_L \int_0^\infty e^{-s\varepsilon} \left(-\int_{-s}^0 a\left(\frac{\tau}{N} + t\right) d\tau + a(t)s \right)^2 ds \leq \Sigma_L \int_0^\infty e^{-s\varepsilon} L^2 \left(\int_{-s}^0 \frac{|\tau|}{N} d\tau \right)^2 ds = \frac{C_3^2}{N^2}$$

for a constant $C_3 > 0$, since $a(\cdot)$ is Lipschitz with constant L and $x \mapsto e^{-x}$ is Lipschitz continuous on $\mathbb{R}_0^+$ with Lipschitz constant 1. Analogously,

$$P_4 \leq \mu_L^2 \left(\int_{\mathbb{R}} \left| e^{-\int_{-s}^0 a(\frac{\tau+Nt}{N})d\tau} - e^{-a(t)s} \right| ds \right)^2 \leq \frac{C_4^2}{N^2}$$

for a constant $C_4 > 0$. Overall, $\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^2} \leq \frac{1}{N}\sqrt{C_3^2 + C_4^2}$. Since $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$, Lemma 5.2 gives a closed form expression for $\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^4}^4$ and $\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\|_{L^4}^4$. Then, calculations analogous to the steps for $p = 2$ show that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for $p = 4$. $\square$

In the following theorem we apply the results from Section 3 to time-varying Lévy-driven CAR(1) processes and provide asymptotic results for different sample moments and observations. In particular, these results provide a first step towards a method of moments based estimation procedure including consistency, as well as results on the asymptotic distribution.

Theorem 5.4. *Let Y_N be a sequence of time-varying Lévy-driven CAR(1) processes as given in (26) and assume that*

(A) the conditions (a) and (b) from Proposition 5.3 hold.

Then, for observations sampled according to (O1) or (O2), we obtain for all $u \in \mathbb{R}^+$ and $k \in \mathbb{N}_0$

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N} \right) Y_N(\tau_i^N) \xrightarrow[N\to\infty]{L^1} E[\tilde{Y}_u(0)] \text{ and for (O1) also} \tag{28}$$

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N} \right) Y_N(\tau_i^N) Y_N\left(\tau_i^N + \frac{k}{N} \right) \xrightarrow[N\to\infty]{L^1} E[\tilde{Y}_u(0)\tilde{Y}_u(k)] \tag{29}$$

with $\tilde{Y}_u$ as given in (27). If we additionally assume that

(B) $\int_{|x|>1} |x|^{2+\varepsilon} \nu(dx) < \infty$ for some $\varepsilon > 0$,

then, the convergence in (29) also holds for (O2). Now, if in addition

(C) $E[L(1)] = 0$,

(D) $\sqrt{m_N} b_N \to 0$, as $N \to \infty$ and

(E) $\sigma(u)^2 > 0$, where $\sigma(u)^2 = \begin{cases} \frac{1}{2}E[\tilde{Y}_u(0)^2] + \sum_{h=1}^\infty E[\tilde{Y}_u(0)\tilde{Y}_u(h)], & \text{if (O1) holds,} \\ E[\tilde{Y}_u(0)^2], & \text{if (O2) holds,} \end{cases}$

then $\sigma(u)^2 < \infty$ and

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N} \right) Y_N(\tau_i^N) \xrightarrow[N\to\infty]{d} \sigma(u)^2 Z, \tag{30}$$

where Z is a standard normally distributed random variable and K_{rect} as defined in (9). Finally, if we additionally assume that

(F) $\int_{|x|>1} |x|^{4+\varepsilon} \nu(dx) < \infty$ *for some $\varepsilon > 0$ and*

(G) $\tilde{\sigma}(u)^2 > 0$, *where*

$$\tilde{\sigma}(u)^2 = \begin{cases} \frac{1}{2} E[\tilde{Y}_u(0)^2 \tilde{Y}_u(k)^2] + \sum_{h=1}^{\infty} E[\tilde{Y}_u(0)\tilde{Y}_u(k)\tilde{Y}_u(h)\tilde{Y}_u(h+k)], & \text{if (O1) holds,} \\ E[\tilde{Y}_u(0)^2 \tilde{Y}_u(k)^2], & \text{if (O2) holds,} \end{cases}$$

then $\tilde{\sigma}(u)^2 < \infty$ and

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) \left(Y_N(\tau_i^N) Y_N\left(\tau_i^N + \frac{k}{N}\right) - Cov(\tilde{Y}_u(0)\tilde{Y}_u(k))\right) \tag{31}$$
$$\xrightarrow[N\to\infty]{d} \tilde{\sigma}(u)^2 Z.$$

Proof. From Proposition 5.3 it follows that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for $p = 2$. If (O1) holds, Theorem 3.5 implies the convergence in (28). Analogous to [13, Corollary 3.4] and since $g(u, \cdot) \in L^2(\mathbb{R})$ for all $u \in \mathbb{R}$, it follows that $\tilde{Y}_u$ is θ-weakly dependent with θ-coefficients $\theta(h)$. Therefore, if (O2) holds, we obtain the convergence in (28) from part (a) of Theorem 3.6.

Define $g : \mathbb{R}^{k+1} \to \mathbb{R}$ as $g(x_1, \ldots, x_{k+1}) = x_1 x_{k+1}$, $\tilde{Z}_u^k(t)$ and $Z_N^k(t)$ as in Section 2.3. As described in Remark 2.8 we have $g \in \mathcal{L}_{k+1}(1, 1)$. From Proposition 2.7 part (a) it follows that $g(\tilde{Z}_u^k(t)) = \tilde{Y}_u(t)\tilde{Y}_u(t-k)$ is a locally stationary approximation of $g(Z_N^k(t)) = Y_N(t)Y_N(t - \frac{k}{N})$ for $p = 1$. If (O1) holds, the convergence in (29) then follows again from Theorem 3.5. In order to prove (29) for (O2), we first note that $E[|Y_u(0)|^{2+\varepsilon}] < \infty$, since $\int_{|x|>1} |x|^{2+\varepsilon} \nu(dx) < \infty$ and $g(u, \cdot) \in L^{2+\varepsilon}(\mathbb{R})$ for all $u \in \mathbb{R}^+$. Proposition 2.7 part (c) implies that $g(\tilde{Z}_u^k(t)) = \tilde{Y}_u(t)\tilde{Y}_u(t-k)$ is θ-weakly dependent. Hence, part (a) of Theorem 3.6 gives the stated convergence in (29) for (O2).

Analogous to the proof of [14, Theorem 3.36], one can show that $\tilde{Y}_u$ has exponentially decaying θ-coefficients, such that DD(ε) holds for any $\varepsilon > 0$. Theorem 3.7 then implies the convergence as stated in (30) for both, (O1) and (O2).

Since $\int_{|x|>1} |x|^{4+\varepsilon} \nu(dx) < \infty$ and $g(u, \cdot) \in L^{4+\varepsilon}(\mathbb{R})$ for all $u \in \mathbb{R}^+$, we have $E[|\tilde{Y}_u(0)|^{4+\varepsilon}] < \infty$ for all $u \in \mathbb{R}^+$. From Proposition 5.3 it follows that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for $p = 4$, such that part (c) of Proposition 2.7 implies that $g(\tilde{Z}_u^k(t))$ is a locally stationary approximation of $g(Z_N^k(t))$ for $p = 2$, where $g(\tilde{Z}_u^k(t)) = \tilde{Y}_u(t)\tilde{Y}_u(t-k)$ is θ-weakly dependent with θ-coefficients $\theta_{\tilde{Z}_u^k}(h) \in \mathcal{O}\left(\theta(h)^{\frac{\varepsilon}{1+\varepsilon}}\right)$. Since $\tilde{Y}_u$ has exponentially decaying θ-coefficients $\theta(h)$, $\theta_{\tilde{Z}_u^k}(h)$ satisfies DD(ε) for any $\varepsilon > 0$. Theorem 3.7 then implies (31) for (O1) and (O2). $\qquad \square$

5.2 Lévy-driven state space and CARMA(p,q) models

Before we give the definition of general time-varying Lévy-driven state space models, we introduce time-varying Lévy-driven CARMA processes (see e.g. [10] for an introduction to time-invariant Lévy-driven CARMA processes). For positive integers $p > q$, let $a_i(t)$, $i = 1, \ldots, p$ and $b_j(t)$, $j = 0, \ldots, q$ be continuous real functions. Then, the polynomials

$$p(t, z) = z^p + a_1(t)z^{p-1} + \ldots + a_{p-1}(t)z + a_p(t) \text{ and}$$
$$q(t, z) = b_0(t) + b_1(t)z + \ldots + b_{q-1}(t)z^{q-1} + b_q(t)z^q,$$

are respectively called autoregressive (AR) and moving average (MA) polynomials. A time-varying Lévy-driven CARMA process is then defined as the solution to the formal differential equation

$$p(t, D)Y(t) = q(t, D)DL(t),$$
(32)

where D denotes the differential operator with respect to time. Since Lévy processes are in general not differentiable, we interpret (32) as usual to be equivalent to the state space representation

$$
\begin{aligned}
Y(t) &= B(t)'X(t) \text{ and} \\
dX(t) &= A(t)X(t)dt + CL(dt),
\end{aligned}
$$
(33)

where $A(t)$ is in companion form, i.e.

$$A(t) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -a_p(t) & -a_{p-1}(t) & \dots & -a_1(t) \end{pmatrix} \in M_{p \times p}(\mathbb{R}) \text{ and}$$

$$B(t) = \begin{pmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_{p-1}(t) \end{pmatrix} \in M_{p \times 1}(\mathbb{R}), \qquad C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in M_{p \times 1}(\mathbb{R}), \quad t \in \mathbb{R}.$$

For a discussion on state space representations of the form of (33) with a Brownian motion as driving noise we refer to [36, Section 2.1.1]. As explained in [8], there exists a solution to the above state space representation which is given by

$$X(t) = \int_{-\infty}^{t} \Psi(t, s)CL(ds) \text{ and } Y(t) = B(t)' \int_{-\infty}^{t} \Psi(t, s)CL(ds)$$
(34)

for $t \in \mathbb{R}$, provided that the integrals exist in L^2. The matrix $\Psi(t, t_0)$ for $t > t_0$ is the unique solution to the homogeneous initial value problem (IVP)

$$\frac{d}{dt}\Psi(t, t_0) = A(t)\Psi(t, t_0) \quad \text{with initial condition} \quad \Psi(t_0, t_0) = \mathbf{1}_p.$$
(35)

In particular, it holds $\Psi(t, t_0) = \Psi(t, s)\Psi(s, t_0)$ for $t_0 < s < t$. For an extensive collection of results on the IVP (35) we refer to [9, Section 3 and 4].

Definition 5.5. *Let $Y = \{Y(t), t \in \mathbb{R}\}$ be a solution to the state space representation (33) in the form of (34). Then, Y is referred to as a time-varying Lévy-driven CARMA(p,q) process.*

The IVP (35) is solved by the Peano-Baker series

$$\Psi(t, t_0) = \mathbf{1}_p + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 + \dots = \sum_{n=0}^{\infty} \mathcal{I}_n,$$

where $\mathcal{I}_0 = \mathbf{1}_p$ and $\mathcal{I}_n = \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \dots \int_{t_0}^{\tau_{n-1}} A(\tau_n)d\tau_n \dots d\tau_2 d\tau_1$. Considering [3, Remark 2], the condition

$$\left[A(t), \int_{t_0}^{t} A(s)ds \right] = 0$$
(36)

26

for all $t > t_0$, i.e. if $A(t)$ and $\int_{t_0}^t A(s)ds$ commute, ensures that the Peano-Baker Series can be expressed as $\Psi(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{t_0}^t A(\tau)d\tau \right)^n = e^{\int_{t_0}^t A(\tau)d\tau}$. If (36) holds, for $t > t_0$ the solution (34) of the state space representation (33) simplifies to

$$X(t) = e^{\int_0^t A(\tau)d\tau} X(t_0) + \int_{t_0}^t e^{\int_s^t A(\tau)d\tau} CL(ds) = \int_{-\infty}^t e^{\int_s^t A(\tau)d\tau} CL(ds) \quad \text{and}$$

$$Y(t) = B(t)'e^{\int_0^t A(\tau)d\tau} X(t_0) + \int_{t_0}^t B(t)'e^{\int_s^t A(\tau)d\tau} CL(ds) = \int_{-\infty}^t B(t)'e^{\int_s^t A(\tau)d\tau} CL(ds).$$

Remark 5.6. *If $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$, i.e. if $A(s)$ and $A(t)$ commute, the commutativity assumption (36) clearly holds. As a matter of fact, the statements are even equivalent (see [30, Exercise 4.8]). Further discussions on the commutativity condition (36) can be found for instance in [37].*

In the following we consider matrix functions $A(t)$, which are not necessarily in companion form and additionally allow C to be time-varying, i.e. $C(t) \in M_{p \times 1}(\mathbb{R})$, $t \in \mathbb{R}$. This leads to the observation and state equation

$$\begin{aligned} Y(t) &= B(t)'X(t) \quad \text{and} \\ dX(t) &= A(t)X(t)dt + C(t)L(dt), \end{aligned} \tag{37}$$

where $A(t) \in M_{p \times p}(\mathbb{R})$ and $B(t), C(t) \in M_{p \times 1}(\mathbb{R})$, $t \in \mathbb{R}$ are arbitrary continuous coefficient functions. A solution of (37) is given by (see [8, Section 4])

$$X(t) = \int_{-\infty}^t \Psi(t, s)C(s)L(ds) \text{ and } Y(t) = B(t) \int_{-\infty}^t \Psi(t, s)C(s)L(ds), \tag{38}$$

provided that the integrals exist in L^2. Again, $\Psi(t, t_0)$ is the unique matrix solution of the IVP (35).

Definition 5.7. *Let $Y = \{Y(t), t \in \mathbb{R}\}$ be a solution to the state space representation (37) in the form of (38). Then, Y is referred to as a time-varying Lévy-driven state space process.*

For an initial time $t_0 \in \mathbb{R}$ a time-varying Lévy-driven state space process can be expressed as

$$X(t) = \Psi(t, t_0) \left(X(t_0) + \int_{t_0}^t \Psi(s, t_0)^{-1} C(s)L(ds) \right).$$

Remark 5.8. *In [33, Corollary 3.4], the authors showed equivalence of the class of time-invariant Lévy-driven CARMA processes and time-invariant Lévy-driven state space processes. The relation of the above classes in the time-varying setting has recently been investigated in [8]. While the classes of time-varying Lévy-driven CARMA processes and time-varying Lévy-driven state space process are not necessarily equivalent for non-continuous coefficient functions (see [8, Proposition 4.6]), the authors showed that a state space process of the form of (37), whose coefficient functions A and C are sufficiently often differentiable, is equivalent to a time-varying Lévy-driven CARMA process of the form (34) if and only if it is instantaneously controllable [8, Proposition 4.8]. For more information on instantaneous controllability we refer to [30, Chapter 9 and 10].*

The following definition from [30, Chapter 6] directly ensures the existence of the stochastic integrals in (34) and (38).

Definition 5.9. *A linear state space model of the form of (37) is called uniformly exponentially stable if there exists $\gamma > 0$ and $\lambda > 0$ such that $\|\Psi(t, t_0)\| \leq \gamma e^{-\lambda(t-t_0)}$ for all $t > t_0$.*

Remark 5.10. *If a linear state space model of the form of (33) or (37) is uniformly exponential stable, then the integrals in (34) or (38) (if C is additionally uniformly bounded) are well-defined, since we assumed that additionally (23) holds.*

5.3 Time-varying Lévy-driven state space models

First, we review conditions from [8] that are sufficient for a sequence of time-varying Lévy-driven state space models to be locally stationary in the sense of Definition 4.2. Then we investigate suitable conditions for these models to to possess a locally stationary approximation. In fact, it will turn out that both conditions show high similarity, as we have already seen it in Section 5.1 To be able to establish local stationarity, we consider a sequence Y_N of time-varying Lévy-driven state space models, i.e.

$$Y_N(t) = \int_{\mathbb{R}} g_N(Nt, Nt - s)L(ds), \text{ with kernel function}$$
$$g_N(Nt, Nt - s) = \mathbb{1}_{\{Nt-s\geq 0\}}B(t)'\Psi_{N,t}(0, -(Nt - s))C\left(\frac{-(Nt - s)}{N} + t\right), \tag{39}$$

where $A(t) \in M_{p\times p}(\mathbb{R})$ and $B(t), C(t) \in M_{p\times 1}(\mathbb{R})$, $t \in \mathbb{R}$, are arbitrary continuous coefficient functions and $\Psi_{N,t}(0, -(Nt - u))$ is the solution to the IVP

$$\Psi_{N,t}(s_0, s_0) = \mathbf{1}_p, \qquad \frac{d}{ds}\Psi_{N,t}(s, s_0) = A\left(\frac{s}{N} + t\right)\Psi_{N,t}(s, s_0).$$

We assume that $A(u)$, $u \in \mathbb{R}^+$ has eigenvalues with strictly negative real part and consider as corresponding locally stationary approximation $\tilde{Y}_u$ the process

$$\tilde{Y}_u(t) = \int_{\mathbb{R}} g(u, t - s)L(ds), \text{ with kernel function}$$
$$g(u, t - s) = \mathbb{1}_{\{t-s\geq 0\}}B(u)'e^{A(u)(t-s)}C(u), \tag{40}$$

where $e^{A(u)(t-s)}$ is the solution to the IVP

$$\Psi_u(s_0, s_0) = \mathbf{1}_p, \qquad \frac{d}{ds}\Psi_u(s, s_0) = A(u)\Psi_u(s, s_0).$$

Proposition 5.11 ([8, Proposition 4.11]). *Let Y_N be a sequence of time-varying Lévy-driven state space models as given in (39). Then, Y_N is locally stationary, if*

(a) the coefficient functions A, B and C are continuous,

(b) $\|B(s)\| < \infty$ for all $s \in \mathbb{R}$, $\sup_{s\in\mathbb{R}} \|C(s)\| < \infty$ and

(c) $\|\Psi_{N,t}(0, s)\| \leq F_t(s)$ for some function $F_t \in L^2(\mathbb{R}^-)$ and all $N \in \mathbb{N}$, $t \in \mathbb{R}$.

In particular, (c) holds if Y_N is uniformly exponential stable with the same γ and λ for all $N \in \mathbb{N}$, since then $\|\Psi_{N,t}(0, s)\| \leq F_t(s) = \gamma e^{\lambda s} \in L^2(\mathbb{R}^-)$. The corresponding limiting local kernel function is given by $g(u, s)$ as defined in (40).

28

Proposition 5.12. *Let Y_N be a sequence of time-varying Lévy-driven state space models as given in (39). Then, $\tilde{Y}_u$ as given in (40) is a locally stationary approximation of Y_N for $p = 2$, if*

(a) *the coefficient functions A, B and C are Lipschitz with constants L_A, L_B and L_C, respectively,*

(b) $\sup_{s \in \mathbb{R}} \|B(s)\| < \infty$ *and* $\sup_{s \in \mathbb{R}} \|C(s)\| < \infty$,

(c) $\|\Psi_{N,t}(0, s)\| \leq F(s)$ *such that* $s \mapsto |s|F(s) \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-)$ *for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$,*

(d) $\|\Psi_{N,t}(0, s) - \Psi_t(0, s)\| \leq \frac{\tilde{F}(s)}{N}$ *such that* $\tilde{F}(s) \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-)$ *for all $t \in \mathbb{R}$,*

(e) $\|\Psi_u(0, s)\| \leq G(s)$ *such that* $G(s) \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-)$ *for all $u \in \mathbb{R}^+$,*

(f) $\|\Psi_u(0, s) - \Psi_v(0, s)\| \leq |u - v|\tilde{G}(s)$ *such that* $\tilde{G}(s) \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-)$.

If we additionally assume that $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$ and $|s|F(s), \tilde{F}(s), G(s), \tilde{G}(s) \in L^1(\mathbb{R}^-) \cap L^4(\mathbb{R}^-)$, then $\tilde{Y}_u$ is also a locally stationary approximation of Y_N for $p = 4$.

Proof. It is clear that $\tilde{Y}_u$ is stationary for all $u \in \mathbb{R}^+$. From [21, Theorem 3.5] it follows, that $\tilde{Y}_u$ is ergodic. Let $S_B = \sup_{s \in \mathbb{R}} \|B(s)\| < \infty$, $S_C = \sup_{s \in \mathbb{R}} \|C(s)\| < \infty$, $\mu_L = \gamma + \int_{|x|>1} x\nu(dx)$ and $\Sigma_L = \Sigma + \int_{\mathbb{R}} x^2 \nu(dx)$. First, we note that

$$\|\tilde{Y}_u(t)\|_{L^2}^2 \leq \Sigma_L S_B S_C \int_{-\infty}^0 G(s)^2 ds + \mu_L S_B^2 S_C^2 \left(\int_{-\infty}^0 G(s)ds\right)^2 < \infty.$$

Now, for $u, v \in \mathbb{R}^+$

$$\|\tilde{Y}_u(t)\|_{L^2}^2 \leq \Sigma_L S_B S_C \int_{-\infty}^0 G(s)^2 ds + \mu_L S_B^2 S_C^2 \left(\int_{-\infty}^0 G(s)ds\right)^2 < \infty.$$

Now, for $u, v \in \mathbb{R}^+$

$$\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\|_{L^2}^2$$
$$= \Sigma_L \int_{\mathbb{R}} \left(\mathbb{1}_{\{t-s \geq 0\}} B(u)' \Psi_u(0, -(t-s))C(u) - \mathbb{1}_{\{t-s \geq 0\}} B(v)' \Psi_v(0, -(t-s))C(v)\right)^2 ds$$
$$+ \mu_L^2 \left(\int_{\mathbb{R}} \mathbb{1}_{\{t-s \geq 0\}} B(u)' \Psi_u(0, -(t-s))C(u) - \mathbb{1}_{\{t-s \geq 0\}} B(v)' \Psi_v(0, -(t-s))C(v)\, ds\right)^2$$
$$=: P_1 + P_2.$$

For P_1 we obtain

$$P_1 = \Sigma_L \int_{-\infty}^t |B(u)' \Psi_u(0, -(t-s))C(u) - B(v)' \Psi_v(0, -(t-s))C(v)|^2 ds$$
$$\leq 3\Sigma_L \int_{-\infty}^0 \left(B(u)' \Psi_u(0, s)C(u) - B(v)' \Psi_u(0, s)C(u)\right)^2$$
$$+ \left(B(v)' \Psi_u(0, s)C(u) - B(v)' \Psi_v(0, s)C(u)\right)^2 + \left(B(v)' \Psi_v(0, s)C(u) - B(v)' \Psi_v(0, s)C(v)\right)^2 ds.$$

Now,

$$\left(B(u)' \Psi_u(0, s)C(u) - B(v)' \Psi_u(0, s)C(u)\right)^2 \leq \|B(u)' - B(v)'\|^2 \|\Psi_u(0, s)C(u)\|^2$$

29

$$\leq S_C^2 L_B^2 (u-v)^2 G(s)^2,$$

$$\left(B(v)'\Psi_v(0,s)C(u) - B(v)'\Psi_v(0,s)C(v)\right)^2 \leq S_B^2 L_C^2 (u-v)^2 G(s)^2 \text{ and}$$

$$\left(B(v)'\Psi_u(0,s)C(u) - B(v)'\Psi_v(0,s)C(u)\right)^2 \leq S_B^2 S_C^2 (u-v)^2 \left\| \Psi_u(0,s) - \Psi_v(0,s) \right\|^2$$

$$\leq S_B^2 S_C^2 (u-v)^2 \tilde{G}(s)^2,$$

such that

$$P_1 \leq 3\Sigma_L (u-v)^2 \int_{-\infty}^0 \left(\left(S_C^2 L_B^2 + S_B^2 L_C^2 \right) G(s)^2 + S_B^2 S_C^2 \tilde{G}(s)^2 \right) ds.$$

For P_2 it holds

$$P_2 \leq \mu_L^2 \bigg(\int_{-\infty}^0 \left| B(u)'\Psi_u(0,s)C(u) - B(v)'\Psi_u(0,s)C(u) \right|$$

$$+ \left| B(v)'\Psi_u(0,s)C(u) - B(v)'\Psi_v(0,s)C(u) \right| + \left| B(v)'\Psi_v(0,s)C(u) - B(v)'\Psi_v(0,s)C(v) \right| ds \bigg)^2$$

$$\leq \mu_L^2 (u-v)^2 \bigg(\int_{-\infty}^0 \left(S_C L_B + S_B L_C \right) G(s) + S_B S_C \tilde{G}(s) ds \bigg)^2.$$

Overall, we obtain $\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\|_{L^2} \leq D_1 |u-v|$. Second,

$$\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^2}^2 = \Sigma_L \int_{\mathbb{R}} \left(\mathbb{1}_{\{Nt-s\geq 0\}} B(t)'\Psi_{N,t}(0, -(Nt-s))C\left(\frac{-(Nt-s)}{N} + t\right) \right.$$

$$\left. - \mathbb{1}_{\{Nt-s\geq 0\}} B(t)'\Psi_t(0, -(Nt-s))C(t) \right)^2 ds$$

$$+ \mu_L^2 \bigg(\int_{\mathbb{R}} \mathbb{1}_{\{Nt-s\geq 0\}} B(t)'\Psi_{N,t}(0, -(Nt-s))C\left(\frac{-(Nt-s)}{N} + t\right)$$

$$- \mathbb{1}_{\{Nt-s\geq 0\}} B(t)'\Psi_t(0, -(Nt-s))C(t) ds \bigg)^2$$

$$=: P_3 + P_4.$$

For P_3 we obtain

$$P_3 = \Sigma_L \int_{-\infty}^0 \left(B(t)'\left(\Psi_{N,t}(0,s)C\left(\frac{s}{N}+t\right) - \Psi_{N,t}(0,s)C(t) + \Psi_{N,t}(0,s)C(t) - \Psi_t(0,s)C(t)\right) \right)^2 ds$$

$$\leq 2\Sigma_L S_B^2 \int_{-\infty}^0 \left(\left\| \Psi_{N,t}(0,s)C\left(\frac{s}{N}+t\right) - \Psi_{N,t}(0,s)C(t) \right\|^2 + \left\| \Psi_{N,t}(0,s)C(t) - \Psi_t(0,s)C(t) \right\|^2 \right) ds$$

$$\leq 2\Sigma_L S_B^2 \int_{-\infty}^0 \left\| \Psi_{N,t}(0,s) \right\|^2 \left\| C\left(\frac{s}{N}+t\right) - C(t) \right\|^2 ds$$

$$+ 2\Sigma_L S_B^2 \int_{-\infty}^0 \left\| C(t) \right\|^2 \left\| \Psi_{N,t}(0,s) - \Psi_t(0,s) \right\|^2 ds$$

$$\leq 2\Sigma_L S_B^2 \int_{-\infty}^0 \left(F(s)^2 \| C\left(\frac{s}{N}+t\right) - C(t) \|^2 \right) ds + 2\Sigma_L S_B^2 S_C^2 \int_{-\infty}^0 \left\| \Psi_{N,t}(0,s) - \Psi_t(0,s) \right\|^2 ds$$

$$\leq \frac{2\Sigma_L S_B^2 L_C^2}{N^2} \int_{-\infty}^0 F(s)^2 s^2 du + \frac{2\Sigma_L S_B^2 S_C^2}{N^2} \int_{-\infty}^0 \tilde{F}(s)^2 ds,$$

since $C(\cdot)$ is Lipschitz with constant L_C. For P_4 it holds

$$P_4 \leq \mu_L^2 S_B^2 \bigg(\int_{-\infty}^0 \left\| \Psi_{N,t}(0,s)C\left(\frac{s}{N}+t\right) - \Psi_{N,t}(0,s)C(t) \right\| + \left\| \Psi_{N,t}(0,s)C(t) - \Psi_t(0,s)C(t) \right\| ds \bigg)^2$$

$$\leq \frac{\mu_L^2 S_B^2}{N^2} \left(L_C \int_{-\infty}^0 F(s)|s|ds + S_C^2 \int_{-\infty}^0 \tilde{F}(s)ds \right)^2.$$

All together $\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^2} \leq \frac{D_2}{N}$ for a constant $D_2 > 0$.

Since $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$, we obtain closed form expressions for $\|Y_N(t) - \tilde{Y}_t(Nt)\|_{L^4}^4$ and $\|\tilde{Y}_u(t) - \tilde{Y}_v(t)\|_{L^4}^4$ from Lemma 5.2. Then, calculations analogous to the steps for $p = 2$ show that $\tilde{Y}_u$ is a locally stationary approximation of Y_N for $p = 4$.

□

Corollary 5.13. *Let Y_N be a sequence of time-varying Lévy-driven state space models as given in (39). Then, $\tilde{Y}_u$ as given in (40) is a locally stationary approximation of Y_N for $p = 2$, if*

(a1) the coefficient functions A, B and C are Lipschitz with constants L_A, L_B and L_C, respectively,

(b1) $\sup_{s \in \mathbb{R}} \|B(s)\| < \infty$ and $\sup_{s \in \mathbb{R}} \|C(s)\| < \infty$,

(c1) $\{A(t)\}_{t \in \mathbb{R}}$ commutes, i.e. $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$,

(d1) $\left\| e^{\nu \int_s^0 A(\frac{\tau}{N} + t)d\tau} \right\| \leq \gamma e^{\nu s \lambda}$, with $\gamma, \lambda > 0$ for all $\nu \in [0, 1]$, $s < 0$, $t \in \mathbb{R}$ and $N \in \mathbb{N}$, and

(e1) $\left\| e^{-\nu A(t)s} \right\| \leq \tilde{\gamma} e^{\nu s \tilde{\lambda}}$, with $\tilde{\gamma}, \tilde{\lambda} > 0$ for all $\nu \in [0, 1]$, $s < 0$ and $t \in \mathbb{R}$,

or if

(a2) the coefficient functions A, B and C are Lipschitz with constants L_A, L_B and L_C, respectively,

(b2) $\sup_{s \in \mathbb{R}} \|B(s)\| < \infty$ and $\sup_{s \in \mathbb{R}} \|C(s)\| < \infty$,

(c2) $\{A(t)\}_{t \in \mathbb{R}}$ commutes, i.e. $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$,

(d2) the eigenvalues $\lambda_j(t)$ of $A(t)$ for $j = 1, \ldots, p$ satisfy $\sup_{t \in \mathbb{R}} \max_{j=1,\ldots,p} \mathfrak{Re}(\lambda_j(t)) < 0$ and

(e2) $A(t)$ is diagonalizable for all $t \in \mathbb{R}$.

If additionally $\int_{\mathbb{R}} x^4 \nu(dx) < \infty$, the conditions (a1)-(d1) and (a2)-(d2) are also sufficient for $\tilde{Y}_u$ to be a locally stationary approximation of Y_N for $p = 4$.

Proof. It is enough to show show the conditions (c)-(f) of Proposition 5.12. Since $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$, it holds $\Psi_{N,t}(t_1, t_0) = e^{\int_{t_0}^{t_1} A(\frac{\tau}{N} + t)d\tau}$ and $\Psi_t(t_1, t_0) = e^{A(t)(t_1 - t_0)}$. Assume now that (a1)-(e1) hold. We obtain

$$\|\Psi_{N,t}(0, s)\| = \|e^{\int_s^0 A(\frac{\tau}{N} + t)d\tau}\| \leq \gamma e^{s\lambda}, \text{ where } |s|\gamma e^{s\lambda} \in L^1(\mathbb{R}^-) \cap L^4(\mathbb{R}^-) \text{ and}$$

$$\|\Psi_u(0, s)\| = \|e^{-A(u)s}\| \leq \tilde{\gamma} e^{s\tilde{\lambda}} \in L^1(\mathbb{R}^-) \cap L^4(\mathbb{R}^-).$$

In order to show (d) and (f), we first note that $e^A - e^B = \int_0^1 e^{\nu B}(A - B)e^{(1-\nu)A}d\nu$ (see [23, page 238]). Then, using (d1) and (e1)

$$\|\Psi_{N,t}(0, s) - \Psi_t(0, s)\| = \left\| e^{\int_s^0 A(\frac{\tau}{N} + t)d\tau} - e^{-A(t)s} \right\|$$

$$= \left\| \int_0^1 e^{\nu \int_s^0 A\left(\frac{\tau}{N}+t\right)d\tau} \left(\int_s^0 A\left(\frac{\tau}{N}+t\right) d\tau + A(t)s \right) e^{-(1-\nu)A(t)s} d\nu \right\|$$

$$\leq \int_s^0 \left\| A\left(\tfrac{\tau}{N}+t\right) - A(t) \right\| d\tau \int_0^1 \gamma e^{\nu\lambda s} \tilde{\gamma} e^{(1-\nu)\tilde{\lambda}s} d\nu$$

$$\leq \int_s^0 L_A \frac{|\tau|}{N} d\tau \gamma\tilde{\gamma} \int_0^1 e^{\nu\hat{\lambda}s} e^{(1-\nu)\hat{\lambda}s} d\nu = \frac{L_A \gamma\tilde{\gamma}}{2N} s^2 e^{\hat{\lambda}s},$$

where $\hat{\lambda} = \min(\lambda, \tilde{\lambda}) > 0$ and $\tilde{F}(s) = \frac{L_A\gamma\tilde{\gamma}}{2} s^2 e^{\hat{\lambda}s} \in L^1(\mathbb{R}^-) \cap L^4(\mathbb{R}^-)$. Finally,

$$\|\Psi_u(0,s) - \Psi_v(0,s)\| = \|e^{-A(u)s} - e^{-A(v)s}\| \leq |s| \, \|A(u) - A(v)\| \int_0^1 \|e^{-\nu A(u)s}\| \|e^{-(1-\nu)A(v)s}\| d\nu$$

$$\leq L_A |s| \, |u-v| \int_0^1 \tilde{\gamma} e^{\nu\tilde{\lambda}s} \tilde{\gamma} e^{(1-\nu)\tilde{\lambda}s} d\nu = |u-v| \, L_A \tilde{\gamma}^2 |s| e^{\tilde{\lambda}s},$$

where $\tilde{G}(s) = L_A\tilde{\gamma}^2 |s| e^{\tilde{\lambda}s} \in L^1(\mathbb{R}^-) \cap L^4(\mathbb{R}^-)$.

For the second part we assume (a2)-(e2) to hold. It suffices to show that (d1) and (e1) hold. Since $\{A(t)\}$ is mutually commutative and (e2) holds, $\{A(t)\}$ is simultaneously diagonalizable (see [24, Theorem 1.3.12]). Therefore, there exists a regular matrix S such that

$$SA\left(\tfrac{s}{N}+t\right) S^{-1} = diag\left(\lambda_1\left(\tfrac{s}{N}+t\right), \ldots, \lambda_p\left(\tfrac{s}{N}+t\right)\right) = D\left(\tfrac{s}{N}+t\right).$$

In the following, we set $-\beta = \sup_{t\in\mathbb{R}} \max_{j=1,\ldots,p} \mathfrak{Re}(\lambda_i(t)) < 0$ and $\|\cdot\|_{Spec}$ denotes the operator norm induced by the euclidean norm. Then,

$$\left\| e^{\nu \int_s^0 A\left(\frac{\tau}{N}+t\right)d\tau} \right\| = \left\| e^{\nu \int_s^0 S^{-1} D\left(\frac{\tau}{N}+t\right)S d\tau} \right\| = \left\| S^{-1} e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)d\tau} S \right\| \leq \|S^{-1}\| \|S\| \left\| e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)d\tau} \right\|$$

$$\leq C \left\| e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)d\tau} \right\|_{Spec} = C \max\left\{ \sqrt{\mu}, \mu \in \sigma\left(\left(e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)d\tau}\right)' \left(e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)d\tau}\right)\right) \right\}$$

$$= C \max\left\{ \sqrt{\mu}, \mu \in \sigma\left(e^{\nu \int_s^0 D\left(\frac{\tau}{N}+t\right)' + D\left(\frac{\tau}{N}+t\right)d\tau}\right) \right\} = C \max_{j=1,\ldots,p} e^{2\nu \int_s^0 \mathfrak{Re}\left(\lambda_j\left(\frac{\tau}{N}+t\right)\right)d\tau}$$

$$\leq C e^{\nu s \beta}$$

for a constant $C > 0$. Analogously we obtain $\|e^{-\nu A(t)s}\| \leq C \|e^{-\nu D(t)s}\|_{Spec} \leq C e^{\nu s \beta}$. $\qquad \square$

The next theorem establishes a number of asymptotic results for different sample moments and observations of time-varying Lévy-driven state space models based on our results from Section 3.

Theorem 5.14. *Let Y_N be a sequence of time-varying Lévy-driven state space models as given in (39) and assume that one of the following sets of conditions holds:*

(A1) conditions (a)-(f) from Proposition 5.12.

(A2) conditions (a1)-(e1) from Corollary 5.13.

(A3) conditions (a2)-(d2) from Corollary 5.13.

Then, we obtain the following results:

32

(a) *For observations sampled according to (O1) or (O2), we obtain for all $u \in \mathbb{R}^+$ and $k \in \mathbb{N}_0$*

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N \to \infty]{L^1} E[\tilde{Y}_u(0)] \text{ and for (O1) also} \tag{41}$$

$$\frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) Y_N\left(\tau_i^N + \frac{k}{N}\right) \xrightarrow[N \to \infty]{L^1} E[\tilde{Y}_u(0)\tilde{Y}_u(k)] \tag{42}$$

with $\tilde{Y}_u$ as given in (40).

(b) *If we additionally assume that either (A2) or (A3) hold and*

(B) $\int_{|x|>1} |x|^{2+\varepsilon} \nu(dx) < \infty$ *for some $\varepsilon > 0$,*

then the convergence in (42) also holds for (O2).

(c) *If in addition to (b) it holds that*

(C) $E[L(1)] = 0$,

(D) $\sqrt{m_N} b_N \to 0$, *as $N \to \infty$ and*

(E) $\sigma(u)^2 > 0$, *where* $\sigma(u)^2 = \begin{cases} \frac{1}{2} E[\tilde{Y}_u(0)^2] + \sum_{h=1}^{\infty} E[\tilde{Y}_u(0)\tilde{Y}_u(h)], & \text{if (O1) holds,} \\ E[\tilde{Y}_u(0)^2], & \text{if (O2) holds,} \end{cases}$

then $\sigma(u)^2 < \infty$ and

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) Y_N(\tau_i^N) \xrightarrow[N \to \infty]{d} \sigma(u)^2 Z, \tag{43}$$

where Z is a standard normally distributed random variable and K_{rect} as defined in (9).

(d) *If in addition to (c) it holds that*

(F) $\int_{|x|>1} |x|^{4+\varepsilon} \nu(dx) < \infty$ *for some $\varepsilon > 0$ and*

(G) $\tilde{\sigma}(u)^2 > 0$, *where*
$$\tilde{\sigma}(u)^2 = \begin{cases} \frac{1}{2} E[\tilde{Y}_u(0)^2 \tilde{Y}_u(k)^2] + \sum_{h=1}^{\infty} E[\tilde{Y}_u(0)\tilde{Y}_u(k)\tilde{Y}_u(h)\tilde{Y}_u(h+k)], & \text{if (O1) holds,} \\ E[\tilde{Y}_u(0)^2 \tilde{Y}_u(k)^2], & \text{if (O2) holds,} \end{cases}$$

then $\tilde{\sigma}(u)^2 < \infty$ and

$$\sqrt{\frac{\delta_N}{b_N}} \sum_{i=-m_N}^{m_N} K_{rect}\left(\frac{\tau_i^N - u}{b_N}\right) \left(Y_N(\tau_i^N) Y_N(\tau_i^N + \frac{k}{N}) - Cov(\tilde{Y}_u(0)\tilde{Y}_u(k))\right) \xrightarrow[N \to \infty]{d} \tilde{\sigma}(u)^2 Z. \tag{44}$$

Proof. We first note that [13, Corollary 3.4] implies that $\tilde{Y}_u$ is θ-weakly dependent. The convergences in (41) and (42) then follow analogous to the proof of Theorem 5.4. The proof of [14, Theorem 3.36] together with (A2) or (A3) imply that the θ-coefficients of $\tilde{Y}_u$ are exponentially decaying. Therefore, we also obtain the convergences in (43) and (44). $\qquad \square$

Remark 5.15. *In ongoing research, the developed asymptotic results from Section 3 are applied to establish a comprehensive asymptotic theory for M-estimators of contrast functions that are based on observations from a sequence of time-varying Lévy-driven state space processes. The contrast functions under investigation are of finite and infinite memory and satisfy suitable regularity conditions that are in the spirit of the function classes $\mathcal{L}_\infty^{p,q}$ and $\mathcal{L}_{k+1}$ from Section 2.3 and 2.4.*
In particular, this includes consistency and asymptotic normality results for a localized least square estimator for time-varying CAR(1) processes as well as consistency results for a localized quasi maximum-likelihood and a localized Whittle estimator for time-varying Lévy-driven state space models.

Acknowledgements

References

[1] ANDREWS, D. W. K. (1988). Laws of large numbers for dependent non-identically distributed random variables. *Econometric Theory* **4** 458–467.

[2] APPLEBAUM, D. (2009). *Lévy processes and stochastic calculus, 2nd ed.* Cambridge University Press, Cambridge.

[3] BAAKE, M. AND SCHLÄGEL, U. (2011). The Peano-Baker series. *Proc. Steklov Inst. Math.* **275** 155–159.

[4] BARDET, J. M., DOUKHAN, P. AND LEÓN, J. R. (2008). Uniform limit theorems for the integrated periodogram of weakly dependent time series and their applications to Whittle's estimate. *J. Time Ser. Anal.* **29** 906–945.

[5] BARDET, J. M., DOUKHAN, P. AND WINTENBERGER, O. (2020). Contrast estimation of general locally stationary processes using coupling. *arXiv:2005.07397.*

[6] BARNDORFF-NIELSEN, O. E. AND SHEPARD, N. (2001). Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63** 167–241.

[7] BENTH, F. E., KLÜPPELBERG, C., MÜLLER, G. AND VOS, L. (2014). Futures pricing in electricity markets based on stable CARMA spot models. *Energy Econ.* **44** 392–406.

[8] BITTER, A., STELZER, R. AND STRÖH, B. (2021). Continuous-time locally stationary time series models. *arXiv:2104.13796.*

[9] BROCKETT, R. W. (1970). *Finite dimensional linear systems.* John Wiley & Sons Inc.

[10] BROCKWELL, P. J. (2014). Recent results in the theory and applications of CARMA processes. *Ann. Inst. Statist. Math.* **66** 647–685.

[11] BROCKWELL, P. J., FERRAZZANO, V. AND KLÜPPELBERG, C. (2013). High-frequency sampling and kernel estimation for continuous-time moving average processes. *J. Time Ser. Anal.* **34** 385–404.

[12] CIALENCO, I. (2018). Statistical inference for SPDEs: an overview. *Stat. Inference Stoch. Process.* **21** 309–329.

[13] CURATO, I. V, AND STELZER, R. (2019). Weak dependence and GMM estimation of supOU and mixed moving average processes. *Electron. J. Stat.* **13** 310–360.

[14] CURATO, I. V., STELZER, R. AND STRÖH, B. (2020). Central limit theorems for stationary random fields under weak dependence with application to ambit and mixed moving average fields. *arXiv:2007.10874.*

[15] DAHLHAUS, R. (1996). On the Kullback-Leibler information divergence of locally stationary processes. *Stoch. Proc. Appl.* **62** 139–168.

[16] DAHLHAUS, R. (2012). Locally stationary processes. In *Handbook of statistics: Time series analysis: Methods and applications* **30** 351–413.

[17] DAHLHAUS, R., RICHTER, S. AND WU, W. B. (2019). Towards a general theory for non-linear locally stationary processes. *Bernoulli* **25** 1013–1044.

[18] DEDECKER, J., AND DOUKHAN, P. (2003). A new covariance inequality and applications. *Stochastic Process. Appl.* **106** 63–80.

[19] DEDECKER, J., AND MERLEVÈDE, F. (2002). Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.* **30** 1044–1081.

[20] DOUKHAN, P., MAYO, N., AND TRUQUET, L. (2008). Weak dependence, models and some applications. *Metrika* **69** 199–225.

[21] FUCHS, F. AND STELZER, R. (2013). Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the supOU stochastic volatility model. *ESAIM-Probab. Stat.* **17** 455–471.

[22] HÄUSLER, E., AND LUSCHGY, H. (2015). *Stable convergence and stable limit theorems.* Springer, Cham.

[23] HIGHAM, N. J. (2008). *Functions of matrices: Theory and computation.* SIAM, Philadelphia.

[24] HORN, R. A. AND JOHNSON C. R. (1990). *Matrix Analysis.* Cambridge University Press, Cambridge.

[25] KALLENBERG, O. (1997). *Foundations of modern probability.* Springer, New York.

[26] KLÜPPELBERG, C., LINDNER, A. AND MALLER, R. (2004). A continuous-time GARCH process driven by a Lévy process: Stationarity and second-order behavior. *J. Appl. Probab.* **41** 601–622.

[27] KOO, B. AND LINTON, O. (2012). Estimation of semiparametric locally stationary diffusion models. *J. Econometrics* **170** 210–233.

[28] LARSSON, E. K. AND MOSSBERG, M. (2004). Fast and approximative estimation of continuous-time stochastic signals from discrete-time data. In *IEEE International Conference on Acoustics, Speech, and Signal Processing, Proceedings, ICASSP 04, 2* 529–532.

[29] MARQUARDT, T. AND STELZER, R. (2007). Multivariate CARMA processes. *Stoch. Proc. Appl.* **117** 96–120.

[30] RUGH, W. J. (1996). *Linear System Theory, 2nd ed.* Prentice-Hall, New Jersey.

[31] SATO, K. I. (2013). *Lévy processes and infinitely divisible distributions, Cambridge studies in advanced mathematics 68.* Cambridge University Press, Cambridge.

[32] SATO, K.-I. (2014). Stochastic integrals with respect to Lévy processes and infinitely divisible distributions. *Sugaku Expositions* **27** 19–42.

[33] SCHLEMM, E. AND STELZER, R. (2012). Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. *Bernoulli* **18** 46–63.

[34] STELZER, R. (2010). Multivariate COGARCH(1,1) processes. *Bernoulli* **16** 80–115.

[35] STRÖH, B. (2021). Statistical inference for continuous-time locally stationary processes using stationary approximations. *Work in progress*

[36] SURULESCU, N. M. (2010). *On some classes of continuous-time series models and their use in financial economics.* PhD thesis, Ruprecht-Karls-Universität Heidelberg.

[37] WU, M.-A. AND SHERIF, A. (1976). On the commutative class of linear time-varying systems. *Int. J. Control.* **23** 433–444.