Semigroups and Evolution Equations

SS 2018

References: Pazy

Engel Nagel

W. A., Heat Kernels.

Chapter 1. Semigroups and their generators

§ 1 C_0-semigroups.

X Banach space.

$\mathcal{L}(X) := \{ S : X \to X : \text{linear, continuous} \}$

$\|S\| = \sup_{\|x\| \leq 1} \|Sx\|$

$\mathcal{L}(X)$ is a Banach space

$S_n \to S$ in $\mathcal{L}(X) :\Leftrightarrow \|S_n - S\| \to 0$

convergence in operator norm

$S_n \longrightarrow S$ <u>strongly</u> $:\Leftrightarrow$ $S_n x \to Sx$ in X $\forall x \in X$

(this is much weaker).

$\mathcal{L}(X)$ is a <u>Banach algebra</u>:

$S_1, S_2 \in \mathcal{L}(X) \Rightarrow S_1 \circ S_2 \in \mathcal{L}(X)$ &

$\|S_1 S_2\| \leq \|S_1\| \, \|S_2\|$.

(1.1) <u>Definition.</u> A <u>C_0-semigroup</u> on X is a mapping $T: [0,\infty) \to \mathcal{L}(X)$ such that

(a) $T(t+s) = T(t)\,T(s)$

(b) $\lim\limits_{t \downarrow 0} T(t)x = x$ $\quad \forall x \in X$

C_0-semigroup = <u>strongly continuous semigroup</u>

Frequently $\quad T(0) := I \quad (T(t))_{t \geq 0} = T$.

(1.2) <u>Properties</u>: 1. ~~1.~~ Put $T(0) = I$

Then $T(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is

strongly continuous

2. $\exists\, M, \omega$ $\quad \| T(t) \| \le M e^{\omega t} \quad (t \ge 0)$

(at most <u>exponential growth</u>).

<u>Recall</u>: <u>uniform boundedness principal.</u>

<u>Proof.</u> a) $\exists\, \tau > 0$ $\quad M = \sup\limits_{0 < t \le \tau} \| T(t) \| < \infty.$

Otherwise, $\exists$ $t_n \downarrow 0$ $\quad \| T(t_n) \| \longrightarrow \infty.$

UBP $\Rightarrow$ $\exists\, x$ $\quad \| T(t_n) x \| \to \infty$ $\quad$ ↯.

b) Let $t > 0$. $\quad \exists!\, n \in \mathbb{N}_0$ $\quad t = n\tau + s$

$0 \le s < \tau$. $\quad \| T(t) \| = \| T(n\tau)\, T(s) \|$

$= \| T(\tau)^n\, T(s) \| \le M \cdot M^n.$

$\omega = \log M$. $\quad M^n = e^{\omega n} \le e^{\omega t}$.

Thus $\| T(t) \| \le M e^{\omega t}$

c) Let $x \in X$, $t > 0$

1st case $t_n \downarrow t \Rightarrow T(t_n)x - T(t)x =$

$T(t_n - t)T(t)x - T(t)x \longrightarrow 0$.

2nd case $t_n \uparrow t$.

$T(t_n)x - T(t)x = T(t_n)(x - T(t - t_n)x)$

$\longrightarrow 0$. □

<u>Lemma.</u> $S_n \to S$ strongly

$x_n \to x \Rightarrow S_n x_n \to Sx$.

Proof. $\sup \| S_n \| < \infty$ UBP

$S_n x_n - Sx = S_n(x_n - x) + S_n x - Sx$ □

<u>Examples of semigroups.</u>

(1.3) <u>Example.</u> $A \in \mathcal{L}(X)$

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

$\| e^{tA} - I \| \longrightarrow 0 \quad (t \downarrow 0)$

(1.4) <u>Equicontinuity Lemma.</u>

$S_n \in \mathcal{L}(X, Y)$, $\|S_n\| \leq M$.

Equivalent.

(i) $\exists X_0$ dense in X such that

$\lim S_n x$ exists $\forall x \in X$

(ii) $\exists S \in \mathcal{L}(X, Y)$ $S_n x \to Sx$

uniformly on compact subsets.

<u>Proof.</u> Assume (i)

a) $\lim_{n\to\infty} S_n x =: Sx$ exists $\forall x \in X$

Let $x \in X$, $\varepsilon > 0$. $\exists x_0 \in X_0$ $\|x - x_0\| \leq \varepsilon$.

$\exists n_0$ $\|S_n x_0 - S_m x_0\| \leq \varepsilon$ $\forall n, m \geq n_0$

$\Rightarrow$ $\|S_n x - S_m x\| \leq \|S_n(x - x_0)\| + \|S_n x_0 - S_m x_0\|$

$\cancel{+ \|(S_n - S_m)x_0}$ $\leq M\varepsilon + \varepsilon$ $n, m \geq n_0$

$\Rightarrow$ $(S_n x)$ CS

$\Rightarrow$ $Sx := \lim S_n x$ exists for all $x \in X$. Clearly $S \in \mathcal{L}(X, Y)$

b) uniformly on compact subsets.

Let $K \subset X$ be compact, $\varepsilon > 0$

$\exists\ y_1, \dots, y_m \qquad K \subset \bigcup_{j=1}^{m} B(y_j, \varepsilon)$

$\exists\ n_0 \quad \forall n \geq n_0 \qquad \| S_n y_j - S y_j \| \leq \varepsilon \quad \forall n \geq n_0$
$\forall j = 1, \dots, m$

Let $x \in K$. $\exists j \quad \|x - y_j\| \leq \varepsilon$. $\Rightarrow$

$$\| S_n x - Sx \| \leq \| (S_n - S)(x - y_j) \| + \| S_n y_j - S y_j \|$$
$$\leq 2M\varepsilon + \varepsilon \quad \forall n \geq n_0 \quad \square.$$

(1.5) ~~Co~~ Proposition. Let $T : (0, \infty) \to \mathcal{L}(X)$ be a semigroup such that $\|T(t)\| \leq M$ $0 < t \leq 1$. If $\exists X_0 \subset X$ dense such that $T(t)x_0 \to x_0$ $(t \downarrow 0)$ $\forall x_0 \in X_0$, then ~~T(t)~~ T is a C_0-semigroup.

Rk. $T : (0, \infty) \to \mathcal{L}(X)$ semigroup $:\Leftrightarrow$

$T(t+s) = T(t)T(s)$.

Proof. Let $t_n \downarrow t \Rightarrow T(t_n)x_0 \to x_0 = I x_0$
$\forall x_0 \in X_0$ Equicontinuity lemma $\Rightarrow T(t_n)x \to x$
$\forall x \in X$. $\square$

(1.8) <u>Example</u> (diagonal semigroup).

Let $X = \ell^2$, $\lambda_n \in \mathbb{C}$, $\mathrm{Re}\,\lambda_n \leq \omega$.

Let $T(t)x = (e^{\lambda_n t} x_n)_{n \in \mathbb{N}}$.

Then T is a C_0-semigroup.

<u>Proof.</u> a) $\|T(t)x\|^2 = \sum_{n=1}^{\infty} |e^{\lambda_n t} x_n|^2$

$\leq e^{2\mathrm{Re}\,\lambda_n t} \sum |x_n|^2$

$\leq e^{2\omega t} \|x\|^2$

Thus $\|T(t)\| \leq e^{\omega t}$.

b) $X_0 = c_{00} = \{ x \in \ell^2 : \exists n_0 \quad x_n = 0 \text{ if } n \geq n_0 \}$

finitely supporte sequences

c_{00} is dense in X.

$x \in c_{00}$ $\cancel{x}$ $(T(t)x)_k = e^{t\lambda_k} x_k \to x_k$

$t \to 0$ $\Rightarrow$ $T(t)x \to x$ $(t \downarrow 0)$ $\forall x \in c_{00}$.

Proposition 1.6 $\Rightarrow$ claim $\square$

<u>Rk</u> $T(t) \to I$ in $\mathcal{L}(\ell^2)$ as $t \downarrow 0$ (exercise).

iff $\sup_{n \in \mathbb{N}} |\lambda_n| < \infty$.

(1.7) <u>Example (Shift semigroup).</u>

Let $X = L^p(\mathbb{R})$, $1 \le p < \infty$

$(T(t)f)(x) = f(x+t)$.

Then T is a C_0-semigroup.

<u>Proof.</u> a) $\|T(t)f\| = \|f\|$ $\forall f$

Thus $\|T(t)\| \le 1$.

b) Let $f \in C_c(\mathbb{R})$ (continuous vanishing
outside a compact set). Then f is
uniformly continuous; i.e. let $\varepsilon > 0$.
Then $\exists\, \delta > 0$ $|f(x) - f(y)| \le \varepsilon$ if
$|x-y| \le \delta$. Let $a > 0$ such that
$f(x) = 0$ for $|x| > a$. Let $0 < t \le \delta$

$$\|(T(t)f)(x) - f(x)\|_p^p \le$$

$$\int_{\mathbb{R}} |f(x+t) - f(x)|^p = \int_{-a}^{a+1} \varepsilon^p \le (2a+1)\,\varepsilon^p$$

Thus $\|T(t)f - f\|_p \to 0$ $(t \downarrow 0)$.

Proposition 1.5 $\Rightarrow$ claim. $\square$

§ 2 Der Generator einer C_0-Halbgruppe.

Let $T : (0,\infty) \to X$ be a C_0-semigroup. $T(0) = 0$.

(2.1) Definition. The generator A of T is defined by

$$D(A) := \{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \}$$

$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t}.$$

Thus: $D(A) \subset X$ subspace (the domain of A).

$A : D(A) \to X$ is linear.

(2.2) **Proposition.** $x \in D(A) \Rightarrow$

$T(t)x \in D(A)$ & $AT(t)x = T(t)Ax$.

Proof. $\lim_{h \downarrow 0} \frac{1}{h}(T(h)T(t)x - T(t)x)$

$= T(t) \lim_{h \downarrow 0} \frac{1}{h}(T(h)x - x)$

$= T(t)Ax$. □

(2.3) **Cauchy problem.**

$$\begin{cases} \dot{u}(t) = Au(t) & t > 0 \\ u(0) = x \end{cases}$$

classical solution: $u \in C^1([0,\infty); X)$,

$u(t) \in D(A)$ $\forall t \geq 0$.

Theorem. $\forall x \in D(A)$ $\exists!$ u classical solution.

Proof. a) **Existence** $u(t) = T(t)x$

$$\frac{u(t+h) - u(t)}{h} = \frac{T(h)T(t)x - T(t)x}{h}$$

$= AT(t)x$.

b) uniqueness. Let u be a solution,

$t > 0$, $v(s) = T(t-s)u(s)$.

$$\dot v(s) = -AT(t-s)u(s) + T(t-s)\dot u(s)$$

$$= -AT(t-s)u(s) + T(t-s)Au(s)$$

$$= 0.$$

$$\Rightarrow \quad v(0) = v(t) \quad \Rightarrow \quad T(t)x = u(t).$$

$\square$

(2.4) Riemann integral.

X Banach space

$u \in C([a,b]; X)$

$$\|u\|_\infty := \max_{t \in [a,b]} \|u(t)\|$$

π = partition $a = t_1 < \dots < t_n = b$
with intermediate points $s \in [t_j, t_{j+1}]$
(pip)

$$|\pi| = \max |t_j - t_{j-1}|$$

$$S(\pi, u) = \sum_{j=1}^{m} u(s_j)(t_j - t_{j-1})$$

<u>Theorem.</u> $\int_a^b u(t)dt := \lim_{|\pi|\to 0} S(\pi, u)$ exists in X.

(2.5) <u>Proposition.</u> $B \in \mathcal{L}(X, Y)$, + $\Rightarrow$

$$B \int_a^b u(t)dt = \int_a^b Bu(t)\,dt$$

<u>Proof.</u> $B\,S(\pi, u) = S(\pi, Bu)$ $\square$

$X' = \{ x' : X \to \mathbb{K} :$ cont. linear $\} = \mathcal{L}(X, \mathbb{K})$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

<u>dual space</u>

<u>HB</u> $\|x\| = \sup_{\|x'\| \le 1} |\langle x', x\rangle|$

In particular: X' <u>separates</u> X;

(2.6) <u>Corollary.</u> $\langle x', \int_a^b u(t)dt\rangle = \int_a^b \langle x', u(t)\rangle dt.$

(2.7) <u>Corollary:</u> $\| \int_a^b u(t)\,dt\| \le \int_a^b \|u(t)\|\,dt$

<u>Proof.</u> $\| \int_a^b u(t)dt\| = \sup_{\|x'\| \le 1} |\langle x', \int_a^b u(t)dt\rangle|$

$$\leq \sup_{\|x'\| \leq 1} \int_a^b |\langle x', u(t)\rangle| \, dt$$

$$\leq \int_a^b \|u(t)\| \, dt . \quad \square$$

(2.6) <u>Fundamental Theorem.</u>

a) $u \in C([a,b]; X)$, $v(t) = \int_a^t u(s)\, ds$

$\Rightarrow v \in C^1([a,b]; X)$ & $v' = u$.

b) $u \in C^1([a,b]; X)) \Rightarrow$

$$u(b) - u(a) = \int_a^b u'(s)\, ds .$$

<u>Proof of b)</u> $\langle x', u(b)\rangle - \langle x', u(a)\rangle$

$$= \int_a^b \frac{d}{dt} \langle x', u(t)\rangle \, dt$$

$$= \int_a^b \langle x', \dot{u}(t)\rangle \, dt$$

$$= \langle x', \int_a^b \dot{u}(t)\, dt\rangle$$

X' separates X $\Rightarrow$ claim. $\square$

T C_0-semigroup with generator A.

(2.7) <u>Proposition.</u> Let $x, y \in X$. Equivalent.

(i) $x \in D(A)$ & $Ax = y$;

(ii) $\int_0^t T(s) y \, ds = T(t)x - x$.

<u>Proof.</u> (ii) $\Rightarrow$ (i)

$$\frac{1}{t}(T(t)x - x) = \frac{1}{t}\int_0^t T(s) y \, ds$$
$$\longrightarrow T(0)y = y. \quad \text{as } t \downarrow 0.$$

Thus $x \in D(A)$ & $Ax = y$.

(i) $\Rightarrow$ (ii) $\quad u(t) = T(t)x$, $\quad x \in D(A)$

$\Rightarrow \quad \dot{u}(t) = T(t)Ax$

$\Rightarrow \quad T(t)x - x = u(t) - u(0) = \int_0^t T(s) Ax \, ds$

$\square$

(2.8) <u>Definition.</u> An operator A on X is <u>closed</u> if

$D(A) \ni x_n \to x$, $Ax_n \to y$ $\Rightarrow$ $x \in D(A)$ & $Ax = y$.

Remark. Let $D(A) = X$

A closed $\Leftrightarrow$ $A \in \mathcal{L}(X)$

(closed graph theorem).

(2.9) Proposition. The generator of a C_0-semigroup is closed.

Proof. Let $x_n \in D(A)$, $x_n \to x$,

$y_n := Ax_n \to y$

$$\int_0^t T(s) y_n \, ds = T(t) x_n - x_n$$

$$\underset{n \to \infty}{\leadsto} \quad \int_0^t T(s) y \, ds = T(t) x - x$$

(2.8) $\Rightarrow$ $x \in D(A)$, $Ax = y$. $\square$

(2.10) **Lemma.** Let $A : D(A) \to X$ be an operator. Then

$$\|x\|_A := \|Ax\| + \|x\|$$

defines a norm on $D(A)$. Equi:

(i) A is closed;

(ii) $(D(A), \|\ \|_A)$ is complete.

(exercise)

(2.11) **Proposition.** Let $u \in C([a,b]; X)$ s.t. $u(t) \in D(A)$ and $Au \in C([a,b], X)$.

Then ~~$u \in$~~ $\int_a^b u(t)\,dt \in D(A)$ and

$$A \int_a^b u(t)\,dt = \int_a^b Au(s)\,ds$$

Proof.

$$S(\pi_n, u) \in D(A)$$

$$S(\pi_n, u) \to \int_a^b u(t)\,dt$$

$$AS(\pi_n, u) \to y \qquad \square$$

(2.12) Proposition. (Every day formula)

$x \in X \Rightarrow$

(AWF) $\int_0^t T(s)x\,ds \in D(A)$ & $A\int_0^t T(s)x\,ds = T(t)x - x$

German: Allerweltsformel = AWF.

Proof. $\frac{1}{h}\left[T(h)\int_0^t T(s)x\,ds - \int_0^t T(s)x\,ds\right]$

$$= \frac{1}{h}\left[\int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds\right]$$

$$= \frac{1}{h}\left[\int_h^{t+h} T(r)x\,dr - \int_0^t T(r)x\,dr\right]$$

$$= \frac{1}{h}\left[\int_t^{t+h} T(r)x\,dr - \int_0^h T(r)x\,dr\right]$$

$$\longrightarrow T(t)x - x. \quad \square$$

1 # Evolutionsgleichungen

24.04.2017

Erinnerung

Sei T eine C_0-Hgr. auf einem BR X.

Der Generator A von T ist definiert durch

$$D(A) := \{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ existiert} \}$$

$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \forall x \in D(A).$$

Für $x, y \in X$ sind äquivalent:

(a) $x \in D(A)$ und $Ax = y$

(b) $\int_0^t T(s)y \, ds = T(t)x - x. \quad \forall t \geq 0.$

(2.13) **Beispiel** (Diagonalhgr. auf L^p)

Sei $q: \mathbb{R} \longrightarrow \mathbb{C}$ messbar mit nach oben beschränkten Realteil, $p \in [1, \infty)$ und

$$T(t) f \, (x) := e^{t \cdot q(x)} f(x) \quad \forall t \geq 0, x \in \mathbb{R}, f \in L^p(\mathbb{R})$$

Dann ist T C_0-Hgr. mit Generator A wobei

$$D(A) = \{ f \in L^p(\mathbb{R}) : q f \in L^p(\mathbb{R}) \},$$

$$Af = q \cdot f \quad \forall f \in D(A).$$

Beweis: (nur Generator)

Wir zeigen zunächst:

$\forall g \in L^p(\mathbb{R})$: $\left(\int_0^t T(s) g \, ds \right)(x) = \int_0^t e^{sq(x)} g(x) \, ds$ für fast alle $x \in \mathbb{R}$.

Sei $g \in L^p(\mathbb{R})$. Nach Definition des Riemann-Int's. gilt:

$$\int_0^t T(s) g \, ds = \lim_{N \to \infty} \frac{t}{N} \sum_{n=0}^{N-1} (T(s) g) \left(\frac{nt}{N} \right) \quad \text{in } L^p(\mathbb{R}).$$

Nach der „Umkehrung" des Satzes von Lebesgue finden wir eine ~~Teilfolge~~ Indexfolge $(N_k)_{k\in\mathbb{N}}$ mit $N_k \to \infty$, $N_k < N_{k+1}$ $\forall k\in\mathbb{N}$ mit

$$\left(\int_0^t T(s)g\,ds\right)(x) = \lim_{k\to\infty} \frac{t}{N_k}\sum_{n=0}^{N_k-1}\left(T\left(\frac{nt}{N_k}\right)g\right)(x)$$
$$= \lim_{k\to\infty} \frac{t}{N_k}\sum_{n=0}^{N_k-1} e^{\frac{nt}{N_k}q(x)} g(x)$$
$$= \int_0^t e^{sq(x)} g(x)\,ds$$

für fast alle $x\in\mathbb{R}$.

Für $f, g \in L^p(\mathbb{R})$ gilt nun:

$f \in D(A)$, $g = Af$

$\Leftrightarrow \int_0^t T(s)g\,ds = T(t)f - f \quad \forall t\geq 0$

$\Leftrightarrow \int_0^t e^{sq(x)} g(x)\,ds = e^{tq(x)}f(x) - f(x)$ für fast alle $x\in\mathbb{R}$ $\forall t\geq 0$.

$\Leftrightarrow g(x) = q(x)f(x)$ für fast alle $x\in\mathbb{R}$.

„$\Leftarrow$" folgt durch Multiplikation mit e^{sq} und Integration.

„$\Rightarrow$" folgt durch Ableiten in 0 (man kann den Grenzwert

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} e^{sq(x)} g(x)$$

entlang einer Folge nehmen, bekommt also kein Problem mit den Nullmengen). $\square$

(2.14) <u>Beispiel</u> (Shifthgr. auf $C_{ub}(\mathbb{R})$)

Betrachte

$C_{ub}(\mathbb{R}) := \{f : \mathbb{R}\to\mathbb{C}$ beschränkt + glm. stetig$\}$

und $(T(t)f)(x) := f(x+t)$ $\forall x\in\mathbb{R}, t\geq 0, f\in C_{ub}(\mathbb{R})$.

Dann ist T C_0-Hgr. mit Generator A

(3) wobei

$$D(A) = \{ f \in C_{ub}(\mathbb{R}) : f \text{ diffbar und } f' \in C_{ub}(\mathbb{R}) \},$$

$$Af = f' \quad \forall f \in C_{ub}(\mathbb{R}).$$

<u>Beweis:</u> (nur Generator)

Es gilt für $f, g \in C_{ub}(\mathbb{R})$:

$$f \in D(A),\ g = Af$$

$$\Leftrightarrow \int_0^t T(s) g \, ds = T(t) f - f \quad \forall t \geq 0.$$

$$\Leftrightarrow \left(\int_0^t T(s) g \, ds \right)(x) = T(t) f(x) - f(x) \quad \forall t \geq 0\ \forall x \in \mathbb{R}. \quad (*)$$

Die Punktauswertung

$$\delta_x : C_{ub}(\mathbb{R}) \longrightarrow \mathbb{C},\ f \longmapsto f(x)$$

ist $\forall x \in \mathbb{R}$ ein stetiges Funktional, daher gilt nach (2.6):

$$(*) \Leftrightarrow \int_0^t (T(s) g)(x) \, ds = T(t) f(x) - f(x) \quad \forall t \geq 0, x \in \mathbb{R}$$

$$\Leftrightarrow \int_0^t g(x+s) \, ds = f(x+t) - f(x) \quad \forall t \geq 0, x \in \mathbb{R}$$

$$\Leftrightarrow \frac{1}{t} \int_x^{x+t} g(s) \, ds = \frac{f(x+t) - f(x)}{t} \quad \forall t > 0, x \in \mathbb{R}$$

$$\Leftrightarrow f \text{ diffbar und } f' = g$$

„$\Rightarrow$": $t \to 0$: f rechtsseitig diffbar mit stetiger Abl. $\Rightarrow$ f diffbar $\square$

„$\Leftarrow$": Hauptsatz der DIR

Was ist der Generator der Shifthgr. auf $L^2(\mathbb{R})$?

$\rightsquigarrow$ brauchen neuen Ableitungsbegriff.

(2.15) <u>Beobachtung</u>

Für $f \in C^1(\mathbb{R})$ und $g \in C_c^\infty(\mathbb{R}) = C^\infty(\mathbb{R}) \cap C_c(\mathbb{R})$ erhalten wir mit partieller Integration:

$$\int_{\mathbb{R}} f'(x)g(x)\,dx = \underbrace{f(x)g(x)\Big|_{-\infty}^{\infty}}_{=0} - \int_{\mathbb{R}} f(x)g'(x)\,dx$$

und für $f \in C^2(\mathbb{R})$:

$$\int_{\mathbb{R}} f''(x)g(x)\,dx = -\int_{\mathbb{R}} f'(x)g'(x)\,dx = \int_{\mathbb{R}} f(x)g''(x)\,dx.$$

(2.16) Definition

Sei $1 \leq p < \infty$. Dann heißt $f \in L^p(\mathbb{R})$ k-mal schwach differenzierbar, falls $\exists f^{(k)} \in L^p(\mathbb{R})$ mit

$$\int_{\mathbb{R}} f^{(k)}(x)\,g(x)\,dx = (-1)^k \int_{\mathbb{R}} f(x)\,g^{(k)}(x)\,dx$$

für alle $g \in C_c^\infty(\mathbb{R})$. In diesem Fall heißt $f^{(k)}$ k-te schwache Ableitung von f.

$W^{k,p}(\mathbb{R}) := \{ f \in L^p(\mathbb{R}) : f$ ~~k~~-mal schwach diffbar $\forall \ell \leq k \}$ heißt Sobolevraum.

(2.17) Bemerkung

Die schwache Ableitung ist eindeutig, denn $C_c^\infty(\mathbb{R})$ trennt $L^p(\mathbb{R})$, d.h. $\forall f_1, f_2 \in L^p(\mathbb{R}), f_1 \neq f_2 : \exists g \in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} f_1(x)\,g(x)\,dx \neq \int_{\mathbb{R}} f_2(x)g(x)\,dx.$$

(2.18) Beispiel (Shifthgr. auf $L^2(\mathbb{R})$)

Sei

$$T(t)f\,(x) := f(x+t) \quad \forall t \geq 0,\ x \in \mathbb{R},\ f \in L^2(\mathbb{R}).$$

Dann ist T C_0-Hgr. mit Generator A wobei $D(A) = W^{1,2}(\mathbb{R})$,

$$Af = f' \quad \forall f \in D(A).$$

Idee: Führe die Halbgruppe auf eine bekannte zurück.

(2.19) **Lemma**

Sei T C_0-Hgr. mit Generator A auf BR X. Weiter sei $V \in \mathcal{L}(Y,X)$ ein Isomorphismus von einem BR Y nach X (d.h. V ist bijektiv). Durch

$$S(t) := V^{-1} T(t) V \quad \forall t \geq 0$$

wird eine C_0-Hgr. S auf Y definiert mit Generator B wobei

$$D(B) = \{ y \in Y : Vy \in D(A) \},$$
$$By = V^{-1} A V y \quad \forall y \in D(B).$$

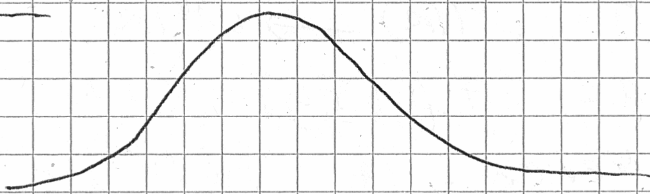
Beweis: Ü A.

(2.20) **Definition**

$$\mathcal{S}(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) : \forall n,m \in \mathbb{N}_0 : \lim_{|x| \to \infty} |x^n f^{(m)}(x)| = 0 \}$$

heißt Schwartzraum.

Bsp.: $f(x) = e^{-x^2}$



(2.21) **Bemerkung**

$\mathcal{S}(\mathbb{R})$ liegt dicht in $L^p(\mathbb{R})$, für $1 \leq p < \infty$.

(2.22) **Definition**

Für $f \in \mathcal{S}(\mathbb{R})$ definieren wir die Fouriertrafo $\mathcal{F}f$ durch

$$\mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y)\, dy \quad \forall x \in \mathbb{R}.$$

(2.23) Satz

Für $f \in \mathcal{S}(\mathbb{R})$ gilt:

(i) $\mathcal{F}f \in \mathcal{S}(\mathbb{R})$

(ii) $(\mathcal{F}f)^{(k)} = \mathcal{F}((-i)^k \cdot \mathrm{id}^k \cdot f) \quad \forall k \in \mathbb{N}$

(iii) $i^k \cdot \mathrm{id}^k \cdot \mathcal{F}f = \mathcal{F}(f^{(k)}) \quad \forall k \in \mathbb{N}$.

(hier ist $\mathrm{id}: \mathbb{R} \to \mathbb{R}, x \mapsto x$)

__Beweis:__ etwa (ii):

Es gilt

$$\mathcal{F}((-i)^k \mathrm{id}^k \cdot f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \cdot (-i)^k \cdot y^k \cdot f(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{d}{dx} e^{-ixy} f(y)\, dy$$

$$\overset{\text{„Lebesgue"}}{=} \frac{d}{dx^k} \mathcal{F}(f)(x) \qquad \forall x \in \mathbb{R}, k \in \mathbb{N}.$$

Rest: siehe z. B. Werner V.2.2 – V.2.5 ∎

(2.24) Theorem

$\mathcal{F}: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ ist bijektiv und besitzt eine unitäre Fortsetzung

$\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$

(d.h. $(\mathcal{F}f \mid \mathcal{F}g) = (f \mid g) \quad \forall f, g \in L^2(\mathbb{R})$).

Weiter gilt $\mathcal{F}(f^{(k)}) = i^k \cdot \mathrm{id}^k \cdot \mathcal{F}f \quad \forall f \in W^{k,2}(\mathbb{R})$.

__Beweis:__ Werner, Abschnitt V.2.

__Beweis__ (von (2.18)):

Betrachte $S(t) := \mathcal{F}\, T(t)\, \mathcal{F}^{-1}$ auf $L^2(\mathbb{R})$, $t \geq 0$.

Dann gilt für $f \in \mathcal{S}(\mathbb{R})$:

$$S(t)\mathcal{F}f\,(x) = (\mathcal{F}\, T(t) f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y+t)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix(y-t)} f(y)\, dy = e^{ixt} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y)\, dy$$

$$= e^{tq(x)} (\mathcal{F}f)(x) \qquad \forall t \geq 0, x \in \mathbb{R}$$

mit $q(x) := ix$ für $x \in \mathbb{R}$.

$\overset{\Rightarrow}{\mathcal{F}\mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R}) \text{ dicht}}$ S ist Diagonalhgr. zu q

(7) $\underset{(2.13)}{\Rightarrow}$ S hat Generator B mit

$$D(B) = \{ f \in L^2(\mathbb{R}) : q \cdot f \in L^2(\mathbb{R}) \},$$

$$B f = q \cdot f \quad \forall f \in D(B)$$

$\overset{(2.19)}{\underset{T(t) = \mathcal{F}^{-1} S(s) \mathcal{F}}{\Rightarrow}}$ T ist stark stetig mit Generator A, wobei

$$D(A) = \{ f \in L^2(\mathbb{R}) : q \cdot \mathcal{F} f \in L^2(\mathbb{R}) \}$$

$$A f = \mathcal{F}^{-1} q \, \mathcal{F} f \quad \forall f \in D(A)$$

Für $f \in W^{1,2}(\mathbb{R})$ gilt nach (2.24):

$$L^2(\mathbb{R}) \ni \mathcal{F}(f') = i \cdot id \cdot \mathcal{F} f = q \, \mathcal{F} f.$$

$\Rightarrow$ $f \in D(A)$ und $A f = \mathcal{F}^{-1} q \, \mathcal{F} f = \mathcal{F}^{-1} \mathcal{F}(f') = f'$

Sei umgekehrt $f \in D(A)$.

$$\Rightarrow - \int_{\mathbb{R}} f(x) \varphi'(x) \, dx = - (f \mid \overline{\varphi}') \overset{(2.24)}{=} - (\mathcal{F} f \mid \mathcal{F} \overline{\varphi}')$$

$$\underset{(2.23)}{=} - (\mathcal{F} f \mid i \cdot id \cdot \mathcal{F} \overline{\varphi}) = (\mathcal{F}^{-1} (\underbrace{i \cdot id \cdot \mathcal{F} f}_{= q \mathcal{F} f \in L^2(\mathbb{R})}) \mid \overline{\varphi})$$

$$= \int_{\mathbb{R}} \mathcal{F}^{-1} (q \cdot \mathcal{F} f)(x) \varphi(x) \, dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R})$$

$\Rightarrow$ $f \in W^{1,2}(\mathbb{R})$ $\square$

28.4.2017

§ 3 The resolvent set of an operator ~~of the generator~~.

X Banach space over $\mathbb{K}$.

Let A be an operator on X.

$$\varrho(A) := \{\lambda \in \mathbb{K} : \lambda - A : D(A) \to X \text{ bij. } \& \ (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

<u>resolvent</u> set. $R(\lambda, A) := (\lambda - A)^{-1}$

(3.1) <u>Remark.</u> a) $\varrho(A) \neq \emptyset \Rightarrow A$ closed

b) A closed $\Rightarrow$

$$\varrho(A) = \{\lambda \in \mathbb{K} : \lambda - A \text{ bijective}\}$$

~~(3.2) Theorem. $\varrho(A)$ is open~~

c) A, B operators

$A \subset B$, $\varrho(A) \cap \varrho(B) \neq \emptyset$

$\Rightarrow A = B$.

(3.2) Resolvent identity.

$$\frac{R(\lambda,A)-R(\mu,A)}{\mu-\lambda} = R(\lambda,A)\,R(\mu,A)$$

$\lambda,\mu\in\rho(A)$ $\lambda\neq\mu$.

(3.3) Proposition. $\rho(A)$ is open in $\mathbb{K}$.

More precisely, $\lambda_0\in\rho(A)$,

$|\lambda-\lambda_0|\;\|R(\lambda_0,A)\|<1 \Rightarrow \lambda\in\rho(A)$

& $\displaystyle R(\lambda,A) = \sum_{n=0}^{\infty}(\lambda_0-\lambda)^n R(\lambda_0,A)^{n+1}$

Proof. Assume $|\lambda-\lambda_0|\;\|R(\lambda_0,A)\| = q<1$

$(\lambda-A) = (\lambda-\lambda_0+\lambda_0-A)$

$= (I-(\lambda_0-\lambda)R(\lambda_0,A))\,\cancel{R(\lambda_0,A)}\,(\lambda_0-A)$

$\Rightarrow R(\lambda,A) = R(\lambda_0,A)\displaystyle\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n R(\lambda_0,A)^n.$

In fact, $S := (\lambda_0-\lambda)R(\lambda_0,A)\in\mathcal{L}(X)$,

$\|S\|<1.$ $\Rightarrow$ $I-S$ invertible.

$$R(\lambda, A) = R(\lambda_0, A)(I-S)^{-1}.$$

Proof. ~~Rf~~ $(\lambda - A) R(\lambda_0, A)(I-S)^{-1}x =$

$(\lambda - \lambda_0 + \lambda_0 - A) R(\lambda_0, A)(I-S)^{-1}x =$

$((\lambda - \lambda_0) R(\lambda_0, A) + I)(I-S)^{-1}x = x.$

$\forall x \in X.$

$x \in D(A) \Rightarrow R(\lambda_0, A)(I-S)^{-1}(\lambda - A)x$

$= (I-S)^{-1} R(\lambda_0, A)(\lambda - A)x$

$= (I-S)^{-1} R(\lambda_0, A)(\lambda - \lambda_0 + \lambda_0 - A)x$

$= (I-S)^{-1}(I-S)x = x.$ □

(3.4) Corollary. Let $\lambda_0 \in \mathbb{K}$.
$\exists \lambda_n \in \rho(A)$, $\lambda_n \to \lambda_0$ &
$|\lambda_n| \leq c \Rightarrow \lambda_0 \in \rho(A).$

~~Proof.~~ ~~dist $(\lambda, \sigma$~~

(3.5) Corollary. Let $\lambda \in \rho(A)$.
Then $\operatorname{dist}(\lambda, \sigma(A)) \geq \|R(\lambda, A)\|^{-1}$

Proof of (3.5). Let $\mu \in \sigma(A)$. Then
$|\lambda - \mu| \, \|R(\lambda, A)\| \geq 1 \Rightarrow (\lambda - \mu) \geq \|R(\lambda, A)\|^{-1}$
$\forall \mu \in \sigma(A).$ □

Here $\sigma(A) := \mathbb{K} \setminus \varrho(A)$ is the spectrum of A.

Proof of (3.4). ~~dist $(\lambda_n, \sigma$~~
$|\lambda_n - \lambda_0| \, \|R(\lambda_n, A)\| \leq c\, |\lambda_n - \lambda_0| < 1$ if n is big enough. Thus $\lambda_0 \in \varrho(A)$. □

(3.6) Yosida approximation. Let A be an operator such that $(\omega, \infty) \subset \varrho(A)$ & $\|\lambda R(\lambda, A)\| \leq M \quad (\lambda > \omega)$.

Then $\overline{D(A)} = X$ iff $\lambda R(\lambda, A)x \to x \quad (\lambda \to \infty) \quad \forall x \in X$.

Pf. „$\Leftarrow$" trivial

„$\Rightarrow$" 1. $x \in D(A)$ $\quad x = \lambda R(\lambda, A)x - R(\lambda, A)Ax$

$\Rightarrow \lambda R(\lambda, A)x - x = R(\lambda, A)Ax \to 0 \quad (\lambda \to \infty)$.

2. Equicontinuity Lemma. □

Assume $\quad \| \lambda R(\lambda, A) \| \leq M \qquad (\lambda > \omega)$

and $\quad \overline{D(A)} = X.$

Define: $\quad A_n = n^2 R(n, A) - n \in \mathcal{L}(X).$

Then $\quad A_n x \longrightarrow Ax \quad$ for all $x \in D(A)$

Proof. $\quad n R(n, A) x \longrightarrow x \qquad n \to \infty$

$$n R(n, A) x - R(n, A) Ax = x$$

$$\Rightarrow \quad R(n, A) Ax = n R(n, A) x - x$$

$$\Rightarrow \quad n R(n, A) Ax = n^2 R(n, A) x - nx$$

$$\longrightarrow Ax \quad (n \to \infty). \quad \square$$

§ 4 The resolvent of the generator.

T C_0-sg, A generator. $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

(4.1) Proposition. (rescaling)

a) $\lambda \in \mathbb{K} \Rightarrow S(t) = e^{\lambda t} T(t)$ defines a C_0-sg. Generator: $B = A + \lambda$, $D(B) = D(A)$.

b) $\alpha > 0$ $S(t) = T(\alpha t)$ C_0-sg generator $B = \alpha A$.

(4.2) Corollary. a) $\lambda \in \mathbb{K}$, $x \in X \Rightarrow$ $\int_0^t e^{-\lambda s} T(s) x \, ds \in D(A)$ &

$$(A - \lambda) \int_0^t e^{-\lambda s} T(s) x \, ds = e^{-\lambda t} T(t) x - x$$

b) $\lambda \in \mathbb{K}$, $x \in D(A) \Rightarrow$

$$\int_0^t e^{-\lambda s} T(s) (\lambda - A) x \, ds = e^{-\lambda t} T(t) x - x$$

$\|T(t)\| \leq Me^{\omega t}$

(4.3) <u>Proposition.</u> Let $\lambda \in \mathbb{K}$, $\mathrm{Re}\lambda > \omega$.

Then $\lambda \in \varrho(A)$ &

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad (1)$$

$\forall x \in X$.

Proof. $Q_t x = \int_0^t e^{-\lambda s} T(s)x \, ds$

$\cancel{\in \mathcal{L}(x}$ $Q_t \in \mathcal{L}(X)$.

Rk. $\|e^{-\lambda t} T(t)x\| \leq Me^{(-\mathrm{Re}\lambda + \omega)t} \|x\|$

$\Rightarrow$ (1) converges and defines

$Q \in \mathcal{L}(X)$.

Moreover $Q_t x \longrightarrow Qx \quad t \to \infty \quad \forall x$

$Q_t x \in D(A)$ & $(\lambda - A)Q_t x = x - e^{-\lambda t}T(t)x$

$\longrightarrow x \quad (t \to \infty)$.

$\lambda - A$ closed $\Rightarrow Qx \in D(A)$ &

$(\lambda - A)Qx = x$. If $x \in D(A)$ then

$Q(\lambda - A)x = \cancel{(\lambda - A)Qx} \lim Q_t(\lambda - A)x$

$= \cancel{(\lambda - A)} \lim (\lambda - A)Q_t x = x$

$\square$

Consequence : $\| R(\lambda, A) \| \leq \frac{M}{\mathrm{Re}\,\lambda - \omega}$.

(4.4) Corollary. $\| T(t) \| \leq 1$
(i.e. contraction semigroup)

$\Rightarrow (0, \infty) \subset \rho(A)$ & $\| \lambda R(\lambda, A) \| \leq 1$

$(\lambda > 0)$.

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§ 5 The Hille-Yosida Theorem.

T contractive $:\Leftrightarrow$ $\|T(t)\| \leq 1$

contraction semigroup.

(5.1) Theorem (Hille-Yosida)

Let A be an operator on X

Equivalent.

(i) A generates a contractive C_0-semigroup.

(ii) (a) $\overline{D(A)} = X$

(b) $(0,\infty) \subset \varrho(A)$ &

$\|\lambda R(\lambda, A)\| \leq 1 \qquad (\lambda > 0)$.

Proof. (i) $\Rightarrow$ (ii) done

(iii) $\Rightarrow$ (i) $\quad A_n = n^2 R(n,A) - nI$

$$e^{tA_n} = e^{-ntI} e^{tn^2 R(n,A)}$$

$$\Rightarrow \quad \|e^{tA_n}\| \leq 1 \qquad (t>0,\ n\in\mathbb{N}).$$

$$e^{tA_n}x - e^{tA_m}x =$$

$$\int_0^t \frac{d}{ds}\left(e^{(t-s)A_n} e^{sA_m}x\right) ds =$$

$$\int_0^t e^{(t-s)A_n}(A_m - A_n) e^{sA_m} x\, ds =$$

$$\int_0^t e^{(t-s)A_n} e^{sA_m} (A_m - A_n)x\, ds$$

$$\Rightarrow \|e^{tA_n}x - e^{tA_m}x\| \leq t\,\|(A_m - A_n)x\|$$

Let $x \in D(A)$. Then $A_n x \to Ax$

(Yosida approximation).

Thus $e^{tA_n}x$ Cauchy for $x\in D(A)$

Equicontinuity Lemma $\Rightarrow$

Define $F_n^\tau : X \longrightarrow C([0,\tau], X)$

by $(F_n^\tau x)(t) = e^{tA_n}x$

F_n^τ is linear $\| F_n^\tau x(t) \| \leq \|x\| \quad \forall t$

$\Rightarrow \| F_n^\tau x \|_\infty \leq \|x\| \quad \Rightarrow \| F_n^\tau \| \leq 1.$

$\| (F_n^\tau - F_m^\tau)(x) \|_\infty \leq \tau \, \| Ax_n - Ax_m \|$

if $x \in D(A)$

Thus $(F_n^\tau x)_{n \in \mathbb{N}}$ converges in $C([0,\tau]; X)$

whenever $x \in D(A)$.

Equicontinuity Lemma $\Rightarrow$ convergence

in $C([0,\tau], X) \quad \forall x \in X$.

Thus $\exists \; T(.) : \mathbb{R}_+ \to \mathcal{L}(X)$ strongly

continuous

$$T(t)x = \lim_{n \to \infty} e^{tA_n}x$$

$\forall$ unif^{ly} on $[0,\tau] \quad \forall x \in X$.

$$T(t)\,T(s)x \overset{(1)}{=} \lim_{n\to\infty} e^{tA_n} e^{sA_n} x$$
$$= \lim e^{(t+s)A_n} x$$
$$= T(t+s)x$$

(1) $y_n = e^{sA_n} x \longrightarrow \cancel{e^s}\, T(s)x$

$\Rightarrow \quad e^{tA_n} y_n \longrightarrow \cancel{e^t}\, T(t)\,T(s)\,x.$

~~Let~~ Thus T is a C_0-sg.

Let B be the generator of T.

Let $x \in D(A)$, $Ax = y$.

$$\int_0^t e^{sA_n} A_n x \, ds = e^{tA_n} x - x$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\int_0^t \cancel{e}\,T(s)y \, ds \qquad\qquad T(t)x - x$$

since ~~dem~~ $e^{sA_n} A_n x \longrightarrow T(s)Ax$ uniformly ~~glm~~

on ~~auf~~ $[0,t]$. ($T_n^t x \longrightarrow T(.)x$ in

$C([0,t],X)$ $\forall x$ $\Rightarrow$ $T_n^t A_n x \longrightarrow T(.)Ax$ in $C([0,t],X)$.)

Char. of the generator $\Rightarrow$ $x \in D(B)$

& $Bx = Ax$.

Thus $A \subset B$. Hence $A = B$. $\square$

Let T be a bounded C_0-sg.

$\|T(t)\| \le M$.

$$\|x\|_0 := \sup_{s \ge 0} \|T(s)x\| .$$

equivalent norm.

$\|T(t)x\|_0 \le \|x\|_0$; i.e. $\|T(t)\|_0 \le 1$.

A generator. $\Rightarrow$ $\|\lambda R(\lambda, A)\|_0 \le 1$.

$\Rightarrow$ $\|[\lambda R(\lambda, A)]^n x\| \le \|[\lambda R(\lambda, A)]^n x\|_0$

$\le \|x\|_0 \le M\|x\|$.

(5.2) <u>Corollary</u>. Let A be an operator. $M \geq 1$. Equi

(i) A generates a C_0-sg T s.t. $\|T(t)\| \leq M$ $(t \geq 0)$;

(ii) (a) $\overline{D(A)} = X$;

(b) $(0,\infty) \subset \rho(A)$;

(c) $\|(\lambda R(\lambda, A))^n\| \leq M$ $\forall n \in \mathbb{N}_0,\ \lambda > 0$.

<u>Proof</u>. (i) $\Rightarrow$ (ii) done

(ii) $\Rightarrow$ (i) $A_n = n^2 R(n, A) - n$.

$$e^{t n^2 R(n,A)} = \sum_{k=0}^{\infty} \frac{t^k n^k}{k!} [n R(n, A)]^k$$

$$\|e^{t n^2 R(n,A)}\| \leq \sum_{k=0}^{\infty} \frac{t^k n^k}{k!} M \leq M e^{tn}$$

$\Rightarrow \|e^{tA_n}\| \leq M$.

Now the proof of (5.1) goes through. □

§ 6 Groups.

(6.1) Definition. A mapping $T : \mathbb{R} \to \mathcal{L}(X)$

is a group if

$$T(t+s) = T(t)\,T(s) \qquad (t, s \in \mathbb{R})$$

$$T(0) = I.$$

and a C_0-semigroup if in addition

$$\lim_{t \downarrow 0} T(t)x = x \qquad \forall x \in X.$$

(6.2) Consequences a) $T(t)^{-1} = T(-t)$. bded [1]

b) $(T(t))_{t \geq 0}$ is a C_0-sg, hence

$$\|T(t)\| \leq Me^{wt} \qquad t \geq 0$$

c) $T : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous

[1] Theorem of the continuous inverse.

Proof. Let $t_n \to t$. Choose $s > 0$ such that $t_n + s \geq 1$. $\Rightarrow$

$$T(t_n)x - T(t)x = T(-s)\left[T(s+t_n)x - T(s+t)x\right]$$

$$\to 0 \quad (n\to\infty) \quad \square$$

d) ~~$(T(-t))_{t\geq 0}$ is a C_0-semigroup.~~

~~Its generator is A.~~

$t \in \mathbb{R} \Rightarrow T(t)D(A) \subset D(A)$ &

$AT(t)x = T(t)Ax \quad (x \in D(A))$

Pf. $$\frac{T(h)T(t)x - T(t)x}{h} = T(t)\frac{T(h)x - x}{h} \to T(t)Ax$$

$h \downarrow 0$

e) $x \in D(A) \Rightarrow \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$

$\forall x \in D(A)$.

Proof. $$\frac{T(t-h)x - T(t)x}{-h} = T(-s)\frac{T(t+s-h) - T(t+s)x}{-h}$$

$$\to T(-s)AT(t+s)x = T(-s)AT(t+s)x \overset{d)}{=} AT(t)x$$

property of C_0-semigroups.

$s+t>0$

$\square$

f) $(T(-t))_{t\geq 0}$ is a C_0-semigroup and $-A$ its generator.

Pf. C_0-sg : clear.

Let B be the generator of $(T(-t))_{t\geq 0}$.

Let $x \in D(A) \overset{e)}{\Rightarrow}$

$$\lim_{t \downarrow 0} \frac{T(-t)x - x}{-t} = Ax$$

$$\Rightarrow \lim_{t \downarrow 0} \frac{T(-t)x - x}{t} = -Ax$$

$$\Rightarrow -A \subset B. \quad \cancel{\Rightarrow A = B} . \cancel{\text{If } x \in D(B)}$$

g) For each $x \in D(A)$ $\exists !$ solution of

$$\begin{cases} u \in C^1(\mathbb{R}, X), & u(t) \in D(A) \quad (t \in \mathbb{R}) \\ \dot{u}(t) = Au(t) & (t \in \mathbb{R}) \\ u(0) = x & \end{cases}$$

Namely: $T(t)x = u(t)$

Pf. existence : e)

uniqueness : from sg case. □

h) $\quad \|T(t)\| \leq M e^{\omega |t|}$ for some $M \geq 0$, $\omega \in \mathbb{R}_+$

Pf. clear for $t \geq 0$ $\quad \|T(t)\| \leq M_+ e^{\omega_+ t}$ $\quad t \geq 0$

$\|T(-t)\| \leq M_- e^{-\omega_- t}$ $\quad t < 0$

$= M_- e^{\omega_- |t|}$ □

i) $\quad \sigma(A) = \{ \lambda \in \mathbb{C} : \cancel{|\lambda|}\ \mathrm{Re}\,\lambda \leq \omega \}$

ω as in (h) and $\quad \|R(\lambda, A)\| \leq \dfrac{M}{|\mathrm{Re}\,\lambda| - \omega}$

Proof. $\mathrm{Re}\,\lambda < -\omega \Rightarrow$ ~~$(\lambda + A)^{-1}$ ex.~~

$\Rightarrow \omega < -\mathrm{Re}\,\lambda \Rightarrow -\lambda \in \varrho(-A) \Rightarrow$

$\lambda \in \varrho(A)$. □

Missing ~~implic~~ inclusion $B \subset -A$

in f)

Let $x \in D(B)$ $\quad \frac{T(-t)x - x}{t} \to Bx = y$.

Let $\lambda \in \varrho(A)$. $\overset{\text{vi}A}{\Rightarrow}$ $\frac{T(-t)R(\lambda,A)x - R(\lambda,A)x}{t}$

$\xrightarrow{t \downarrow 0}$ $R(\lambda, A)y$

d) $\Rightarrow$ $-AR(\lambda, A)x = R(\lambda, A)y$

$\lambda R(\lambda, A)x - AR(\lambda, A)x = x$

$\Rightarrow$ $R(\lambda, A)y = x - \lambda R(\lambda, A)x$ $\Rightarrow x \in D(A)$.

$\square$

Lemma. T Halbgruppe $: (0,\infty) \to \mathcal{L}(X)$

$t_0 > 0$.

a) $T(t_0)$ inj. $\Rightarrow$ $T(t)$ inj. $\forall t > 0$

b) $T(t_0)$ surj. $\Rightarrow$ $T(t)$ surj. $\forall t > 0$

Pf. a) 1. Let $0 < t < t_0$, $T(t)x = 0$

$\Rightarrow 0 = T(t_0 - t)\,T(t)x = T(t_0)x \Rightarrow x = 0$.

2. $T(nt_0)$ inj. $T(t_0)^n x = 0 \Rightarrow T(t_0)^{n-1} x = 0$

$\dots \Rightarrow T(t_0)x = 0 \Rightarrow x = 0$

3. Let $0 < t$ be arbitrary. Choose $nt_0 > t$. 1. & 2. $\Rightarrow T(t)$ inj.

b) 1. Let $0 < t < t_0$. Let $y \in X$.

$\exists x \in X$ $T(t_0)x = y$ $\Rightarrow T(t)\,T(t_0 - t)x = y$

2. $T(nt_0)$ surj.

3. Let $t > 0$ be arbitrary. $\exists n$ $nt_0 > t$

1.82. $\Rightarrow T(t)$ surj. $\square$

(6.3) <u>Theorem.</u> Let T be a C_0-sg with generator A. Equ.

(i) $\exists\ \mathcal{U}$ a C_0-group s.t. $T(t) = \mathcal{U}(t)$ $(t \geq 0)$

(ii) $\exists\ t_0 > 0$ $T(t_0)$ is bijective;

(iii) $-A$ generates a C_0-sg.

<u>Proof.</u> (ii) $\Rightarrow$ (i) $\mathcal{U}(t) := T(t)$ for $t \geq 0$, $\mathcal{U}(-t) := T(t)^{-1}$ for $t \geq 0$. (cf. Lemma).

Claim: $\mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1)\,\mathcal{U}(t_2)$ $\forall\, t_1, t_2 \in \mathbb{R}$

Clear if $t_1 > 0, t_2 > 0$ or $t_1 < 0, t_2 < 0$.

a) Assume $t_1 + t_2 > 0$, $t_1 < 0$

Then $t_2 > 0$ and $\mathcal{U}(t_2) =$
$T(t_2 - (t_1+t_2))\, T(t_1+t_2) = T(-t_1)\, T(t_1+t_2)$

$\Rightarrow$ $\mathcal{U}(t_1+t_2) = \mathcal{U}(t_1+t_2) = T(t_1)\,\mathcal{U}(t_2)$

$= \mathcal{U}(t_1)\,\mathcal{U}(t_2)$.

b) Assume $t_1 + t_2 < 0$, $t_2 > 0$. Then
$\mathcal{U}(t_1+t_2) = \mathcal{U}(-t_1-t_2)^{-1} \overset{a)}{=} (\mathcal{U}(-t_1)\,\mathcal{U}(-t_2))^{-1} = \mathcal{U}(t_2)\,\mathcal{U}(t_1)$

By our definition $\mathcal{U}$ is a C_0-group. ~~and~~

$\cancel{A}$

(i) $\Rightarrow$ (iii) see consequences.

(iii) $\Rightarrow$ (ii). Denote by S the C_0-sg generated by $-A$. Let $x \in D(A)$.

Then $\frac{d}{dt} T(t)S(t)x = A\,T(t)S(t)x + T(t)(-AS(t)x) = 0$. Thus $T(t)S(t)x \equiv \text{const} = T(0)S(0)x = x$

Thus $T(t)S(t) = I$. Similarly $S(t)T(t) = I$

$\Rightarrow$ (i). $\square$

As application of the HY we show the following.

(6.4) <u>Theorem.</u> Let A be the generator of an isometric C_0-group. Then A^2 generates a contractive C_0-semigroup.

$$D(A^2) := \{x \in D(A) : Ax \in D(A)\}$$

<u>Proof.</u> We know that $(0,\infty) \subset \rho(\pm A)$ & $\|\lambda R(\lambda, A)\| \leq 1$, $\|\lambda R(\lambda, -A)\| \leq 1$

$R(\lambda, -A) = (\lambda + A)^{-1}$.

Let $x \in D(A^2)$, $\lambda > 0$

$(\lambda^2 - A^2)x = (\lambda - A)(\lambda + A)x$.

$\Rightarrow \lambda^2 \in \rho(A)$ & $R(\lambda^2, A^2) = R(\lambda, -A) R(\lambda, A)$.

In fact, $R(\lambda, -A) R(\lambda, A)(\lambda^2 - A^2)x = x$

$(x \in D(A^2))$.

Conversely, let $y \in X$, $x = R(\lambda, -A) R(\lambda, A) y$

$\Rightarrow x \in D(A)$ & $\lambda x + Ax = R(\lambda, A) y$

$\Rightarrow x \in D(A^2)$ & $\underbrace{(\lambda - A)(\lambda + A)x}_{(\lambda^2 - A^2)x} = y$.

$\| \lambda^2 R(\lambda^2, A^2) \| \leq \| \lambda R(\lambda, -A) \| \, \| \lambda R(\lambda, A) \| \leq 1$.

Let $y \in X$ $\quad \lambda^2 R(\lambda^2, A^2) y =$

$\lambda R(\lambda, -A) \, \lambda R(\lambda, A) y \to y \quad \lambda \to \infty \quad \square$

<u>Rq</u>: $\quad S_n x \to Sx \quad \forall x \in X$

$x_n \to x \Rightarrow S_n x_n \to Sx$.

<u>Pf.</u> $\quad S_n x_n - Sx = S_n(x_n - x) + S_n x - Sx$

$\to 0 \quad \square$

Example. $X = L^p(\mathbb{R})$, $1 \le p < \infty$

$(S(t)f)(x) = f(x+t)$ isometric C_0-group

generator B

$D(B) = \cancel{\{ f \in} \, W^{1,p}(\mathbb{R})$

Thus
$D(B^2) = W^{2,p}(\mathbb{R})$ $\quad B^2 f = f''$.

Let T be the sg generated by B^2.

ÜA $\quad (T(t)f)(x) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} f(y)\,dy.$

ÜA

1. Sei A ein Operator mit $\varrho(A) \neq \emptyset$, $S \in \mathcal{L}(X)$. Äqu.

(i) $\exists \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$

(ii) $x \in D(A) \Rightarrow Sx \in D(A) \,\&\, ASx = SAx$

(iii) $\forall \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$.

2. S. 47

<u>Lemma.</u> T Halbgruppe : $(0,\infty) \to \mathcal{L}(X)$

$t_0 > 0$.

a) $T(t_0)$ inj. $\Rightarrow$ $T(t)$ inj. $\forall t > 0$

b) $T(t_0)$ surj. $\Rightarrow$ $T(t)$ surj. $\forall t > 0$

<u>Pf.</u> a) 1. Let $0 < t < t_0$, $T(t)x = 0$

$\Rightarrow$ $0 = T(t_0 - t)\,T(t)x = T(t_0)x$ $\Rightarrow$ $x = 0$.

2. $T(nt_0)$ inj. $T(t_0)^n x = 0$ $\Rightarrow$ $T(t_0)^{n-1} x = 0$

$\dots \Rightarrow$ $T(t_0)x = 0$ $\Rightarrow$ $x = 0$

3. Let $0 < t$ be arbitrary. Choose $nt_0 > t$. 1. & 2. $\Rightarrow$ $T(t)$ inj.

b) 1. Let $0 < t < t_0$. Let $y \in X$.
$\exists x \in X$ $T(t_0)x = y$ $\Rightarrow$ $T(t)\,T(t_0 - t)x = y$

2. $T(nt_0)$ surj.

3. Let $t > 0$ be arbitrary. $\exists n$ $nt_0 > t$

1.82. $\Rightarrow$ $T(t)$ surj. $\square$

(6.3) <u>Theorem.</u> Let T be a C_0-sg with generator A. Equ.

(i) $\exists\, \mathcal{U}$ a C_0-group s.t. $T(t) = \mathcal{U}(t)$ $(t \geq 0)$

(ii) $\exists\, t_0 > 0$ $T(t_0)$ is bijective;

(iii) $-A$ generates a C_0-sg.

<u>Proof.</u> (ii) $\Rightarrow$ (i) $\mathcal{U}(t) := T(t)$ for $t \geq 0$, (cf. Lemma).

$\mathcal{U}(-t) := T(t)^{-1}$ for $t \geq 0$.

Claim: $\mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1)\,\mathcal{U}(t_2)$ $\forall\, t_1, t_2 \in \mathbb{R}$

Clear if $t_1 > 0, t_2 > 0$ or $t_1 < 0, t_2 < 0$.

a) Assume $t_1 + t_2 > 0$, $t_1 < 0$

Then $t_2 > 0$ and $\cancel{T}\, \mathcal{U}(t_2) =$

$T(t_2 - (t_1 + t_2))\, T(t_1 + t_2) = T(-t_1)\, T(t_1 + t_2)$

$\Rightarrow \cancel{T(t}\, \mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1 + t_2) = T(t_1)\, \mathcal{U}(t_2)$

$= \mathcal{U}(t_1)\, \mathcal{U}(t_2)$.

b) Assume $t_1 + t_2 < 0$, $t_1 > 0$. Then

$\mathcal{U}(t_1 + t_2) = \mathcal{U}(-t_1 - t_2)^{-1} \overset{a)}{=} \left(\mathcal{U}(-t_1)\,\mathcal{U}(-t_2)\right)^{-1} = \mathcal{U}(t_1)\,\mathcal{U}(t_2)$.

By our definition $\mathcal{U}$ is a C_0-group. ~~and~~

$\cancel{A}$

(i) $\Rightarrow$ (iii) see consequences.

(iii) $\Rightarrow$ (ii). Denote by S the C_0-sg generated by $-A$. Let $x \in D(A)$.

Then $\frac{d}{dt} T(t)S(t)x = A\,T(t)S(t)x + T(t)(-AS(t)x) = 0$. Thus $T(t)S(t)x \equiv \text{const} = T(0)S(0)x = x$.

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As application of the HY we show the following.

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$\Rightarrow \lambda^2 \in \rho(A)$ & $R(\lambda^2, A^2) = R(\lambda, -A)\, R(\lambda, A)$.

In fact, $R(\lambda, -A)\, R(\lambda, A)\, (\lambda^2 - A^2)\, x = x$

$(x \in D(A^2))$.

Conversely, let $y \in X$, $x = R(\lambda, -A)\, R(\lambda, A) y$

$\Rightarrow$ $x \in D(A)$ & $\lambda x + Ax = R(\lambda, A) y$

$\Rightarrow$ $x \in D(A^2)$ & $\underset{(\lambda^2 - A^2)x}{\underset{\Vert}{(\lambda - A)(\lambda + A)x}} = y$.

$\| \lambda^2 R(\lambda^2, A^2) \| \leq \| \lambda R(\lambda, -A) \| \, \| \lambda R(\lambda, A) \| \leq 1$.

Let $y \in X$ $\quad \lambda^2 R(\lambda^2, A^2) y =$

$\lambda R(\lambda, -A)\, \lambda R(\lambda, A) y \to y$ $\quad \lambda \to \infty$ $\quad \square$

Rq: $\quad S_n x \to Sx \quad \forall x \in X$

$x_n \to x \quad \Rightarrow \quad S_n x_n \to Sx$.

Pf. $\quad S_n x_n - Sx = S_n(x_n - x) + S_n x - Sx$

$\to 0$ $\square$

<u>Example.</u> $X = L^p(\mathbb{R})$, $1 \leq p < \infty$

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generator B

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B^2.

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ÜA

1. Sei A ein Operator mit $\varrho(A) \neq \emptyset$, $S \in \mathcal{L}(X)$. Äqu.

(i) $\exists \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$

(ii) $x \in D(A) \Rightarrow Sx \in D(A)$ & $ASx = SAx$

(iii) $\forall \lambda \in \varrho(A) \quad R(\lambda, A) S = S R(\lambda, A)$.

2. S. 47

8.5.2017

§ 7 Dissipative operators.

We start by some operator theory.

(7.1) Lemma (a priori estimate).

Let $\Lambda \subset \mathbb{C}$ be open, connected, $M : \Lambda \to (0,\infty)$ continuous. A an operator such that

(a) $\| \lambda x - Ax \| \geq M(\lambda) \|x\|$ $(\lambda \in \Lambda,\ x \in D(A))$

(b) $\exists \lambda_0 \in \Lambda$ $(\lambda_0 - A)$ surjective.

Then $\Lambda \subset \rho(A)$ & $\|R(\lambda, A)\| \leq \frac{1}{M(\lambda)}$ for all $\lambda \in \Lambda$.

Pf. 1. $\Lambda_0 := \rho(A) \cap \Lambda$ is open in Λ

2. ~~As~~ $\lambda \in \Lambda_0 \Rightarrow \|R(\lambda, A)\| \leq \frac{1}{M(\lambda)}$.

3. Λ_0 is closed in Λ. Let $\lambda_n \in \Lambda_0$, $\lambda_n \to \mu$, $\mu \in \Lambda$. Then $\|R(\lambda_n, A)\| \leq \frac{1}{M(\lambda_n)}$

Since $\lambda_n \to \mu$, $M(\lambda_n) \to M(\mu) > 0$

$\Rightarrow$ $\sup \| R(\lambda_n, A) \| < \infty$

$\Rightarrow$ $\mu \in \varrho(A)$.

4. $\Lambda_0 \neq \emptyset$ since $\lambda_0 \in \Lambda_0$. $\square$

5.5.2017

(7.2) <u>Definition</u>. An operator A is <u>dissipative</u> if

$$\|x - tAx\| \geq \|x\| \quad \forall t > 0,\ x \in D(A)$$

$$\left[\Leftarrow \quad \|\lambda x - Ax\| \geq \lambda \|x\| \quad \forall \lambda > 0,\ x \in D(A) \right]$$

A is <u>m-dissipative</u> if in addition

~~(7.3)~~ $\exists \lambda_0 > 0$ s.t. $\lambda_0 - A$ is surjective.

(7.3) <u>Theorem</u> (Lumer-Phillips). Let A be an operator. Equ.

(i) A generates a contraction C_0-sg

(ii) A is m-diss. & dd.

Since $\lambda_n \to \mu$, $M(\lambda_n) \to M(\mu) > 0$,

it follows that $\sup_{n\in\mathbb{N}} \|R(\lambda_n, A)\| < \infty$

$\Rightarrow \mu \in \varrho(A)$.

4. $\Lambda_0 \neq \emptyset$ since $\lambda_0 \in \Lambda_0$ $\square$

(7.2) Definition. An operator A is dissapative if $\|x - tAx\| \geq \|x\|$ $\forall t > 0$, $x \in D(A)$

[$\Leftrightarrow$ $\|\lambda x - Ax\| \geq \lambda \|x\|$ $\forall \lambda > 0$, $x \in D(A)$],

A is m-dissipative if in addition

$\exists \lambda_0 > 0$ s.t. $(\lambda_0 - A) D(A) = X$

range condition.

(7.3) Theorem (Lumer - Phillips).

Let A be an operator. Equi:

(i) A generates a contractive C_0-sg

(ii) A is cld & ~~diss~~ m-dissipative.

(7.4) <u>Rk.</u> A m-diss $\Leftrightarrow$ $(0,\infty) \subset \rho(A)$

& $\|\lambda R(\lambda, A)\| \leq 1$

follows from (7.1)

(7.4) & HY $\Rightarrow$ (7.3).

(7.5) <u>Hilbert space:</u>

<u>Proposition.</u> $X = H$. Equ.:

(i) A is dissipative

(ii) $\mathrm{Re}\,(Ax \mid x) \leq 0 \quad \forall x \in D(A)$.

<u>Proof.</u> (ii) $\Rightarrow$ (i) $\|x - tAx\|^2 =$

$(x - tAx \mid x - tAx) = \|x\|^2 - t\,2\,\mathrm{Re}\,(x \mid Ax) + t^2 \|Ax\|^2$

$\geq \|x\|^2$

(i) $\Rightarrow$ (ii) $\quad -t\,2\,\mathrm{Re}\,(x \mid Ax) + t^2 \|Ax\|^2 \geq 0 \quad (t > 0)$

$\Rightarrow -2\,\mathrm{R}(x \mid Ax) + t \|Ax\|^2 \geq 0 \quad t \downarrow 0$

$\Rightarrow$ claim. $\square$

Weak convergence

(7.6) Recall. X Banach space

a) $x_n \rightharpoonup x \iff \langle x', x_n \rangle \longrightarrow \langle x', x \rangle$

$\forall x' \in X'$.

(weak convergence)

b) $S \in \mathcal{L}(X)$.

$\exists!\ S' \in \mathcal{L}(X') \qquad \langle Sx, x' \rangle = \langle x, S'x' \rangle$

c) $x_n \rightharpoonup x \Rightarrow Sx_n \rightharpoonup Sx$

Proof. $\langle Sx_n, x' \rangle = \langle x_n, S'x' \rangle$

$\longrightarrow \langle x, S'x' \rangle = \langle Sx, x' \rangle$. $\square$

d) $x_n \to x \Rightarrow x_n \rightharpoonup x \Rightarrow \sup\limits_{n \in \mathbb{N}} \|x_n\| < \infty$

e) X reflexive $\iff \Big[\|x_n\| \leq c \Rightarrow \exists\, x \in X$

& ss $x_{n_k} \rightharpoonup x \Big]$

Examples: $L^p(\Omega)$ reflexive $1 < p < \infty$.

$L^1(\Omega)$ is not refl. $\emptyset \neq \Omega \subset \mathbb{R}^d$ open

$C(K)$ is not refl. if $\# K = \infty$.

each Hilbert space is reflexive

(7.7) <u>Proposition.</u> X reflexive

$$(\omega, \infty) \subset \rho(A)$$

$$\| \lambda R(\lambda, A) \| \le M \qquad (\lambda > \omega)$$

$$\Rightarrow \quad \overline{D(A)} = X$$

<u>Proof.</u> Let $x \in X$.

$$\exists \ \lambda_n \to \infty \qquad \lambda_n R(\lambda_n, A) x \rightharpoonup y$$

Let $\mu \in \rho(A)$ $\overset{c)}{\Rightarrow}$

$$\lambda_n R(\mu, A) R(\lambda_n, A) x \longrightarrow R(\mu, A) y$$

$$\|$$

$$\frac{\lambda_n}{\lambda_n - \mu} R(\mu, A) x - \frac{\lambda_n}{\lambda_n - \mu} R(\lambda_n, A) x$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$R(\mu, A) x \qquad\qquad\qquad 0$$

$$\Rightarrow \quad R(\mu, A) x = R(\mu, A) y$$

$$\Rightarrow \quad x = y.$$

Thus $x \in \overline{D(A)}^{w} \neq \overline{D(A)}$. $\cancel{\square}$

See below. $\square$

Recall. $C \subset X$ convex.

$$x_n \in C, \quad x_n \to x \quad \Rightarrow \quad x \in \overline{C}.$$

(Hahn-Banach).

(7.8) Theorem. H Hilbert space. A an operator on H. Equ.

(i) A generates a contractive C_0-sg

(ii) A is m-dissipative

(Lumer-Phillips).

Remark. a) H Hilbert space. A dissipative. Then

A ~~is~~ m-dissipative $\Leftrightarrow$ A maximal dissipative

$\overset{!}{\Leftrightarrow}$ $[A \subset B$ B dissipative $\Rightarrow A = B]$

b) false in Banach spaces.

Let $x \in X$, ~~$x \neq 0$~~. Then $\exists x' \in X'$

$\|x'\| \leq 1$, $\quad \langle x'_{x}, x \rangle = \|x\|$ $\quad$ (HB).

$J(x) := \{ x' \in X' : \|x'\| \leq 1, \ \langle x', x \rangle = \|x\| \}$.

(7.9) Proposition. ~~Ex~~ A operator. Equ:

(i) A is diss.

(ii) $\forall x \in D(A)$ $\ \exists x' \in J(x)$

$$\mathrm{Re} \langle x', Ax \rangle \leq 0.$$

Rk. $X = H$ Hilbert.

$x \neq 0 \implies J(x) = \{ \frac{x}{\|x\|} \}$

$$\langle x \mid \frac{x}{\|x\|} \rangle = \frac{\|x\|^2}{\|x\|} = \|x\|. \quad \square$$

Proof. Only (ii) $\implies$ (i)

~~$\|x\|$~~. Let $x \in D(A)$, $x' \in J(x)$ s.t.

$\mathrm{Re} \langle x', Ax \rangle \leq 0$

$$\|x\| = \langle x', x \rangle \leq \mathrm{Re} \langle x', x - tAx \rangle$$

$$\leq \|x - tAx\|. \quad \square$$

(7.10) <u>Proposition.</u> Let A be the generator of a contractive C_0-sg. Then A is <u>strictly dissipative</u>; i.e.

$$\forall x \in D(A) \quad \forall x' \in J(x)$$

$$\operatorname{Re} \langle x', Ax \rangle \leq 0 .$$

<u>Rk.</u> more is true: ~~remains~~ A diss & dd $\Rightarrow$ strictly diss. (without proof)

<u>Pf of (7.10).</u> Let $x \in D(A)$, $x' \in J(x)$

$$\operatorname{Re}\langle Ax, x' \rangle = \lim_{t \downarrow 0} \left\langle \frac{T(t)x - x}{t}, x' \right\rangle$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[\langle T(t)x, x' \rangle - \langle x, x' \rangle \right]$$

$$\leq \overline{\lim_{t \downarrow 0}} \frac{1}{t} \left(\|T(t)x\| - \|x\| \right) \leq 0 . \quad \square$$

<u>Conclusion.</u>

(7.11) <u>Theorem.</u> Let A be dd. Equ.:

(i) A generates a contractive C_0-sg.

(ii) a) $\forall x \in D(A) \quad \exists x' \in \cancel{D(A)}\, J(x) \quad \operatorname{Re}\langle Ax, x' \rangle \leq 0$

b) $\exists \lambda_0 > 0 \quad (\lambda_0 - A)D(A) = X .$

(7.12) Closable operators.

Rk. Let $G \subset X \times X$

$\exists$ an operator A such that $G = G(A)$

$\Leftrightarrow$ $(0, y) \in G \Rightarrow y = 0$.

Proposition. Let A be an operator. Equi:

(i) $\exists$ an operator $\bar{A}$ s.t. $G(\bar{A}) = \overline{G(A)}$

(ii) $x_n \in D(A)$, $x_n \to 0$, $Ax_n \to y \Rightarrow y = 0$.

In that $\bar{A}$ is ~~cl~~ called the closure of A.

Clear:

$$D(\bar{A}) = \{ x \in X : \exists x_n \in D(A),\ x_n \to x,\ (Ax_n) \text{ converges} \}$$

$$\bar{A}x = \lim_{n \to \infty} Ax_n$$

(7.13) Proposition. Let A be dissipative and ~~clos~~ dd. Then A is closable and $\bar{A}$ is dissipative

<u>Proof.</u> $x_n \to 0$ $\quad Ax_n \to y$.

Let $z \in D(A)$. Then

$$\| (x_n + tz) \cancel{+} tA(x_n+tz) \| \geq \| x_n + tz \| \qquad t>0$$

$$\| x_n + tz - tAx_n - t^2 z \| \geq \| x_n + tz \|$$

$n \to \infty \quad \Rightarrow \quad t \| z - \cancel{Ax_n}^{y} - tz \| \geq t \|z\|$

$t \downarrow 0$

$\Rightarrow \quad \| z - \cancel{Ax_n}^{y} - tz \| \geq \|z\|$

$\Rightarrow \quad \| z - y \| \geq \|z\|$ $\quad \boxtimes$.

$z \to y \quad \Rightarrow \quad \|y\| \leq 0$

Let $x \in D(\bar{A})$, $\bar{A}x = y$.

$\Rightarrow \exists x_n \in D(A) \quad x_n \to x, \quad Ax_n \to y = \bar{A}x$

$n \to \infty$

$\| x_n \cancel{+} - tAx_n \| \geq \|x_n\|$

$\| x - t\bar{A}x \| \geq \|x\|$. $\quad \square$

(7.14) <u>Theorem</u> (Lumer–Phillips).

Let A be dissipative and dd.

~~Assume~~ $\exists \lambda_0 > 0$ s.t.

$(\lambda_0 - A) D(A)$ is dense in X.

Then $\bar{A}$ generates a contractive C_0-semigroup.

<u>Proof.</u> $\bar{A}$ is dissipative and dd.

Let $y \in X$. $\exists\, x_n \in D(A)$ $\lambda_0 x_n - Ax_n \to y$

$\|\lambda_0(x_n - x_m)\| \leq \|\lambda_0(x_n - x_m) - A(x_n - x_m)\|$

$\longrightarrow 0$ as $n, m \to \infty$.

Thus $x := \lim x_n$ exists.

$\Rightarrow$ $\cancel{\lambda_0 x_n - Ax_n} \longrightarrow$

$Ax_n = -(\lambda_0 x_n - Ax_n) + \lambda_0 x_n$

$\longrightarrow -y + \lambda_0 x$

$\Rightarrow$ $x \in D(\bar{A})$ & $\bar{A}x = -y + \lambda_0 x$

$\lambda_0 x - \bar{A}x = y$. $\square$

10.5.2017

8. Elliptic operators

$\Omega \subset \mathbb{R}^d$ open.

(8.1) Weak derivatives

Let $u \in L^1_{loc}(\Omega)$

a) Let $j \in \{1, \dots d\}$, $f \in L^1_{loc}(\Omega)$

$D_j u = f \quad :\Longleftrightarrow$

$$-\int_\Omega \partial_j \varphi \, u = \int_\Omega f \varphi \qquad \forall \varphi \in C_c^\infty(\Omega)$$

b) Let $j \in \{1, \dots, d\}$; $E \subset L^1_{loc}(\Omega)$

$D_j u \in E \quad :\Leftrightarrow \quad \exists f \in E \quad D_j u = f.$

f

$D_j u =:$ the weak derivative of f

Rk (consistence) Let $u \in C^1(\Omega)$. Then

$\partial_j u = D_j u.$

(8.2) Weak Laplacian. $u \in L^1_{loc}(\Omega)$

a) $f \in L^1_{loc}(\Omega)$

$$\Delta u = f \quad :\Leftrightarrow \quad \int_\Omega u\, \Delta\varphi = \int_\Omega f\varphi$$
$$\forall \varphi \in C_c^\infty(\Omega)$$

b) $\Delta u \in E \quad :\Leftrightarrow \quad \exists f \in E \quad \Delta u = f.$

(8.3) Sobolev space. $H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega)\}$ is a Hilbert space

for

$$(u \mid v)_{H^1} = \int_\Omega u\bar{v} + \sum_{j=1}^d \int D_j u\, \overline{D_j v}.$$

Thus

$$\|u\|^2_{H^1} = \|u\|^2_{L^2} + \sum_{j=1}^d \|D_j u\|^2_{L^2}$$

Rk (convergence) $u_n \to u$ in $H^1(\Omega)$

$\Leftrightarrow$ $u_n \to u$ & $D_j u_n \to D_j u$

in $L^2(\Omega)$ for $n \to \infty$ $\quad j = 1, \dots, d.$

$$H^1_0(\Omega) := \overline{C_c^\infty(\Omega)}^{\cancel{L^2(\Omega)}\, H^1(\Omega)} .$$

Definition (Dirichlet Laplacian).

Let $D(\Delta^D) := \{ u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega) \}$

$\Delta^D u := \Delta u$.

Theorem. Δ^D generates a contractive C_0-semigroup T on $L^2(\Omega)$.

$\mathbb{K} = \mathbb{R}$

Proof. a) Δ^D is dissipative.

$$(\Delta^D u \mid u) = \int_\Omega (\Delta u) u \, dx$$

$$\int \Delta u \, \varphi = \cancel{\sum_j \int D_j} \int u \, \Delta\varphi$$

$$= \sum_j \int u \, D_j D_j \varphi$$

$$= -\sum_j \int D_j u \, D_j \varphi \qquad \varphi \in C_c^\infty(\Omega)$$

$C_c^\infty(\Omega)$ dense in $H_0^1(\Omega) \Rightarrow$

$$\int \Delta u \, u = - \sum_{j=1}^{d} \int |D_j u|^2 \, dx \leq 0$$

$\forall u \in D(\Delta^D)$.

b) range condition : $\lambda_0 = 1$.

Let $f \in L^2(\Omega)$. Find $u \in H_0^1(\Omega)$ s.t.

$u - \Delta u = f$.

$L\varphi = \int_\Omega f \varphi \, dx$ defines $L \in (H_0^1(\Omega))'$

Riesz - Fréchet $\Rightarrow$ $\exists ! \, u \in H_0^1(\Omega)$

$(u | \varphi)_{H^1} = L\varphi \quad \forall \varphi \in H_0^1(\Omega)$; i.e.

$$\int_\Omega u\varphi + \sum_{j=1}^{d} \int D_j u \, D_j \varphi = \int_\Omega f \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$$

$\Leftrightarrow \forall \varphi \in C_c^\infty(\Omega)$

$$\Rightarrow \int_\Omega u\varphi - \int_\Omega u \, \Delta\varphi = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

$\Rightarrow u - \Delta u = f$. $\square$

8.5

Let $a_{ij} \in L^\infty(\Omega)$ s.t.

$$\sum a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^d$$

a.e.

$$D(A) := \{ u \in H_0^1(\Omega) \quad \sum_{i,j=1}^{d} D_i a_{ij} D_j u \in L^2(\Omega) \}$$

$$Au = \sum_{i,j=1}^{d} D_i(a_{ij} D_j u)$$

Hier : $\sum_{i,j=1}^{d} D_i(a_{ij} D_j u) \in L^2(\Omega)$

$:\Leftrightarrow \exists f \in L^2(\Omega)$

$$-\int \sum_i \sum_{j=1}^{d} a_{ij} D_j u \, \partial_i \varphi \, dx = \int_\Omega f\varphi \, dx$$

$\forall \varphi \in \mathcal{D}(\Omega) := C_c^\infty(\Omega)$.

<u>Theorem.</u> A generates a contractive C_0-semigroup on $L^2(\Omega)$.

We assume here $a_{ij} = a_{ji}$.

ÜA <u>Proof.</u> a) dissipative: $u \in D(A)$

$$(Au \mid \varphi) = -\sum_i \int \sum_j a_{ij} D_j u \, \partial_i \varphi \, dx$$

$\forall \varphi \in \mathcal{D}(\Omega)$ $\Longrightarrow$

$$(Au \mid u) = -\sum_i \int \sum_j a_{ij} D_j u \, D_i u \, dx$$

$$\leq 0 \qquad \forall u \in D(A).$$

b) ~~Let~~ $[u, v] = \alpha \int uv \, dx +$

$$\int \sum_{i,j=1}^{d} a_{ij}(x) D_i u \, D_j v \, dx$$

defines an equivalent scalar product on ~~$L^2(\Omega)$~~ $H^1_0(\Omega)$. In fact

$$[u, u] \geq \int_\Omega |u|^2 dx + \alpha \int_\Omega |\nabla u|^2 d$$

$$|[u, v]| \leq \|u\|_{L^2} \|v\|_{L^2} + c \sum_{j=1}^{d} \|D_j u\| \, \|D_j v\|$$

$\leq \frac{1}{2}$

$$\|u\|^2_{H^1}\|v\|^2_{H^1} = \left(\|u\|^2_{L^2} + \sum_j \|D_j u\|^2\right) \cdot \left(\|v\|^2_{L^2} + \sum_j \|D_j v\|^2\right)$$

$$\geq |[u, v]|.$$

Thus $\alpha \|u\|^2_{H^1} \leq |[u \mid u]| \leq c \|u\|^2_{H^1}$ -

Let $f \in L^2(\Omega)$, $L\varphi = \int_\Omega f\varphi$

Riesz – Fréchet $\Rightarrow$ $\exists!\ u \in H^1_0(\Omega)$

$$[u, v] = \int f v \qquad \forall v \in H^1_0(\Omega)$$

$$\Rightarrow \int u\varphi + \int \sum_i \sum_j a_{ij} D_i u\, D_j \varphi = \int f\varphi$$

$$\forall \varphi \in \mathcal{D}(\Omega)$$

$$\Rightarrow \quad u - Au = f \quad , u \in D(A). \qquad \square$$

<u>Rk.</u> If the a_{ij} are not symmetric one uses the Lax Milgram Lemma instead of Riesz – Fréchet.

9 Adjoint ~~& the surjective LP Thm.~~

<u>Definition.</u> Let A be a d.d operator on X, X' the dual space of X. The adjoint A' of A is defined by

$$D(A') = \{ x' \in X' : \exists y' \in X' \quad \langle Ax, x' \rangle = \langle x, y' \rangle \ \forall x \in D(A) \}$$

$A'x' = y'$. <u>Rk. Clearly:</u> $(\lambda - A)' = \lambda - A'$.

(9.1) <u>Lemma.</u> Let $\lambda \in \varrho(A)$. Then $\lambda \in \varrho(A')$ and

$$R(\lambda, A') = R(\lambda, A)'$$

iiA <u>Proof.</u> Let $x' \in X'$. Claim $R(\lambda, A)'x' \in D(A')$ and $(\lambda - A')R(\lambda, A)'x' = x'$.

$$\langle R(\lambda, A)'x', (\lambda - A)x \rangle = \langle x', R(\lambda, A)(\lambda - A)x \rangle$$
$$= \langle x', x \rangle \quad \forall x \in D(A)$$

$\Rightarrow \quad R(\lambda, A)'x' \in D((\lambda - A)')$ and

$(\lambda - A)' R(\lambda, A)'x' = x'$.

Conversely, let $x' \in D((\lambda - A)') = D(A')$

$\langle R(\lambda, A)'(\lambda - A')x', y\rangle = \langle(\lambda - A)'x', R(\lambda, A)y\rangle$

$= \langle x', (\lambda - A)R(\lambda, A)y\rangle = \langle x', y\rangle \quad \forall\, y \in X$.

$\Rightarrow \quad R(\lambda, A)'(\lambda - A')x' = x'$. □

(9.2) Theorem. Let X be reflexive, A the generator of a C_0-sg T on X. Then $(T(t)')_{t \geq 0}$ is a C_0-sg and A' its generator.

Rk. Clearly, $(T(t)')_{t \geq 0}$ is a sg.

Pb.: strong continuity.

Proof. Replacing A by $A - \omega$ we may assume $\|T(t)\| \leq M$.

Passing to an equivalent norm we may assume $\|T(t)\| \le 1$.

$\Rightarrow \quad \|\lambda R(\lambda, A)\| \le 1$

$= \quad \|\lambda R(\lambda, A')\| \le 1$

$D(A')$ is dense : $\cancel{\lambda R(\lambda}$ Let

$x \in D(A)$ s.t. $\langle x, x' \rangle \quad \forall x' \in D(A')$

$\Rightarrow \quad \langle x, R(\lambda, A)' x' \rangle = 0 \quad \forall x' \in X'.$

$\Rightarrow \quad \langle R(\lambda, A) x, x' \rangle = 0 \quad \forall x' \in X'$

$\Rightarrow \quad R(\lambda, A) x = 0 \quad \Rightarrow x = 0.$

Rk (9.3) $\Rightarrow$ $D(A')$ is dense in X'.

LP $\Rightarrow$ A' generates a C_0-sg. $(S(t))_{t \geq 0}$ on X'. ~~Let $x' \in D(A')$~~

For $x \in X$, $x' \in X'$, $\lambda > 0$

$$\langle R(\lambda, A) x, x' \rangle = \int_0^\infty e^{-\lambda t} \langle T(t) x, x' \rangle \, dt$$

$$\|$$

$$\langle x, R(\lambda, A') x' \rangle = \int_0^\infty e^{-\lambda t} \langle x, S^*(t) x' \rangle \, dt$$

uniqueness Thm $\Longrightarrow$ $\langle T(t) x, x' \rangle = \langle x, S(t) x' \rangle$

$\forall t > 0 \quad \Rightarrow \quad T(t)' = S(t). \quad \square$

<u>Uniqueness Theorem.</u> Let $f : (0,\infty) \longrightarrow \mathbb{K}$ be measurable & bounded. If $\exists\, \lambda_0 \geq 0$ s.t.

$$\int_0^\infty e^{-\lambda t} f(t)\, dt = 0 \qquad \forall\, \lambda > \lambda_0,$$

then $f(t) = 0$ a.e.

10.5.2017

15.5.2017

§ 10 The surjective LP Theorem

(10.1) **Lemma (kernel)** $\lambda \in \rho(A)$, $x \in X$.

Then

$$x \in \ker A \iff \lambda R(\lambda, A) x = x.$$

Pf. "$\Rightarrow$" $\lambda R(\lambda, A) x - R(\lambda, A) A x = x$

"$\Leftarrow$" $\lambda R(\lambda, A) x \cancel{- A R(\lambda, A) x} = x$

Thus $\Rightarrow$ $x \in D(A)$ & $\lambda x = (\lambda - A) x$

$\Rightarrow Ax = 0$. □

(10.2) **Recall**: **w^*-convergence.**

a) $x_n', x' \in X'$

$x_n' \xrightarrow{*} x' \iff \langle x_n', x \rangle \to \langle x', x \rangle$ $\forall x \in X$

$\Rightarrow \sup \|x_n'\| < \infty$.

b) <u>Theorem</u> <u>(Alaoglu Bourbaki).</u>

X separable, $\|x_n'\| \le c$

$\Rightarrow \exists$ ss $\exists\, x' \in X'$ $\quad x_{n_k}' \xrightarrow{*} x'$.

c) $\quad S \in \mathcal{L}(X)$

$x_n' \xrightarrow{*} x' \quad \Rightarrow \quad S'x_n' \xrightarrow{*} S'x'$

<u>Pf.</u> $\langle S'x_n', x\rangle = \langle x_n', Sx\rangle \to \langle x', Sx\rangle$
$= \langle S'x', x\rangle$.

(10.3) <u>Theorem</u> <u>(LP: surjective version).</u>

Let A be diss., dd, surj.
Then A is m-diss. & $0 \in \rho(A)$.

(10.4) <u>Theorem.</u> $\mathcal{L}_s(X,Y) := \{ S \in \mathcal{L}(X,Y)$ surjective$\}$ is open in $\mathcal{L}(X,Y)$

(10.5) <u>Kernel - separation lemma.</u>

Let A be ~~an~~ dd operator such that $\|\lambda R(\lambda, A)\| \le M$ $\lambda \in (0, \delta]$,

$\delta > 0$.

Let $x \in \ker A$, $x \neq 0$. Then

$\exists x' \in \ker A'$, $\langle x', x \rangle \neq 0$

Pf. Assume that X is sapable (for convenience) otherwise nets.

Let $x \in \cancel{X}$ $\ker A$, $x \neq 0$

Let $x_0' \in X'$ $\langle x_0', x \rangle = 0$

$\exists \lambda_n \downarrow 0$ $\exists x' \in X'$ $\lambda_n R(\lambda_n, A)' x_0' \xrightarrow{*} x'$

$$\langle x', x_0' \rangle = \lim \langle \lambda_n R(\lambda_n, A)' x_0', x^* \rangle$$
$$= \lim \langle x_0', \lambda_n R(\lambda_n, A) x \rangle$$
$$= \lim \langle x_0', x \rangle = \langle x_0', x \rangle \neq 0$$

Thus $x' \neq 0$. Let $\mu \in \varrho(A) \Rightarrow$

$$\lambda_n R(\mu, A)' R(\lambda_n, A)' x_0' \xrightarrow{*} R(\mu, A)' x'$$

$$\|$$

$$\frac{\lambda_n}{\mu - \lambda_n} R(\mu, A)' x_0' - \frac{\lambda_n}{\mu - \lambda_n} R(\lambda_n, A)' x_0'$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad - \frac{1}{\mu} x' \quad (10.1) \Rightarrow \text{claim}$$

□

<u>Pf of (10.3).</u> $\bar{A}$ is surj. & diss.

$\bar{A} \in \mathcal{L}(D(\bar{A}), X)$ surj. $\overset{(10.4)}{\Longrightarrow}$ $\exists \lambda > 0$

$\lambda - \bar{A}$ is surj. $\Rightarrow$ $\bar{A}$ is m-diss.

Assume $\exists x_0 \in \ker \bar{A}$, $x_0 \neq 0$.

$\overset{(10.5)}{\Longrightarrow}$ $\exists x_0' \in \ker \bar{A}'$, $x_0' \neq 0$.

$\Rightarrow$ $\langle Ax, x_0' \rangle = 0$ $\forall x \in D(A)$

A surj. $\Rightarrow$ $x_0' = 0$ ↯.

Thus $\bar{A}$ is bij. $\Rightarrow$ $0 \in \rho(\bar{A})$.

Let $x \in D(\bar{A})$. $\exists x_0 \in D(A)$

$Ax_0 = \bar{A}x$ $\Rightarrow$ $x_0 - x \in \ker \bar{A} = \{0\}$

$\Rightarrow$ $x = x_0 \in D(A)$. □

Supplements to Alaoglu.

Definition (nets). Let $(I, \preccurlyeq)$ be an ordered set which is directed, i.e.
$\forall i_1, i_2 \in I \ \exists i_3 \in I \quad i_1 \preccurlyeq i_3, \ i_2 \preccurlyeq i_3$.

A family $(x_i)_{i \in I}$ is called a net.

Let $x \in X$, $\lim\limits_{I} x_i = x$: $\Leftrightarrow$
$\forall \varepsilon > 0 \ \exists i_0 \quad \|x - x_i\| < \varepsilon$;

~~$w^*-\lim x_i$~~

Let $x_i' \in X'$, $x' \in X'$

$$w^*-\lim_{I} x_i' = x' \Leftrightarrow \lim_{I} \langle x_i', x \rangle = \langle x', x \rangle .$$

$$\Leftrightarrow \forall \varepsilon > 0 \ \exists i_0 \ \forall i \succcurlyeq i_0$$
$$|\langle x_i', x \rangle - \langle x', x \rangle| \leq \varepsilon .$$

Theorem (Alaoglu). Let X be a Banach space. Each bounded net in X' has a w^*-convergent subnet.

Definition. Let $(x_i)_{i\in I}$ be a net.

Let J be directed, $\phi : J \to I$ s.t.

(a) $j_1 \leq j_2 \Rightarrow \phi(j_1) \leq \phi(j_2)$

(b) $\forall i \in I \quad \exists j \in J \quad \phi(j) \geq i$.

Then $(x_{\phi(j)})_{j\in J}$ is called a subnet of $(x_i)_{i\in I}$.

Example: $X = \ell^\infty$, $\langle e_n', x\rangle = x_n$

for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$.

Thus $e_n' \in X'$, $\|e_n'\| = 1$ $(n \in \mathbb{N})$

There is no w^*-convergent ~~subnet~~ sequence of

$(e_n')_{n\in\mathbb{N}}$.

Proof. Let $n_k < n_{k+1}$. Define $x \in \ell^\infty$

by $x_n = \begin{cases} (-1)^k & \text{if } n = n_k \\ 0 & \text{if } n \notin \{n_k : k \in \mathbb{N}\} \end{cases}$.

Then $\langle e_{n_k}', x\rangle = (-1)^k$ does not converge □

However, $(e_n')_{n\in\mathbb{N}}$ possesses a w^*-convergent subnet.

10. The Dirichlet Laplacian on $C_0(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be open, bdd & <u>Dirichlet regular</u>; i.e.

$\forall g \in C(\partial\Omega) \quad \exists u \in C(\bar{\Omega}) \cap C^2(\Omega)$

$$\Delta u = 0 \quad \text{in } \Omega$$
$$u_{|\partial\Omega} = g.$$

<u>Example</u>: a) Ω has Lipschitz boundary

b) $\Omega \subset \mathbb{R}^2$ is simply connected.

$$C_0(\Omega) := \{ u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0 \}$$

<u>Definition</u>. The operator Δ_0 on $C_0(\Omega)$ given by

$$D(\Delta_0) := \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}$$
$$\Delta_0 u = \Delta u$$

is ~~at~~ called the <u>Dirichlet Laplacian</u>.

(10.1) <u>Theorem.</u> Δ_0 generates a contractive C_0-semigroup T.

Moreover, $T(t) \geq 0$ for all $t > 0$.

Let $0 < \rho \in \mathcal{D}(\mathbb{R}^d)$, $\int \rho = 1$,

$\operatorname{supp} \rho \subset B(0,1)$. $\rho_n(x) = c_n \rho(nx)$

s.t. $\int \rho_n(x)\, dx = 1$. Then $\rho_n \in \mathcal{D}(\mathbb{R}^d)$,

$\operatorname{supp} \rho_n \subset B(0, 1/n)$.

(11.2) <u>Lemma.</u> Let $f \in C(\mathbb{R}^d)$,

$$\rho_n * f(x) := \int_{|y| < \frac{1}{n}} f(y)\, \rho_n(x-y)\, dy \,.$$

Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ &

$$\| \rho_n * f - f \|_{C(K)} \longrightarrow 0 \quad (n \to \infty)$$

$\forall K \subset \mathbb{R}^d$ compact.

<u>Proof.</u> a) $\partial_j (\rho_n * f) = \partial_j \rho_n * f$

b) $\rho_n * f(x) = \int_{|y|<\frac{1}{n}} f(x-y)\, \rho_n(y)\, dy$.

Let $K_1 = \{ K + \overline{B}(0,1)$ compact.

Let $\varepsilon > 0$ $\exists n_0$ $|f(x-y) - f(x)| \leq \varepsilon$

$\forall x \in K$, $|y| \leq \frac{1}{n_0}$

$\Rightarrow |\rho_n * f(x) - f(x)| \leq \int |f(x-y) - f(x)|\, \rho_n(y)\, dy$

$\leq \varepsilon$ $n \geq n_0$ □

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(11.3) <u>Rk.</u> $v \in C^2(\Omega)$, $x_0 \in \Omega$ $v(x_0) = \max_{y \in \Omega} v(y)$

$\Rightarrow \Delta v(x_0) \leq 0$

<u>Pf.</u> $\sup_{|t| \leq \delta} v(x_0 + t e_j) = v(x_0)$.

$\Rightarrow 0 \geq \frac{d^2}{dt^2}\Big|_{t=0} v(x_0 + t e_j) = \partial_j^2 v(x_0)$

$\Rightarrow \Delta v(x_0) = \sum_{j=1}^{d} \partial_j^2 v(x_0) \leq 0$ □

17.5.2017

10. The Dirichlet Laplacian on $C_0(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be open, bdded &

Dirichlet regular; i.e.

$\forall g \in C(\partial\Omega) \quad \exists u \in C(\bar\Omega) \cap C^2(\Omega)$

$$\Delta u = 0 \quad \text{in } \Omega$$

$$u_{|\partial\Omega} = g.$$

Example: a) Ω has Lipschitz boundary

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Definition. The operator Δ_0 on $C_0(\Omega)$

given by

$$D(\Delta_0) := \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}$$

$$\Delta_0 u = \Delta u$$

is ~~ct~~ called the <u>Dirichlet Laplacian</u> on $C_0(\Omega)$.

(10.1) <u>Theorem.</u> Δ_0 generates a contractive C_0-semigroup T.

Moreover, $T(t) \geq 0$ for all $t > 0$.

Let $0 < \rho \in \mathcal{D}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho = 1$,

$\operatorname{supp} \rho \subset B(0,1)$. $\rho_n(x) = c_n \rho(nx)$

s.t. $\int \rho_n(x)\,dx = 1$. Then $\rho_n \in \mathcal{D}(\mathbb{R}^d)$,

$\operatorname{supp} \rho_n \subset B(0, 1/n)$.

(11.2) <u>Lemma.</u> Let $f \in C(\mathbb{R}^d)$,

$$\rho_n * f(x) := \int_{\mathbb{R}^d} f(y)\,\rho_n(x-y)\,dy$$

Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ &

$$\| \rho_n * f - f \|_{C(K)} \longrightarrow 0 \quad (n \to \infty)$$

$\forall K \subset \mathbb{R}^d$ compact.

<u>Proof.</u> a) $\partial_j(\rho_n * f) = \partial_j \rho_n * f$

b) $\rho_n * f(x) = \int_{|y|<\frac{1}{n}} f(x-y)\,\rho_n(y)\,dy$.

Let $K_1 = \{ K + \overline{B}(0,1)$ compact.

Let $\varepsilon > 0$ $\exists n_0$ $|f(x-y) - f(x)| \leq \varepsilon$

$\forall x \in K$, $|y| \leq \frac{1}{n_0}$

$\Rightarrow |\rho_n * f(x) - f(x)| \leq \int |f(x-y) - f(x)| \rho_n(y)\,dy$

$\leq \varepsilon$ $n \geq n_0$ $\square$

15.5.2017

(11.3) <u>Rk.</u> $v \in C^2(\Omega)$, $x_0 \in \Omega$ $v(x_0) = \max_{y \in \Omega} v(y)$

$\Rightarrow \Delta v(x_0) \leq 0$

<u>Pf.</u> $\sup_{|t| \leq \delta} v(x_0 + t e_j) = v(x_0)$.

$\Rightarrow 0 \geq \frac{d^2}{dt^2}\Big|_{t=0} v(x_0 + t e_j) = \partial_j^2 v(x_0)$

$\Rightarrow \Delta v(x_0) = \sum_{j=1}^{d} \partial_j^2 v(x_0) \leq 0$ $\square$

(11.4) Lemma. Let $u \in C_0(\Omega)$ such that

$$m := \max_{x \in \Omega} u(x) > 0.$$

Assume that $\Delta u \in C(\bar{\Omega})$.

Then $\exists\, x_0 \in \Omega$ s.t. $u(x_0) = m$ &

$\Delta u(x_0) \leq 0$.

Proof. $C_0(\Omega) \subset C(\mathbb{R}^d)$

$u_n = u * \rho_n \longrightarrow u$ in $C(\bar{\Omega})$.

~~Let~~ ~~K~~. Let $x_n \in \bar{\Omega}$ such that $u_n(x_n) = \max_{x \in \bar{\Omega}} u_n(x)$.

We may assume that $x_n \to x_0 \in \bar{\Omega}$

Since $u_n \to u$ uniformly,

$u_n(x_n) \longrightarrow m$.

~~Observe that $u_n(x) = \int \rho_n(y) u(x-y)\, dy$~~

~~$\leq m \quad \forall x \in \bar{\Omega},\ n \in \mathbb{N}$.~~

$$u(x_0) = \underbrace{(u(x_0) - u(x_n))}_{\downarrow\, 0} + \underbrace{(u(x_n) - u_n(x_n))}_{\downarrow\, 0} +$$

$$u_n(x_n) \longrightarrow m.$$

Thus $u(x_0) = m$.

<u>Claim</u>: $\Delta u_n(x_n) \longrightarrow \Delta u(x_0)$.

[In fact, let $n_0 \in \mathbb{N}$ such that $\overline{B}(x_n, \frac{1}{n_0}) \subset \Omega$ $\forall n \geq n_0$. Then for $n \geq n_0$ $\varphi(y) = \rho_n(x_n - y)$ defines $\varphi \in \mathcal{D}(\Omega)$. Thus

$$\begin{aligned}\Delta u_n(x_n) &= \int \Delta\rho_n(x_n - y)\, u(y)\, dy \\ &= \int \Delta\varphi(x_n - y)\, u(y)\, dy \\ &= \int \varphi(x_n - y)\, \Delta u(y)\, dy \\ &= (\rho_n * \Delta u)(x_n).\end{aligned}$$

The proof of (11.2) shows that $\rho_n * \Delta u \longrightarrow \Delta u$ uniformly on compact subsets of Ω.

Thus

$$\Delta u_n(x_n) = (\rho_n * \Delta u)(x_n)$$

$$= (\rho_n * \Delta u)(x_n) - \Delta u(x_n) + \Delta u(x_n)$$

$$\longrightarrow \Delta u(x_0) \qquad (n \to \infty).\ ]$$

By (11.3) $\Delta u_n(x_n) \leq 0$. Thus

$\Delta u(x_0) \leq 0$.

(11.5) <u>Lemma</u>. Let $u \in D(\Delta_0)$, $\lambda > 0$,

$\lambda u - \Delta_0 u = f$.

If $f(x) \leq 1$ $\forall x \in \Omega$, then

$\lambda u(x) \leq 1$ $\forall x \in \Omega$.

<u>Proof</u>. <u>1st case</u>: $u \leq 0$ trivial

<u>2nd case</u> $m = \sup_{x \in \Omega} u(x) > 0$.

(11.4) $\Rightarrow$ $\exists x_0 \in \Omega$ $u(x_0) = m$, $\Delta u(x_0) \leq 0$

$$\Rightarrow \lambda u(x_0) \leq \lambda u(x_0) \underbrace{- \Delta u(x_0)}_{\geq 0} = f(x_0) \leq 1$$

$\Rightarrow$ $\lambda u(x) \leq \lambda u(x_0) \leq 1$ $\forall x \in \Omega$. $\square$

(11.6) <u>Lemma</u>. Δ_0 ist dissipativ.

<u>Beweis</u>. Sei $u \in D(\Delta_0)$, $\lambda > 0$

$\lambda u - \Delta u = f$.

$\|f\|_\infty = 1$. Claim $\lambda \|u\|_\infty \leq 1$.

<u>1st case</u>: $\exists x_0 \quad f(x_0) = \|f\|_\infty$.

$\Rightarrow \quad \lambda u(x) \leq 1 \qquad \forall x \in \Omega$.

<u>2nd case</u> $\quad \lambda u - \Delta u = f \geq -1$

$\Rightarrow \quad \lambda(-u) - \Delta(-u) \leq +1$

$\Rightarrow \quad -u \leq 1 \quad \Rightarrow \quad u \geq 1$.

~~As~~ Thus $\|\lambda u - \Delta u\|_\infty \leq 1 \Rightarrow \lambda \|u\|_\infty \leq 1$. $\square$

(11.7) Fundamental solution of the Laplace equation.

$$E(x) = \begin{cases} \frac{1}{2}|x| & d=1 \\ \frac{1}{2\pi} \log |x| & d=2 \\ \frac{1}{-(d-2)\sigma_d} \frac{1}{|x|^{d-2}} & d \geq 3 \end{cases}$$

$$\sigma_d = |\partial B|$$

Then $E \in L^1_{loc}(\mathbb{R}^d)$ &

$$\Delta E = \delta_0$$

i.e. $\int_{\mathbb{R}^d} E \, \Delta\varphi = \varphi(0) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$

Fundamental solution:

Let $f \in C_c(\mathbb{R}^d)$

$u = E * f$

Then $u \in C^1(\mathbb{R}^d)$ &

$$\Delta u = f$$

(i.e $\int_{\mathbb{R}^d} u \, \Delta\varphi = \int_{\mathbb{R}^d} f\varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$

10.5

Proof of Theorem 11.1.

a) Δ_0 is dissipative.

b) $\mathcal{D}(\Omega) \subset D(\Delta_0)$ and $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ $\Rightarrow$ ~~D(Δ~~ Δ_0 is d.d.

c) Δ_0 is surjective.

Let $f \in C_0(\Omega) \subset C_c(\mathbb{R}^d)$. $\Rightarrow$

$u = E * f \in C^1(\mathbb{R}^d)$ $\quad \Delta u = f$.

Let $g = u_{|\partial\Omega}$. $\exists\ w \in C^2(\Omega) \cap C(\bar{\Omega})$

$\Delta w = 0$, $w_{|\partial\Omega} = g$.

Let $v = u - w \in C_0(\Omega)$

Then $\Delta v = \Delta u - \Delta w = \Delta u = f$. □

The surj. LP Theorem implies the claim. □

(11.8) Lemma. $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$

Proof. Let $f \in C_0(\Omega)$, $\varepsilon > 0$

$K = \{x : |f(x)| \geq \varepsilon\} \subset \Omega$ compact.

Choose $\varphi \in \mathcal{D}(\Omega)$ such that

$\cancel{0 \leq \varphi \leq}$ $1_K \leq \varphi \leq 1_\Omega$.

Then $\varphi \cdot f \in C_c(\Omega)$

$|(f - \varphi \cdot f)(x)| = 0 \quad x \in K$

$|(f - \varphi f)(x)| \leq \varepsilon |1-\varphi| \leq \varepsilon \; (x \notin K)$.

Thus $C_c(\Omega)$ is dense in $C_0(\Omega)$.

b) Let $f \in C_c(\Omega)$. Let $f_n = \varrho_n * f$.

Then $\operatorname{supp} f_n \subset \operatorname{supp} f + \operatorname{supp} \varrho_n$

$\subset \operatorname{supp} f + B(0, \frac{1}{n})$

$\subset K$

Thus $f_n \in \mathcal{D}(\Omega)$ if $n \geq n_0$ □

(11.9) Bemerkung. (11.6) $\Rightarrow$ ~~Δ_0 is~~

$\lambda R(\lambda, \Delta_0)$ is submarkovian; i.e.

$f \leq 1 \Rightarrow \lambda R(\lambda, \Delta_0) f \leq 1$

In particular, $f \geq 0 \Rightarrow R(\lambda, \Delta_0) f \geq 0$

17. 5. 2017

(11.4) Lemma

Sei $u \in C_0(\Omega)$ mit $C := \max_{x \in \Omega} u(x) > 0$.

Ang. $\Delta u \in C(\Omega)$

$\Rightarrow \exists x_0 \in \Omega : u(x_0) = C, \ \Delta u(x_0) \leq 0.$

Beweis:

Wegen $C_0(\Omega) \hookrightarrow C(\mathbb{R}^d)$ können wir u als Funktion in $C(\mathbb{R}^d)$ auffassen.

Es gilt dann

$$u_n = u * \varrho_n \longrightarrow u \text{ in } C(\overline{\Omega}).$$

Sei $x_n \in \overline{\Omega}$ mit $u_n(x_n) = \max_{x \in \overline{\Omega}} u_n(x)$.

Wir finden Teilfolge $(x_{n_k})_{k \in \mathbb{N}}$ mit

$\lim_{k \to \infty} x_{n_k} = x_0 \in \overline{\Omega}$.

$\Rightarrow C = \|u\| = \lim_{k \to \infty} \|u_{n_k}\| = \lim_{k \to \infty} u_{n_k}(x_{n_k})$

und daher

$$\begin{aligned} |u(x_0) - C| &\leq |u(x_0) - u(x_{n_k})| + |u(x_{n_k}) - u_{n_k}(x_{n_k})| \\ &\quad + |u_{n_k}(x_{n_k}) - C| \\ &\leq |u(x_0) - u(x_{n_k})| + \|u - u_{n_k}\|_\infty + |u_{n_k}(x_{n_k}) - C| \to 0, \end{aligned}$$

also $u(x_0) = C$.

Insbesondere gilt $x_0 \in \Omega$.

Wir zeigen $\lim_{k \to \infty} \Delta u_{n_k}(x_{n_k}) = \Delta u(x_0)$.

Wähle $k_0 \in \mathbb{N}$ mit $B(x_0, \frac{1}{k_0}) \subseteq \Omega$.

Wir finden dann $\tilde{k}$ mit $n_k \geq 2k_0 \ \forall k \geq \tilde{k}$.

Es gilt dann

$$\operatorname{supp} \varrho_{n_k} \subseteq \overline{B(0, \tfrac{1}{2k_0})} \qquad \forall k \geq \tilde{k},$$

also gilt für jedes $x \in B(x_0, \frac{1}{2k_0})$ und

$\varphi_{n_k}^x(y) := \varrho_{n_k}(x-y) \quad \forall y \in \mathbb{R}^d$:

$\operatorname{supp} \varphi_{n_k}^x \subseteq \overline{B(x_0, \frac{1}{k_0})} \subseteq \Omega \quad \forall k \geq \tilde{k}$,

d.h. $\varphi_{n_k}^x|_\Omega \in \mathcal{D}(\Omega)$.

Es folgt:

$$\Delta u_{n_n}(x) \overset{\text{Leb.}}{=} \int_{\mathbb{R}^d} \Delta g_{n_n}(x-y)\, u(y)\, dy$$

$$= \int_\Omega \Delta \varphi^x_{n_n}(y)\, u(y)\, dy$$

$$= \int_\Omega \varphi^x_{n_n}(y)\, \Delta u(y)\, dy$$

$$= \int_\Omega g_{n_n}(x-y)\, \Delta u(y)\, dy$$

$$= (g_{n_n} * \Delta u)(x) \quad \forall x \in \overline{B(x_0, \tfrac{1}{2n_0})} \subseteq \Omega.$$

Wie in der Vorlesung folgt:

$$\Delta u_{n_n}\big|_{\overline{B(x_0,\frac{1}{2n_0})}} \longrightarrow \Delta u\big|_{\overline{B(x_0,\frac{1}{2n_0})}} \quad \text{gleichmäßig} \quad \Big(\Delta u\big|_{\overline{B(x_0,\frac{1}{2n_0})}} \text{ ist stetig!}\Big)$$

Folglich:

$$|\Delta u_{n_n}(x_{n_n}) - \Delta u(x_0)|$$

$$\leq |\Delta u_{n_n}(x_{n_n}) - \Delta u(x_{n_n})| + |\Delta u(x_{n_n}) - \Delta u(x_0)|$$

$$\overset{n \text{ groß}}{\leq} \underbrace{\Big\| \Delta u_{n_n}\big|_{\overline{B(x_0,\frac{1}{2n_0})}} - \Delta u\big|_{\overline{B(x_0,\frac{1}{2n_0})}} \Big\| + |\Delta u(x_{n_n}) - \Delta u(x_0)|}_{\xrightarrow[n\to\infty]{} 0}.$$

Also: $\Delta u(x_0) = \lim_{n\to\infty} \Delta u_{n_n}(x_{n_n}) \leq 0$. $\square$

10 (11.8) **Definition** ~~Rem.~~: $T \in \mathcal{L}(C_0(\Omega))$ **submarkovian** $:\Leftrightarrow$

$$f \leq 1 \Rightarrow Tf \leq 1$$

Consequence: a) $T \geq 0$, i.e. $f \geq 0 \Rightarrow Tf \geq 0$

b) T^k submarkovian $\forall k \in \mathbb{N}$

Proof. a) $f \leq 0 \Rightarrow f \leq \lambda 1 \quad \forall \lambda > 0$

$\Rightarrow \frac{f}{\lambda} \leq 1 \quad \forall \lambda > 0 \Rightarrow \frac{1}{\lambda} Tf \leq 1 \quad \forall \lambda > 0$

$\Rightarrow Tf \leq 0.$

$f \geq 0 \Rightarrow -f \leq 0 \Rightarrow -Tf \leq 0 \Rightarrow Tf \geq 0.$ □

(11.9) **Proposition.** Let T be a C_0-semigroup on $C_0(\Omega)$ with generator A.

Equivalent:

(i) T is submarkovian

(ii) $\exists \lambda_0 > \omega \quad \lambda R(\lambda, A)$ submarkovian $\forall \lambda \geq \lambda_0$

(i) $\Rightarrow$ (ii) $\quad \|T(t)\| \leq M e^{\omega t}$

$$\lambda > \omega \qquad \lambda R(\lambda, A) f = \int_0^\infty \lambda e^{-\lambda t} T(t) f \, dt$$

$$f \leq 1 \Rightarrow T(t) f \leq 1 \Rightarrow$$

$$\lambda R(\lambda, A) f \leq \int_0^\infty \lambda e^{-\lambda t} dt = 1.$$

(ii) $\Rightarrow$ (i) $\quad T(t) f = \lim e^{tA_n} f$

e^{tA_n} submarkovian?

$$A_n = n\,(n R(n, A) - I)$$

$$f \leq 1 \Rightarrow (n R(n, A))^k \leq 1$$

$$\Rightarrow e^{tA_n} = e^{-nt} \sum \frac{t^k n^k}{k!} (n R(n, A))^k f$$

$$\leq 1. \quad \square$$

Conclusion.

(11.10) <u>Theorem</u>. The semigroup T generated by Δ_0 is submarkovian.

§ 12 Perturbation

(12.1) Example. Let $B \in \mathcal{L}(X)$. Then

$B - \|B\| I$ is diss.

Proof. $\|e^{tA}\| = e^{-t\|B\|} \|e^{tB}\|$
$= e^{-t\|B\|} \|\sum_{k=0}^{\infty} \frac{t^k B^k}{k!}\| \leq 1.$ □

(12.2) Proposition. A m-diss., $B \in \mathcal{L}(X)$ diss

$\Rightarrow$ $A + B$ m-diss.

If $\overline{D(A)} = X$, then $A+B$ gen. a contractive C_0-sg.

Pf. a) Let $x \in D(A)$. $\exists x' \in J(x)$

$\mathrm{Re} \langle Ax, x' \rangle \leq 0$. (7.10) $\Rightarrow$

B is strictly m-diss $\Rightarrow$

$\mathrm{Re} \langle Bx, x' \rangle \leq 0$ $\Rightarrow$

$\mathrm{Re} \langle Ax + Bx, x' \rangle \leq 0$.

Thus $A + B$ diss.

b) Let $\lambda > 0$, $\lambda > \|B\|$

$$\lambda - A - B = (I - BR(\lambda, A))(\lambda - A).$$

$$\Rightarrow \quad \|BR(\lambda, A)\| \leq \frac{\|B\|}{\lambda} < 1$$

$\Rightarrow$ $\lambda - A - B$ surj. $\square$

(12.3) __Theorem.__ A generator of a C_0-sg T, $B \in \mathcal{L}(X) \Rightarrow A + B$ generator of a C_0-sg.

__Proof.__ a) First case: $\|T(t)\| \leq M$.

Then ~~not~~ $\|T(t)\|_0 \leq 1$ for equ. norm.

(12.2) $\Rightarrow$ $A + B - \|B\|_0$ generates a contraction s.g. $\Rightarrow$ $A + B$ gen. a sg. S

$\|S(t)\|_0 \leq e^{\|B\|_0 t}$

Second case $\quad \|T(t)\| \le M e^{\omega t}$

1st case $\Rightarrow$ $A - \omega + B$ generates a

C_0 - sg $\Rightarrow$ $A + B$ generator. □

(12.4) Exercise: A generator of T on X

$\mathcal{U} : X \to Y$ isomorphism.

$\Rightarrow$ $S(t) = \mathcal{U}\, T(t)\, \mathcal{U}^{-1}$ $\quad C_0$-sg

Generator: $\quad D(B) = \{ y \in Y : \mathcal{U}^{-1} y \in D(A) \}$

$\qquad By = \mathcal{U} A \mathcal{U}^{-1} y$.

(12.5) Exercise. A generator of T on X.

$D(A)$ with graph norm

Then $T_1(t) = T(t)_{|D(A)}$ is a C_0-sg.

Its generator is A_1 given by

$$D(A_1) = D(A^2)$$

$$A_1 x = Ax .$$

(12.6) Exercise. (converse of 12.5).

Let A be an operator on X with $\varrho(A) \neq \emptyset$.
If A_1 generates a C_0-sg on $D(A)$, then A generates a C_0-sg on X

(12.7) Theorem. Let A be the gen. of a C_0-sg on X, $B \in \mathcal{L}(D(A))$. Then $A+B$ generates a C_0-sg on X

Proof. $A_1 + B$ generates a C_0-sg on $D(A)$.
~~It suffices~~ Obviously, $A_1 + B = (A+B)_1$
It suffices to show that $\varrho((A+B)_1) \neq \emptyset$
$\exists\, \lambda_0$ (ω, M) s.t. $\|R(\lambda, A_1)\| \leq M \frac{1}{\lambda - \omega}$. $(\lambda > \omega)$
Choose $\lambda_0 > \omega$ s.t.

$$\|R(\lambda_0, A_1)\|_{\mathcal{L}(D(A))} \, \|B\|_{\mathcal{L}(D(A))} < 1$$

Then $(I - R(\lambda_0, A_\varepsilon)B_\varepsilon) : D(A) \to D(A)$

is ~~bijective~~ invertible. $\Rightarrow$

$(\lambda_0 - A - B) = (\lambda_0 - A)(I - R(\lambda_0, A_\varepsilon)B)$:

$D(A) \to X$ is bijective. and

$(\lambda_0 - A - B) = (I - R(\lambda_0, A_\varepsilon)B)^{-1} R(\lambda_0, A)$

$\in \mathcal{L}(X, D(A)) \subset \mathcal{L}(X)$. Thus $\lambda_0 \in \rho(A)$.

$\square$

(12.8) <u>Definition.</u> Let A be a closed operator. $D \subset D(A)$ <u>core</u> $:\Leftrightarrow$

$\overline{A_{|D}} = A$

$\Leftrightarrow$ D is dense in $(D(A), \| \, \|_A)$.

(12.9) <u>Proposition</u> <u>(uniqueness)</u>

Let A be generator of T, B of S, $D \subset D(A)$ a core of A.

$A_{|D} \subset B \Rightarrow A = B$ & $S(t) = T(t)$

<u>Proof.</u> $A_{|D} \subset B$ & $\Rightarrow A = \overline{A_{|D}} \subset B$.

Let $x \in D(A)$, $u(t) = T(t)x$ solves

$\dot{u}(t) = Au(t) = Bu(t)$, $u(0) = x_0$ $\Rightarrow$ $u(t) = S(t)x$.

Thus $T(t)x = S(t)x$ $\forall x \in D(A)$

$\overline{D(A)} = X \Rightarrow T(t) = S(t)$.

(12.10) <u>Theorem.</u> T sg with generator A.

$D_0 \subset D(A)$ not a core. $\Rightarrow$

$\exists \infty$ many generators extending

$A_0 = A|_{D_0}$

<u>Pf.</u> ~~Let~~ HB $\exists \varphi \in D_0^{\perp} D(A)'$

$\varphi \neq 0$, $\varphi|_{D_0} = 0$ $\quad Bx = \langle \varphi, x \rangle u$

$u \in D(A)$ $\quad A + B$ Generator

$(A+B)|_{D_0} = A_0$ $\quad \square$

(12.11) <u>Proposition.</u> A Generator of T,

$D_0 \subset D(A)$, $\quad \overline{D_0}^X = X$

$T(t) D_0 \subset D_0$

$\Rightarrow$ D_0 is a core.

<u>Proof.</u> Let ~~D_0~~ $A_0 \subset B$ generator of S

~~with~~ Let $x_0 \in D_0$, $\quad u(t) = \cancel{S}T(t)x$ sol.

of $\quad \dot{u}(t) = Bu(t)$ $\quad \Rightarrow u(t) = S(t)x_0$

$u(0) = \cancel{x}x_0$ $\quad \Rightarrow S(t) = T(t)$ on D_0

$\Rightarrow S \equiv T$. $\square$

24.5.2017

§ 13 Selfadjoint operators

H Hilbert

(13.1) Remark

Let $S \in \mathcal{L}(H)$

a) $\exists!\ S^* \in \mathcal{L}(H)$ s.t.

$(Sx \mid y) = (x \mid S^* y) \quad \forall x, y \in H$

b) S s.a. $:\Leftrightarrow S = S^*$

$\Leftrightarrow (Sx \mid y) = (x \mid Sy)$

$\forall x, y \in H$. $:\Leftrightarrow S$ symmetric

(13.2) Definition. A operator on H, $\overline{D(A)} = H$

a) $D(A^*) = \{ y \in H : \exists z \in H$

$(Ax \mid y) = (x \mid z) \quad \forall x \in D(A) \}$

$A^* y = z$.

Thus $(Ax \mid y) = (x \mid A^* y) \quad \forall x \in D(A), y \in D(A^*)$

A^* is the adjoint of A.

b) A is <u>symmetric</u> if

$$(Ax|y) = (x|Ay) \quad \forall x,y \in D(A)$$

~~Remark. a) A generator of T ⟹ A* generator of T*(t)~~
~~b) A = A symmetric ⟺ T(t) = T(t)*~~

(13.3) <u>Lemma.</u> A dd. Then

(i) A sym.

$\Updownarrow$

(ii) $A \subset A^*$

<u>Proof.</u> (i) ⟹ (ii) $y \in D(A) \Rightarrow$

$(Ax|y) = (x|Ay) \quad \forall x \in D(A)$

$\Rightarrow y \in D(A^*)$ & $A^*y = Ay$.

(ii) ⟹ (i) Let $y \in D(A)$. Then $y \in D(A^*)$

& $A^*y = Ay \Rightarrow \quad \forall x \in D(A)$

$(Ax|y) = (x|A^*y) = (x|Ay)$. □

(13.4) <u>Lemma.</u> Let A be dd. Then A^* is closed.

<u>Pf.</u> $y_n \in D(A^*)$ $y_n \to y$, $A^*y_n \to z$

$$x \in D(A) \qquad \begin{array}{c}(Ax|y_n) = (x|A^*y_n) \to (x|z) \\ \downarrow \\ (Ax|y)\end{array}$$

$\Rightarrow y \in D(A^*)$ $A^*y = z$. □

(13.5) <u>Corollary.</u> A dd sym. $\Rightarrow$

A closable

Pf. $A \subset A^*$, A^* closed. $\square$

(13.6) <u>Polarization identity.</u> V ~~H. unitary~~

~~A~~ vector space, $\mathbb{K} = \mathbb{C}$.

$a : V \times V \to \mathbb{C}$ sesquilinear; i.e.

$a(\cdot, y) : V \to \mathbb{C}$ linear,

$a(x, \cdot) : V \to \mathbb{C}$ antilinear.

Here $\varphi : V \to \mathbb{C}$ <u>antilinear</u> $:\Leftrightarrow$ $\bar{\varphi}$ linear.

$$a(x) := a(x,x)$$

$$a(x,y) = \frac{1}{4}\left\{ a(x+y) - a(x-y) + i\,a(x+iy) - i\,a(x-iy) \right\}$$

<u>Corollary.</u> a symmetric $:\Leftrightarrow$

$a(x,y) = \overline{a(y,x)}$ $\Leftrightarrow$

$a(x) \in \mathbb{R}$ $\forall x \in V$.

~~(13.7) Corollary. A operator on H, $\mathbb{K} = \mathbb{C}$~~

~~A symmetric $\Leftrightarrow$ $(Ax \mid x) \in \mathbb{R}$ $\forall x \in D(A)$.~~

<u>Definition</u>. An operator A is symmetric if

$$(Ax \mid y) = (x \mid Ay) \quad \forall x, y \in D(A).$$

$\mathbb{K} = \mathbb{C}$

(13.7) <u>Corollary</u>. Equivalent

(i) A symmetric;

(ii) $(Ax \mid x) \in \mathbb{R} \quad \forall x \in D(A)$;

(iii) $\pm iA$ dissipative.

(13.8) <u>Proposition</u>. Let T be a C_0-sg with ~~Equ.~~ generator A.

a) $(T(t)^*)_{t \geq 0}$ is a sg and A^* its generator;

b) Equivalent:

(i) $T(t) = T(t)^* \quad \forall t \geq 0$

(ii) $A = A^*$

(iii) A symmetric

Pf. b) (iii) $\Rightarrow$ (ii)

A sym. $\to$ $A \subset A^*$

$\Rightarrow$ $A = A^*$ since both are generators.

(13.9) Definition. unitary group :=

C_0-group of unitary operators

$(\mathcal{U}(t))_{t \in \mathbb{R}}$

Consequence: 1. $\mathcal{U}(t)^* = \mathcal{U}(-t)$.

2. B the generator $\Rightarrow$ $-B = B^*$.

(13.10) Theorem. Let A be an operator on H.

Equ: (i) iA generates a unitary group

(ii) a) A symmetric

b) $\pm i - A$ surj. (range condition)

(iii) A dd & $A = A^*$

$\Leftrightarrow$: (A is selfadjoint)

(iv) $\pm iA$ is m-dissipative.

<u>Proof.</u> (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (i') clear.

(ii) $\Rightarrow$ (iii) 1. $A \subset A^*$

2. $i - A^*$ injective:

$ix - A^*x = 0 \quad \Rightarrow$

$(iy + Ay \mid x) = (y \mid -ix + A^*x) = 0$

$\forall y \in D(A)$. But $R(i + A) = H \quad \Rightarrow x = 0$

3. $\pm iA$ is m-diss. $\Rightarrow$ dd

4. Let $x \in D(A^*)$ $\exists y \in D(A)$

$iy - Ay = ix - A^*x$, $u := x - y$

$(i - A^*)u = 0 \Rightarrow u = 0 \Rightarrow x = y \in D(A)$.

(iii) $\Rightarrow$ (ii) $A = A^* \Rightarrow A$ closed & symmetric

$\Rightarrow \pm iA$ diss

$\Rightarrow \|\pm ix - Ax\| \geq \|x\|$ (1)

$\Rightarrow R(\pm i - A)$ closed.

Suppose $R(\pm i - A) \neq H$

$\Rightarrow \exists z \neq 0 \quad (\pm ix - Ax \mid z) = 0$

$\forall x \in D(A)$

$\Rightarrow \mp iz - Az = 0$

$\overset{(1)}{\Rightarrow} z = 0$. $\square$

$\boxed{\text{Ü}A}$ A s.a. $\Longleftrightarrow$

A dd, symmetric, closed and

$\pm i - A^*$ injective

Equi:

(13.11) <u>Corollary</u>. A dd-symmetric

(i) $\bar{A}$ is sa

(ii) $(\pm i - \cancel{\bar{}} A) D(A)$ is dense

<u>Pf.</u> $\boxed{\text{Ü}A}$.

<u>Rk.</u> A is <u>essentially s.a.</u> $:\Longleftrightarrow$ A dd, symmetric & $\bar{A}$ s.a.

(13.12) <u>Example</u> (multiplication operator)

(Ω, Σ, μ) measure space, $H = L^2(\Omega)$

$m : \Omega \to \mathbb{R}$ measurable

$A_m f = m f$

$D(A_m) = \{ f \in L^2 : m f \in L^2 \}$

Then A_m is selfadjoint.

$U(t) f = e^{itm} f$

defines a unitary C_0 group.

Its generator is $i A_m$.

(13.13) <u>Spectral Theorem.</u>

Let A be selfadjoint on H

Then $\exists\ \Phi : H \to L^2(\Omega, \Sigma, \mu)$

where (Ω, Σ, μ) is a measure space

& $\exists\ m : \Omega \to \mathbb{R}$ measurable

such that

$$\begin{array}{ccc} D(A) & \xrightarrow{A} & H \\ \downarrow & & \downarrow \Phi \\ D(A_m) & \xrightarrow{A_m} & L^2(\Omega, \Sigma, \mu) \end{array}$$

$$D(A_m) = \Phi D(A) \qquad \Phi^{-1} A_m \Phi = A .$$

Let $\mathcal{U}$ be the unitary group generated by iA. Then

$$\Phi\, \mathcal{U}(t) \Phi^{-1} g = e^{itm} g$$

$$\begin{array}{ccc} H & \xrightarrow{\mathcal{U}(t)} & H \\ \Phi \downarrow & & \downarrow \Phi \\ L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ g & \longmapsto & e^{itm} g . \end{array}$$

①

§ 14 Hille's proof

(14.1) Proof of HY

Let A be d.d., $(0,\infty) \subseteq \varrho(A)$, $\|\lambda R(\lambda, A)\| \leq 1 \; \forall \lambda > 0$.

Define

$$V_n(t) := \left(I - \tfrac{t}{n} A\right)^{-n} = \left(\tfrac{n}{t}\right)^n R\left(\tfrac{n}{t}, A\right)^n \quad \forall t > 0 \; \forall n \in \mathbb{N}$$

and $V_n(0) := I \quad \forall n \in \mathbb{N}$.

Then $\|V_n(t)\| \leq 1 \quad \forall n \in \mathbb{N}, t \geq 0$.

Moreover: $\forall n \in \mathbb{N} \; \forall x \in X$: $V_n(t)x = \left(\tfrac{n}{t}\right)^n R\left(\tfrac{n}{t}, A\right)^n x \xrightarrow{t \downarrow 0} x$ (*)

since $\lambda R(\lambda, A) x \xrightarrow{\lambda \to \infty} x$ by (3.6).

By (3.3) (Neumann series representation of $R(\lambda, A)$)

$$(0,\infty) \longrightarrow \mathcal{L}(X), \; \lambda \longmapsto R(\lambda, A)^n$$

is differentiable with

$$\frac{d}{d\lambda} R(\lambda, A)^n = (-1)\, n \cdot R(\lambda, A)^{n+1}$$

$$\Rightarrow \frac{d}{d\lambda} \lambda^n R(\lambda, A)^n = n \lambda^{n-1} R(\lambda, A)^n - n \cdot \lambda^n \cdot R(\lambda, A)^{n+1}$$
$$= ((\lambda - A) - \lambda)\, n \cdot \lambda^{n-1} R(\lambda, A)^{n+1}$$
$$= -A n \lambda^{n-1} R(\lambda, A)^{n+1} \quad \forall n \in \mathbb{N}, \lambda > 0$$

$$\left.\begin{aligned} &\Rightarrow V_n \text{ is differentiable on } (0,\infty) \text{ with} \\ &\frac{d}{dt} V_n(t) = -A\, n \cdot \left(\tfrac{n}{t}\right)^{n-1} R\left(\tfrac{n}{t}, A\right)^{n+1} \cdot \left(-\tfrac{n}{t^2}\right) \\ &\qquad = A \left(\tfrac{n}{t}\right)^{n+1} R\left(\tfrac{n}{t}, A\right)^{n+1} \end{aligned}\right\} (**)$$

(*) + (**) $\Rightarrow$ V_n is strongly continuous $\forall n \in \mathbb{N}$.

Moreover:

$$V_n(t)x - V_m(t)x = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \frac{d}{ds} V_m(t-s) V_n(s) x \, ds$$

$$= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \underbrace{\left[-V_m'(t-s) V_n(s) x + V_m(t-s) V_n'(s) x \right]}_{(***)} ds$$

with

$$(***) = -A\left(I - \tfrac{t-s}{m} A\right)^{-m-1} \left(I - \tfrac{s}{n} A\right)^{-n} x$$
$$+ \left(I - \tfrac{t-s}{m} A\right)^{-m} A \left(I - \tfrac{s}{n} A\right)^{-n-1} x$$

$$= (I - \tfrac{t-s}{m} A)^{-m-1} [(I - \tfrac{s}{n} A) - (I - \tfrac{t-s}{m} A)] (I - \tfrac{s}{n} A)^{-n-1} Ax$$

$$= (\tfrac{t-s}{m} - \tfrac{s}{n}) (I - \tfrac{t-s}{m} A)^{-m-1} (I - \tfrac{s}{n} A)^{-n-1} A^2 x$$

$\forall t > 0,\ x \in D(A^2),\ n, m \in \mathbb{N}$.

2

$$\Rightarrow \quad \| V_n(t)x - V_m(t)x \| \le \int_0^t \frac{s}{n} + \frac{t-s}{m} \, ds \cdot \|A^2 x\|$$

$$= \frac{t^2}{2} \left(\frac{1}{n} + \frac{1}{m}\right) \|A^2 x\| \le \frac{T^2}{2} \left(\frac{1}{n} + \frac{1}{m}\right) \|A^2 x\|$$

$\forall t \in (0, T],\ x \in D(A^2),\ n, m \in \mathbb{N},\ T > 0$.

$D(A^2)$ dicht in X

$\Rightarrow$ $(V_n(\cdot)x)_{n \in \mathbb{N}}$ is Cauchy sequence in $C([0,T], X)$ $\forall x \in X$ $\forall T > 0$.

$\Rightarrow \exists\ T(t)x := \lim_{n \to \infty} V_n(t)x \quad \forall t \ge 0,\ \forall x \in X$ and the limit is uniform on $[0,T]$ $\forall T > 0$.

In particular: $t \mapsto T(t)x$ is continuous $\forall x \in X$. and $\|T(t)\| \le 1\ \forall t \ge 0$

Moreover:

(a) $T(0) = I$

$$\left(\tfrac{n}{t}\right)^n R(\tfrac{n}{t}, A)^n A x = A \left(\tfrac{n}{t}\right)^n R(\tfrac{n}{t}, A)^n x \qquad \forall x \in D(A)$$

$$\downarrow_{n \to \infty}$$

$$T(t)Ax$$

$\overset{A \text{ closed}}{\Rightarrow}$ (b) $T(t)x \in D(A)$ and $T(t)Ax = AT(t)x \quad \forall t \ge 0$

$$(**) \Rightarrow \quad V_n(t)x - x = \int_0^t \left(\tfrac{n}{s}\right)^{n+1} R(\tfrac{n}{s}, A)^{n+1} A x \, ds$$

$$= \int_0^t \left(\tfrac{n}{s}\right) R(\tfrac{n}{s}, A) V_n(s) A x \, ds$$

$$\downarrow_{n \to \infty}$$

$$T(t)x - x = \int_0^t T(s) A x \, ds \qquad \forall x \in D(A),\ t > 0.$$

$$\Rightarrow (c) \quad \frac{d}{dt} T(t)x = \cancel{\ldots}\ AT(t)x \quad \forall t > 0,\ x \in D(A)$$

3

(14.2) Lemma

Let A be d.d. + closed and

$T: [0,\infty) \to \mathcal{L}(X)$ strongly cont. with $\|T(t)\| \leq 1$ $\forall t \geq 0$

satisfying (a), (b), (c).

$\Rightarrow$ T is contractive C_0-sgr. with generator A.

Proof:

Given $x \in D(A)$ $u(t) := T(t)x$ $\forall t \geq 0$ is the unique solution of

$$(CP) \begin{cases} u \in C^1([0,\infty), X), u(t) \in D(A) \ \forall t \geq 0 \\ \dot{u}(t) = Au(t) \quad \forall t > 0 \\ u(0) = x. \end{cases}$$

(uniqueness is proved as in (2.3)).

Now let $s > 0$ and $v(t) := T(t+s)x$ for $x \in D(A)$.

$\Rightarrow$ v is solution of (CP) with $v(0) = T(s)x$.

$\Rightarrow$ $v(t) = T(t)T(s)x$.

$\xRightarrow{D(A) \text{ dense}}$ $T(t)T(s) = T(t+s)$ $\forall t,s \geq 0$.

Let B the generator of T.

$\xRightarrow{(c)}$ $A \subseteq B$

Since $D(A)$ is dense and invariant, it is a core for B [see (12.11)].

$\Rightarrow$ $\overline{D(A)}^{\|\cdot\|_B} = D(B)$ $\xRightarrow{A \text{ closed}}$ $A = B$. $\square$

(14.3) Corollary (of (14.1))

Let A be the generator of a contractive C_0-sgr. T.

$\Rightarrow$ $T(t)x = \lim\limits_{n\to\infty} \frac{n}{t} R(\frac{n}{t}, A)x$

uniformly on $[0,J]$ $\forall x \in X$ $\forall J > 0$.

4

§ 15 Numerical range

Let H be a complex Hilbert space.

(15.1) Def.

Let A be an operator on H.

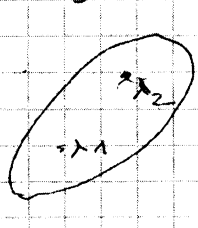
$W(A) := \{(Ax|x) : x \in D(A), \|x\| = 1\} \subseteq \mathbb{C}$

is the *numerical range of A*.

(15.2) Example

$H = \mathbb{C}^2$, $A \in \mathcal{L}(H)$, $\sigma(A) = \{\lambda_1, \lambda_2\}$

$\Rightarrow$ $W(A)$ is a (possibly degenerate) elliptical disc with foci λ_1, λ_2.



Pf: Exercise (maybe).

(15.3) Proposition

$W(A)$ is convex.

Proof: Let $w_1, w_2 \in W(A)$, $w_i = (Ax_i|x_i)$, $i = 1,2$.

Set $H_1 := \operatorname{span}\{x_1, x_2\} \cong \mathbb{C}^2$ and let $P : H \longrightarrow H$ the orthogonal proj. onto H_1.

Then $B := PA|_{H_1} \in \mathcal{L}(H_1)$ and

$w_i = (Ax_i|x_i) = (Ax_i|Px_i) = (Bx_i|x_i)$, $i = 1,2$,

$\overset{(15.2)}{\Rightarrow}$ $\lambda w_1 + (1-\lambda) w_2 \in W(B)$ $\forall \lambda > 0$

Let $\lambda \in (0,1)$ and $y \in H_1$ with $(By|y) = \lambda w_1 + (1-\lambda) w_2$, $\|y\| = 1 = \|Py\|$.

$\Rightarrow$ $(By|y) = (PAy|y) = (Ay|Py) = (APy|Py)$

$\Rightarrow$ $\lambda w_1 + (1-\lambda) w_2 \in W(A)$. $\square$

~~[illegible]~~

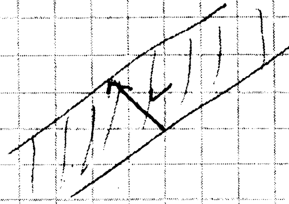
(15.4) Proposition

Let $\Gamma \subseteq \mathbb{C}$ be closed + convex. Then

(a) $\mathbb{C} \setminus \Gamma$ is connected or

(b) Γ is a closed strip, i.e. $\exists L \subseteq \mathbb{C}$ line, $c \geq 0$:

(5) $\Gamma = \bigcup_{t \in [0,c]} (L + t \cdot v)$



with v normed vector orthogonal to L.

($c = 0$: Γ = line)

Proof:

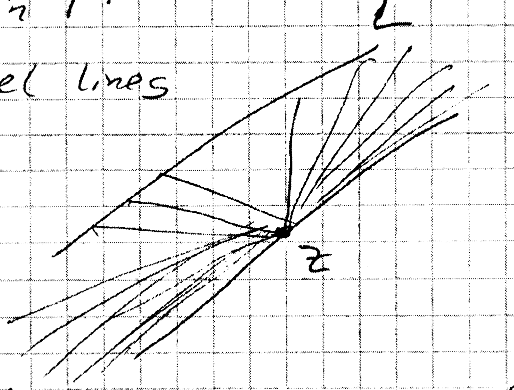
Suppose $\mathbb{C} \setminus \Gamma$ is not connected.

1) Assume $\exists$ line $L \subseteq \Gamma$.

$\Rightarrow$ (Γ closed) $\forall z \in \Gamma$: the line parallel to L through z also is contained in Γ

$\Rightarrow$ Γ is union of parallel lines

$\Rightarrow$ (Γ convex) Γ is a strip.



2) Suppose that no line is contained in Γ.

First case: Γ is unbounded.

Choose $x_0 \in \Gamma$

$\Rightarrow \forall n \in \mathbb{N}: \exists x_n \in \Gamma: |x_n - x_0| \geq n$

Set $v_n := \frac{x_n - x_0}{|x_n - x_0|} \quad \forall n \in \mathbb{N}$

$|v_n| = 1 \ \forall n \in \mathbb{N}$ $\Rightarrow$ $\exists$ subsequence $(v_{n_k})_{k \in \mathbb{N}}$ with

$$\lim_{k \to \infty} v_{n_k} = v \in \mathbb{C}, \quad |v| = 1.$$

For each $\tilde{n} \in \mathbb{N}$ we have

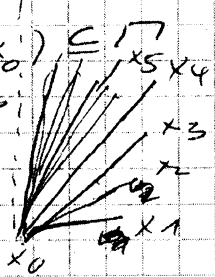
$$\lim_{k \to \infty} x_0 + n_{\tilde{n}} v_{n_k} = x_0 + n_{\tilde{n}} v$$

Since $x_0 + n_{\tilde{n}} v_{n_k} \in x_0 + [0,1] \cdot (x_{n_k} - x_0) \subseteq \Gamma$

we obtain: $x_0 + n_{\tilde{n}} v \in \Gamma \quad \forall \tilde{n} \in \mathbb{N}$

(since Γ is closed)

$\Rightarrow$ Γ contains a halfline.



6

Define

$L(x) := \{y \in \mathbb{R} : x + iy \in \Gamma\}$

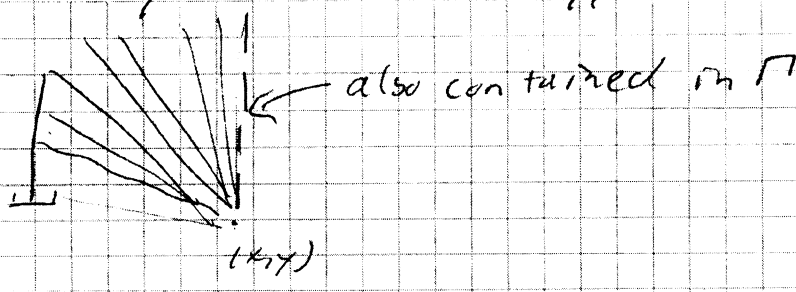
By rotating we may assume

$L(x_0) = [b, \infty)$

for some $b \in \mathbb{R}$.

~~⇒ ... := {x ∈ ℝ ∃ y ∈ ℝ : x + iy ∈ Γ}~~

$\Rightarrow \quad \forall\, x + iy \in \Gamma : \quad x + i[y, \infty) \in \Gamma$



Set $Q := \{x \in \mathbb{R} : \exists y \in \mathbb{R} : x + iy \in \Gamma\}$

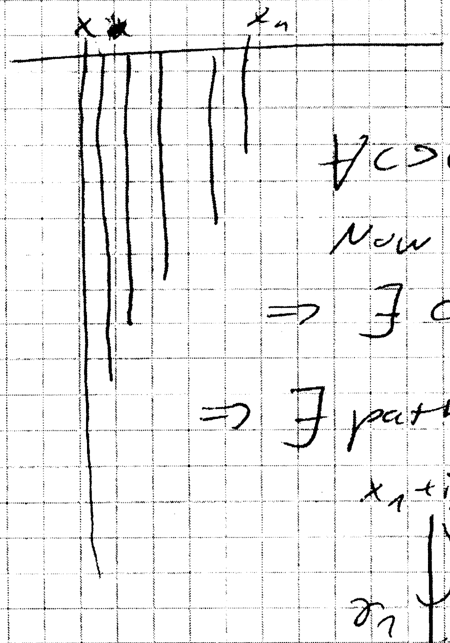
and

$$\beta(x) := \begin{cases} \min\{y \in \mathbb{R} : x + iy \in \Gamma\}, & x \in Q \\ \infty, & x \notin \end{cases}$$

Then

$$\Gamma = \bigcup_{x \in Q} (x + i[\beta(x), \infty)).$$

Suppose $\exists x_n \in Q$ with $x_n \to x$, $\beta(x_n) \to -\infty$.



$\Rightarrow x + i\mathbb{R} \subseteq \Gamma$ ↯

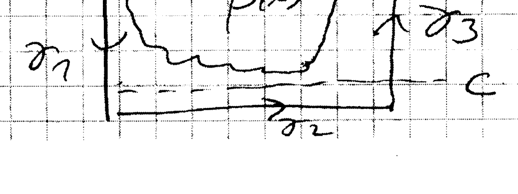
Consequently:

$\forall c > 0 : \inf\{\beta(x) : x \in [-c, c]\} > -\infty$

Now let $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C} \setminus \Gamma$.

$\Rightarrow \exists\, c \in \mathbb{R} : \beta(x) \geq c \quad \forall x \in [x_1, x_2]$

$\Rightarrow \exists$ path connecting $x_1 + iy_1$ and $x_2 + iy_2$:

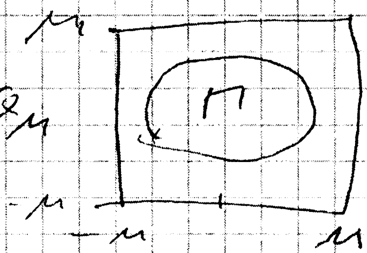


$\Rightarrow \mathbb{C} \setminus \Gamma$ connected ↯.

7

Second case: Γ is bounded

Choose $M > 0$ with



$\Gamma \subseteq [-M, M] + i[-M, M] =: Q_M$

Clearly $\mathbb{C} \setminus Q_M$ is connected.

Take $x \in \mathbb{C} \setminus \Gamma$ and consider the line

$$L := x + i\mathbb{R}.$$

If there are $t_1, t_2 > 0$ with

$x + it_1 \in \Gamma$

and $x - it_2 \in \Gamma$,

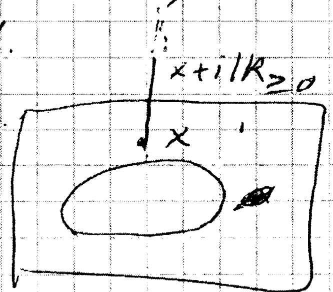
then also $x \in \Gamma$, a contradiction.

Consequently $x + i\mathbb{R}_{\geq 0} \subseteq \mathbb{C} \setminus \Gamma$ or

$x - i\mathbb{R}_{\geq 0} \subseteq \mathbb{C} \setminus \Gamma$.

$\Rightarrow$ There is a line segment ~~connecting~~ from x to $\mathbb{C} \setminus Q_M$.

$\Rightarrow \mathbb{C} \setminus \Gamma$ is connected. $\square$



(15.5) **Lemma**

$\|\lambda x - Ax\| \geq \operatorname{dist}(\overline{W(A)}, \lambda) \cdot \|x\| \quad \forall x \in D(A),\ \lambda \in \mathbb{C}$

Proof

Let $x \in D(A) \setminus \{0\},\ \lambda \in \mathbb{C}$.

$$\Rightarrow \operatorname{dist}(\overline{W(A)}, \lambda) \cdot \|x\|^2 \leq \left(\lambda - \left(A\tfrac{x}{\|x\|} \middle| \tfrac{x}{\|x\|}\right)\right)\|x\|^2$$
$$= (\lambda x | x) - (Ax | x)$$
$$\leq \|\lambda x - Ax\| \cdot \|x\| \quad \square$$

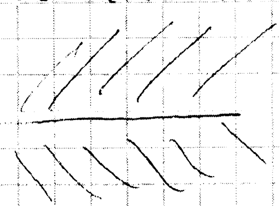
(15.6) **Proposition**

Let D be a connected component of $\mathbb{C} \setminus \overline{W(A)}$. If $\exists \lambda_0 \in D : \lambda_0 - A$ surjective then $D \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{W(A)})} \quad \forall \lambda \in D.$$

Pf.: Apply (15.4) to $\Gamma = \overline{W(A)}$ and use (15.5).

(15.7) **Corollary**



Let A be symmetric.

1. Then a) $\sigma(A) \subseteq \mathbb{R}$ or

b) $\sigma(A) \subseteq \{z \in \mathbb{C}\ \operatorname{Im} z \geq 0\}$ or

c) $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$ or

d) $\sigma(A) = \mathbb{C}$.

2. A is s.a. $\Leftarrow$ $\sigma(A) \subseteq \mathbb{R}$.

29.05.17 **Proof**: A symmetric $\Rightarrow \overline{W(A)} \subseteq \mathbb{R}$.

① $\overline{W(A)} = \mathbb{R}$.

If $\exists \lambda_0 \in \mathbb{C} : \operatorname{Im}\lambda_0 > 0 : \lambda_0 - A$ surj $\Rightarrow$ c)

" " $\operatorname{Im}\lambda_0 < 0$ " $\Rightarrow$ b)

If both of the above $\Rightarrow$ a)

② $\overline{W(A)} \subsetneq \mathbb{R}$.

If $\exists \lambda \in \mathbb{C}\setminus\overline{W(A)} : \lambda - A$ surj $\Rightarrow$ a)

Else : $\mathbb{C} \setminus \mathbb{R} \subseteq \sigma(A) \xRightarrow{\sigma(A)\text{ closed}} \sigma(A) = \mathbb{C} \Rightarrow$ d)

For 2. apply (13.10).

Remark A s.a. $\Rightarrow \overline{\operatorname{conv}(A)} = \overline{W(A)}$. 9

(15.8) **Corollary**

$A \in \mathcal{L}(H) \Rightarrow \sigma(A) \subseteq \overline{W(A)}$

Proof: $\overline{W(A)}$ is bounded

$\Rightarrow \overline{W(A)}$ cannot contain a line

$\Rightarrow \mathbb{C} \setminus \overline{W(A)}$ connected.

Let $|\lambda| > \|A\|$. $\Rightarrow \lambda \in \varrho(A)$

$\underset{(15.6)}{\Rightarrow} \mathbb{C} \setminus \overline{W(A)} \subseteq \varrho(A)$

$\Rightarrow \sigma(A) \subseteq \overline{W(A)}$. $\square$

(15.9) **Proposition**

~~Let $\overline{W(A)} \neq H$~~

If A is d.d. and $\overline{W(A)} \neq \mathbb{C}$,

then A is closable (and $\overline{W(\bar{A})} = \overline{W(A)}$).

proof:

Take $\lambda \in \mathbb{C} \setminus \overline{W(A)}$. By the Hahn-Banach sep. theorem for convex sets (applied to the Bspace $\mathbb{C}$) we find a halfplane which contains $\overline{W(A)}$ but not λ.

Replacing A bei $c + e^{i\theta} A$ we may assume that

$$\overline{W(A)} \subseteq \{(x+iy) : x \leq 0\}$$

$\Rightarrow \operatorname{Re}(Ax|x) \leq 0 \quad \forall x \in H$.

$\Rightarrow A$ is dissipative

$\Rightarrow A$ is closable $\square$

2.6.2017

Chapter 2 Holomorphic Semigroups

§16 Holomorphic functions.

$\Omega \subset \mathbb{C}$ open, X complex Banach space.

(16.1) Definition: $f : \Omega \to X$

holomorphic $:\Leftrightarrow$

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in \Omega$.

Consequence: a) $x' \circ f : \Omega \to \mathbb{C}$ hol.

$\forall x' \in X'$ & $(x' \circ f)' = x' \circ f'$.

b) $\sup\limits_{z \in K} \|f(z)\| < \infty \quad \forall K \subset \Omega$

compact.

(16.2) <u>Theorem</u> Let $f : \Omega \to X$ be [locally] bounded. Let $\Phi \subset X'$ be separating. If

$$\varphi \circ f : \Omega \to \mathbb{C}$$

is hol. $\forall \varphi \in \Phi$, then f is holomorphic

<u>Pf.</u> Green Book Appendix

(16.3) <u>Corollary.</u> Let $T : \Omega \to \mathcal{L}(X, Y)$ such that for all $x \in X$, $y' \in Y'$ $\langle T(\cdot)x, y' \rangle$ is holomorph. Then T is holomorphic

<u>Pf.</u> $\Phi = \{ \varphi_{x,y'} : x \in X, y' \in Y' \}$

$\varphi_{x,y'}(S) = \langle Sx, y' \rangle \qquad (S \in \mathcal{L}(X, Y))$

$\square$

(16.4) Lemma Let $g_n : \Omega \to \mathbb{C}$ be hol., $|g_n(z)| \le M$, $\forall n \in \mathbb{N}, z \in \Omega$,

$$g(z) = \lim_{n\to\infty} g_n(z) \quad \forall z \in \Omega.$$

Then g is holomorphic.

Proof. Let $\overline{B}(z_0, r) \subset \Omega \Rightarrow$

$$g_n(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{g_n(w)}{w-z}\, dw$$

$\forall z \in B(z_0, r)$ $\quad n \to \infty \Rightarrow$

$$g(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{g(w)}{w-z}\, dw$$

$\Rightarrow g$ hol. on $B(z_0, r)$

W.l.g. $z_0 = 0$ $\quad \frac{1}{w-z} = \frac{1}{w} \frac{1}{1-\frac{z}{w}}$

$= \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$. $\Rightarrow g(z) =$

$$\sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw}_{a_n} z^n \qquad \square$$

(16.5) <u>Proposition</u>. Let $T_n : \Omega \to \mathcal{L}(X, Y)$ be holomorphic, $\|T_n(z)\| \leq M$ $\forall z \in \Omega$, $n \in \mathbb{N}$. Assume that

$$T(z)x := \lim_{n \to \infty} T_n(z)x$$

exists $\forall x \in X$.

Then

$$T : \Omega \to \mathcal{L}(X, Y)$$

is holomorphic.

<u>Proof.</u> a) $T(z) \in \mathcal{L}(X, Y)$ ~~by the~~ clear.

~~(Banach Steinhaus Thm)~~

b) Let $y' \in Y'$, $x \in X$ $\overset{(16.4)}{\Rightarrow}$

$\langle T(\cdot)x, y' \rangle : \Omega \to \mathbb{C}$ holomorphic

(16.3) $\Rightarrow$ T is hol. $\square$

(16.6) <u>Uniqueness Theorem.</u> Ω connected

$f, g : \Omega \to X$ hol.

$\exists\ z_k \in \Omega$, $\lim z_k = z_0 \in \Omega$,

$z_k \neq z_0\ \forall k$, $f(z_k) = g(z_k)\ \forall k$

$\Rightarrow f(z) = g(z)\quad \forall z \in \Omega$

§ 17 Holomorphic semigroups

$\theta \in [0, \pi)$

$\Sigma_\theta = \{ r e^{i\alpha} : r > 0 , |\alpha| < \theta \}$.

(17.1) Definition A C_0-sg T is holomorphic if $\exists\, \theta \in (0, \pi/2)$ & $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$, a hol extension of T s.t.

$$\sup_{\substack{z \in \Sigma_\theta \\ |z| \le 1}} \| \tilde{T}(z) \| < \infty .$$

Consequences.

a) $\tilde{T}(z_1 + z_2) = \tilde{T}(z_1) \tilde{T}(z_2)$

Pf. 1st case. $z_2 = t \in (0, \infty)$.

Then $\tilde{T}(z_1 + t) = \tilde{T}(z_1) \tilde{T}(z_2)$ if $z_1 \in (0, \infty)$

uniqueness thm $\Rightarrow \forall z_1 \in \Sigma_\theta$.

<u>2nd case</u> Let $z_1 \in \Sigma_\theta$. Then

$\tilde{T}(z_1 + z_2) = \tilde{T}(z_1)\tilde{T}(z_2)$ if $z_2 \in (0,\infty)$

by case 1. Uniqueness Theorem

$\Rightarrow \forall z_2 \in \Sigma_\theta$.

b) $\exists M, \omega$ $\quad \|\tilde{T}(z)\| \leq M e^{(\mathrm{Re}\, z)\omega}$

$z \in \Sigma_\theta$.

<u>Proof</u> $M := \sup\limits_{\substack{z \in \Sigma_\theta \\ |z| \leq 1}} \|\tilde{T}(z)\|$

Let $z = re^{i\alpha} \in \Sigma_\theta$. $\exists !\, n \in \mathbb{N}$,

$r = [n, n+1)$. $\|\tilde{T}(z)\| = \|\tilde{T}((r-n)e^{i\theta})$

$\tilde{T}(ne^{i\theta})\| \leq M\|(\tilde{T}(e^{i\theta}))^n\| \leq M M^n$

$\leq M M^r = M e^{\omega' r} = M e^{|z|\omega'}$ $\quad \omega' = \ln M$

~~But for $|\mathrm{Im}\, z| \leq$~~ But $|z| = r$

$= |r \cos\alpha| \frac{1}{|\cos\alpha|} \leq \mathrm{Re}\, z \frac{1}{\cos\theta}$

$\square$

$\omega = \frac{\omega'}{\cos\theta}$.

c) $\lim\limits_{\substack{z \in \Sigma_\theta \\ z \to 0}} \tilde{T}(z)x = x \qquad \forall x \in X$

<u>Pf</u> 1. $x = T(t)y \qquad t > 0.$

Then $\tilde{T}(z)x = \tilde{T}(z+t)y \underset{z \to 0}{\to} \tilde{T}(t)y = x$

2. $\{T(t)y : t > 0, y \in X\}$ dense in X

Equicontinuity Lemma $\Rightarrow$ claim. $\square$

d) $\tilde{T}(z)X \subset D(A)$ and

$\frac{d}{dz}\tilde{T}(z) = A\tilde{T}(z)$

<u>Pf.</u> Let $z \in \Sigma_\theta$. Then

$$\tilde{T}(z)' = \lim_{t \downarrow 0} \frac{\tilde{T}(z+t) - \tilde{T}(z)}{t}$$

$$= \lim_{t \downarrow 0} \frac{T(t)\tilde{T}(z) - \tilde{T}(z)}{t} \qquad \square$$

e.g.) Let $z \in \Sigma_\theta$. Then $T_z(t) = \tilde{T}(tz)$

defines a C_0-sg. Generator: $z \cdot A$

<u>Proof.</u> $B :=$ generator of T_z.

1. Let ~~y~~ $x = \tilde{T}(w)y$, ~~s>0~~ $w \in \Sigma_\theta$

$$\frac{T_z(t)x - x}{t} = z\,\frac{\tilde{T}(w + tz)y - \tilde{T}(w)y}{tz}$$

$$\longrightarrow z A \tilde{T}(w) y \qquad (t \downarrow 0)$$

by f)

Thus $x \in D(B)$ & $Bx = zAx$.

2. $D = \operatorname{span}\{\tilde{T}(w)y : w \in \Sigma_\theta\}$ ~~s>0~~

is dense in X, invariant by T_z,

and $D \subset D(B)$. Thus D is a

core of B.

But $D \subset D(A)$ & $ABx = zAx$

$(x \in D)$. Thus $B = \overline{zA|_D} = zA$. $\square$

2.6.2017

7.6.2017

§ 18 Holomorphic contraction semigroups

X

(18.1) Definition. Let $\theta \in (0, \frac{\pi}{2}]$.

a) An operator A is θ-m-diss. if
$\Sigma_\theta \subset \rho(A)$ & $\|\lambda R(\lambda, A)\| \le 1 \quad \forall \lambda \in \Sigma_\theta$

b) A C_0-sg T is a θ-contractive if it has a contractive hol extension
$\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$

c) A hol. contraction sg is a C_0-sg T having a contractive hol. extension to some sector Σ_θ, where $0 < \theta \le \frac{\pi}{2}$.

(18.2) Theorem. Let A be a dd operator $\theta \in (0, \pi/2)$. Equ:

(i) A is θ-m-diss.

(ii) A generates a θ-contractive C_0-sg

(iii) $e^{\pm i\theta}A$ generates a contractive C_0-sg.

(iv) $e^{\pm i\theta}A$ is diss. & $wI - A$ surj. for some $w \in \Sigma_{\theta+\frac{\pi}{2}}$.

~~Pf.~~

(7.1) Recall: $\Lambda \subset \mathbb{C}$ open, connected, $d : \Lambda \to (0,\infty)$ continuous

A an operator.

(a) $\| \lambda x - Ax \| \geq d(\lambda) \|x\|$

(b) $\exists \lambda_0 \in \Lambda$ $(\lambda_0 - A)$ surj.

$\Rightarrow \Lambda \subset \rho(A)$.

Recall: A m-diss $\Rightarrow \mathbb{C}_+ \subset \rho(A)$.

Pf of (18.2). (i) $\Rightarrow$ (ii) Let $z \in \Sigma_\theta$.

$\lambda > 0$. $\lambda(\lambda - zA)^{-1} = \frac{\lambda}{z}\left(\frac{\lambda}{z} - A\right)^{-1}$

$\Rightarrow$ zA m-diss. $\Rightarrow$ zA generates a C_0-sg T_z, $\|T_z(t)\| \leq 1$

Hille $\Rightarrow$ $T_z(1) = \lim_{n\to\infty} (I - \frac{zA}{n})^{-n}$ strongly.

Since $z \mapsto (I - \frac{zA}{n})^{-n}$ is holomorphic

also $z \mapsto T_z(1)$ is holomorphic.

Thus $z \mapsto \tilde{T}(z) := T_z(1) : \Sigma_\theta \to \mathcal{L}(X)$

is holomorphic. If $t > 0$, then

$\tilde{T}(t) = T_t(1)$ ~~$= T(t)$~~ $= s\text{-}\lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} = T(t)$.

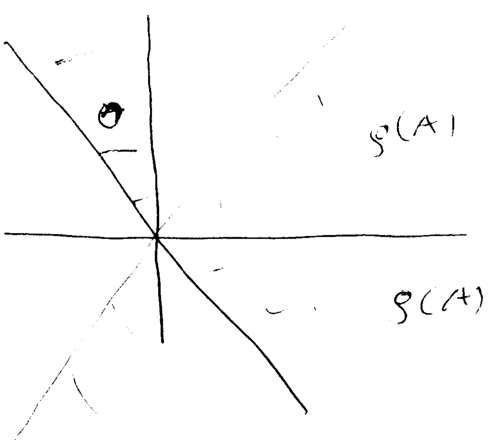
This proves (ii).

(ii) $\Rightarrow$ (iii) (17.1) e).

(iii) $\Rightarrow$ (iv) $\operatorname{Re}\lambda > 0 \Rightarrow \lambda \in \varrho(e^{\pm i\theta}A) = e^{\pm i\theta}\varrho(A)$

$\Rightarrow e^{\mp i\theta}\lambda \in \varrho(A)$

$\Rightarrow \Sigma_{\theta + \frac{\pi}{2}} \subset \varrho(A)$.



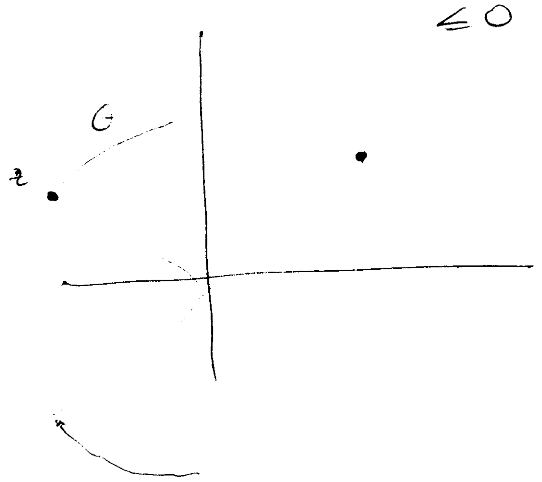
$e^{i\alpha}A$ is diss. for $|\alpha| \le \theta$. In fact,

let $|\alpha| \le \theta$, $x \in D(A)$, $x' \in J(x)$. Then

$\operatorname{Re} \langle e^{\pm i\theta} A x, x' \rangle \le 0$ (since $e^{\pm i\theta}A$ diss)

$\Rightarrow$ $\operatorname{Re} \langle e^{\pm i\theta} Ax, x' \rangle \le 0$ $\Rightarrow$ $\operatorname{Re} e^{i\alpha} \langle Ax, x' \rangle$

≤ 0. $\Rightarrow$



$\| \lambda \cancel{e^{i\alpha}} x - e^{i\alpha} A x \| \ge \lambda \|x\|$

$\lambda > 0$

$\| \lambda e^{-i\alpha} x - Ax \| \ge \lambda \|x\|$

$\Rightarrow$ $\| \mu x - Ax \| \ge |\mu| \, \|x\|$

$\mu \in \Sigma_\theta$

$1 - A$ surj. $\Rightarrow$ $\Sigma_\theta \subset \rho(A)$ &

$\| \mu R(\mu, A) \| \le 1$ $\forall \mu \in \Sigma_\theta$.

i.e. A is θ-m-dissipative. □

Exercise: Show that $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$

has a unique strongly continuous extension to Σ_θ. Moreover, $(\tilde{T}(te^{\pm i\theta}))_{t\geq 0}$ is a C_0-sg and $e^{\pm i\theta}A$ its generator.

§ 19 Sectorial operators and forms.

H complex Hilbert space, $\theta \in [0, \pi/2)$

(19.1) Theorem. A an operator on H

Equ.

(i) $-A$ generates a θ-contractive C_0-sg

(ii) a) $(Ax \mid x) \in \Sigma_\theta \quad \forall x \in D(A)$ &

b) $I + A$ is surjective

Proof. This is (18.2) since

$e^{\pm i\theta}(-A)$ diss $\Longleftrightarrow$ Re $(e^{\pm i\theta} Ax \mid x) \geq 0$

$\forall x \in D(A) \Longleftrightarrow (Ax \mid x) \in \Sigma_\theta \quad \forall x \in D(A)$. □

Rk. a) $\Longleftrightarrow$ $W(A) \subset \Sigma_\theta$.

$W(A) \subset \Sigma_0 \Longleftrightarrow$ A symmetric.

(19.2) <u>Definition</u>. A an operator on H.

a) Let $\theta \in [0, \pi/2)$. A is θ <u>sectorial</u> if $(Ax|x) \in \Sigma_\theta$ $\forall x \in D(A)$;

A is <u>m-θ-sectorial</u> if in addition $(I+A)$ is surjective

b) A is <u>sectorial</u> if $\exists\, \theta \in [0, \pi/2)$ such that A is θ sectorial

c) A <u>m-sectorial</u> $:\Leftrightarrow$ $\exists\, \theta \in [0, \pi/2)$ s.t. A is m-θ sectorial, i.e. θ-sectoriel and $(I+A)$ surjective.

(19.3) ~~Lax Milgram~~ Propositi
<u>Replacing (19.1)</u>

$-A$ generates a ~~Poolomo~~ θ'-contractive C_0-sg $\Leftrightarrow$ A is m-θ-sectorial

$\theta = \frac{\pi}{2} - \theta'$ or

<u>Lumer Phillips complex:</u>
<u>Corollary</u>. A m-sectorial $\Leftrightarrow$ $-A$ generates a contractive hol. sg

(19.4) <u>Forms.</u>

Let $a : D(a) \times D(a) \to \mathbb{C}$ be sesquilinear, $D(a)$ a vector space.

a symmetric $:\Leftrightarrow$ $a(u,v) = \overline{a(v,u)}$

$\Rightarrow$ $a(u) \in \mathbb{R}$ $\forall u \in D(a)$.

$a^*(u,v) = \overline{a(v,u)}$ is also ~~set~~ sesquilinear.

$$h(u,v) := \frac{a + a^*}{2}$$

$$k(u,v) := \frac{a - a^*}{2i}$$

are symmetric

$a = h + ik$ $\quad$ $\operatorname{Re} a := h$, $\operatorname{Im} a := k$

a <u>sectorial</u> $:\Leftrightarrow$ $\exists\, \Theta \in [0, \pi/2)$

$a(u) \in \Sigma_\theta$ $\forall u \in D(a)$

$\Leftrightarrow$ $\exists c > 0$ $\quad \frac{|k(u)|}{\cancel{h(u)}} \leq c\, h(u)$ $\quad u \in D(a)$

a <u>accretive</u> $:\Leftrightarrow$ $\operatorname{Re} a(u) \geq 0$ $\forall u \in D(a)$

(19.5) Polarization:

$$a : D(a) \times D(a) \to \mathbb{C} \quad \text{form}$$

1. $a(u,v) = \frac{1}{4}\left[a(u+v) - a(u-v) + i\,a(u+iv) - i\,a(u-iv)\right]$

2. a sym. $\Rightarrow$ $a(u+iv),\ a(u-iv) \in \mathbb{R}$ $\Rightarrow$

$$\mathrm{Re}\, a(u,v) = \frac{1}{4}\left[a(u+v) - a(u-v)\right]$$

(19.6) Cauchy-Schwarz. $a, h : V \times V \to \mathbb{C}$ forms. h sym. & accretive. &

$$|a(u)| \leq M\, h(u)^{\cancel{*}} \qquad (u \in V)$$

a) a sym. $\Rightarrow$

$$|a(u,v)| \leq M\, h(u)^{1/2}\, h(v)^{1/2}$$

Pf. a) a sym.

$$|a(u,v)| = \operatorname{Re} e^{i\theta} a(u,v)$$

$$= \operatorname{Re} a(\tilde{u}, v) \qquad \tilde{u} = e^{i\theta} u$$

$$= \frac{1}{4}\left[a(\tilde{u}+v) - a(\tilde{u}-v)\right]$$

$$\leq \frac{1}{4} M \left[h(\tilde{u}+v) + h(\tilde{u}-v)\right]$$

$$= \frac{1}{4} M \left[2h(\tilde{u}) + 2h(v)\right]$$

$$= \frac{M}{2}\left[h(u) + h(v)\right]$$

$$u \mapsto \sqrt{\varepsilon}\, u \qquad v \mapsto \frac{1}{\sqrt{\varepsilon}} v$$

$$|a(u,v)| = \frac{M}{2}\left[\varepsilon h(u) + \frac{1}{\varepsilon} h(v)\right]$$

$$\varepsilon = \frac{h(v)^{1/2}}{h(u)^{1/2}} \qquad = \frac{M}{2}\, 2\, h(v)^{1/2} h(u)^{1/2}.$$

b) a arbitrary $\qquad a = a_1 + i a_2 \qquad a_j$ sym.

~~$|a(u,v)|$~~ $\quad |a_1(u)| \leq |a(u)| \leq M\, h(u)$

$|a_2(u)| \leq M h(u)$

a) $\Rightarrow$ ~~$|a(u)| \leq ?$~~

$$|a(u,v)| \leq |a_1(u,v)| + |a_2(u,v)|$$

$$\leq 2M\, h(u)^{1/2} h(v)^{1/2} \qquad \text{by a)}. \quad \square$$

7.6.2017

Reminder:

A form $a: D(a) \times D(a) \longrightarrow \mathbb{C}$ is sectorial if $\exists\, \theta \in [0, \frac{\pi}{2}): \; a(u) \in \Sigma_\theta \; \forall u \in D(a)$.

($\Leftrightarrow \exists\, c > 0: \; |(\operatorname{Im} a)(u)| \leq c \cdot (\operatorname{Re} a)(u) \; \forall u \in D(a)$)

(19.7) **Definition**

Let H be a complex Hilbert space.

A sectorial form $a: D(a) \times D(a) \longrightarrow \mathbb{C}$ is a sectorial form on H if $D(a)$ is a subspace of H.

(19.8) **Consequence**

(i) $(u|v)_a := (\operatorname{Re} a)(u,v) + (u|v)_H$ for $u, v \in D(a)$ defines a scalar product on $D(a)$.

(ii) $\|u\|_a := (u|u)_a = ((\operatorname{Re} a)(u) + \|u\|_H^2)^{\frac{1}{2}}$ for $u \in D(a)$ defines a norm on $D(a)$.

(iii) a is continuous with respect to $\|\cdot\|_a$, i.e., $\exists\, M \geq 0: \; |a(u,v)| \leq M \cdot \|u\|_a \cdot \|v\|_a \; \forall u, v \in D(a)$.

Proof: (i), (ii) are obvious

Since a is accretive, $\operatorname{Re} a$ is symmetric and $\exists\, c > 0$:

$$|a(u)|^2 = (\operatorname{Re} a(u))^2 + (\operatorname{Im} a(u))^2$$
$$\leq \operatorname{Re} a(u))^2 + c^2 (\operatorname{Re} a(u))^2 = (1 + c^2) \operatorname{Re} a(u)^2$$

the Cauchy-Schwartz inequality (19.6) implies

$$|a(u,v)| \leq \underbrace{2\sqrt{1+c^2}}_{=: M} (\operatorname{Re} a(u))^{\frac{1}{2}} (\operatorname{Re} a(v))^{\frac{1}{2}}$$
$$\leq M \cdot \|u\|_a \cdot \|v\|_a \quad \forall u, v \in D(a) \; \square$$

(19.9) Definition

A sectorial form a on a Hilbert space H is... ②

(a) ... closed if $(D(a), \|\cdot\|_a)$ is complete.

(b) ... densely defined if $D(a)$ is dense in H.

(19.10) Definition

Let a be a d.d., closed, sectorial form on a Hilbert space H. We define the operator A associated with a by

$$D(A) := \{x \in D(a) : \exists y \in H : a(x,v) = (y|v)_H \ \forall v \in D(a)\}$$

$$Ax := y \text{ for } y \in H \text{ with } \text{" " "}, \ x \in D(A).$$

Notation: $A \sim a$

(19.11) Remark

A is well-defined since $\overline{D(a)}^H = H$.

(19.12) Theorem

Let a be a d.d., closed, sectorial form on a Hilbert space H, $A \sim a$.

$\Rightarrow$ A is m-sectorial

($\Leftrightarrow$ $-A$ generates a contractive hol. sgr.)

For the proof we need the Lax-Milgram-Theorem.

(19.13) Definition

Let V be a complex Hilbert space. An anti-linear mapping $L : V \longrightarrow \mathbb{C}$ is continuous if ~~[illegible]~~ $\exists c \geq 0 : |Lv| \leq c\|v\| \ \forall v \in V$.

We set $V^* := \{L : V \longrightarrow \mathbb{C} : L \text{ antilinear} + \text{continuos}\}$

(19.14) Example

Let $y \in V$ and define $L_y v := (y|v)_V$ $\forall v \in V$.

$\Rightarrow L \in V^*$

(19.15) Lax-Milgram-Theorem

Let V be a complex Hilbert space, $a : V \times V \longrightarrow \mathbb{C}$ a sesquilinear form which is continuous and coercive (i.e. $\exists \alpha > 0 : \operatorname{Re} a(u) \geq \alpha \|u\|_V^2 \; \forall u \in V$).

Then: $\exists B \in \mathcal{L}(V)$ invertible with

$$a(u,v) = (Bu|v) \quad \forall u,v \in V.$$

In particular: $\forall L \in V^* \; \exists w \in V$ with

$$Lv = a(w,v) \quad \forall v \in V.$$

Proof:

Let $u \in V$. By Riesz-Fréchet there is a unique $Bu \in V$ with

$$a(u,v) = (Bu|v) \quad \forall v \in V.$$

and $B : V \longrightarrow V$ is clearly linear + injective.

* B is continuous: $\exists M \geq 0$ such that

$$\|Bu\|^2 = (Bu|Bu) = a(u, Bu) \leq M \cdot \|u\| \cdot \|Bu\| \quad \forall u \in V.$$

$\Rightarrow \|Bu\| \leq M \cdot \|u\| \quad \forall u \in V.$

* ~~Bu~~ rg B is closed:

$$\alpha \|u\|^2 \leq \operatorname{Re} a(u) = \operatorname{Re} (Bu|u) \leq \|Bu\| \cdot \|u\|$$

$\Rightarrow \alpha \|u\| \leq \|Bu\| \quad \forall u \in V$

This shows that B is closed.

* rg B is dense:

Take $v \in \operatorname{rg}(B)^{\perp}$. $\Rightarrow 0 = (Bu|v) \; \forall u \in V$

$\Rightarrow a(v) = (Bv|v) = 0 \Rightarrow v = 0$

Now take $L \in V^*$.

$\overset{\text{Riesz-Fréchet}}{\Longrightarrow}$ $\exists u \in V$ with

$Lv = (u|v)_V \quad \forall v \in V$.

$\Rightarrow Lv = (BB^{-1}u|v)_V = a(\underbrace{B^{-1}u}_{=:w}, v) \ \forall v \in V. \ \square$

<u>Proof (of (19.12))</u>:

Let $x \in D(A)$. $\Rightarrow (Ax|x) = a(x) \in \Sigma_\theta$ for some $\theta \in [0, \frac{\pi}{2})$.

$\Rightarrow A$ sectorial.

We now show: $I + A$ is surjective.

Let $y \in H$ and define

~~$L_y v := (y|v)_H \ \forall v \in D(a)$~~

$L_y v := (y|v)_H \quad \forall v \in D(a)$

$\Rightarrow L_y$ is antilinear and continuous since

$|L_y v| \le \|y\|_H \cdot \|v\|_H \le \|y\|_H \cdot \|v\|_a$

$\forall v \in D(a)$.

$\tilde{a}(u,v) := a(u,v) + (u|v)_H \quad \forall u, v \in D(a)$

defines a continuous + coercive form on $(D(a), \|\cdot\|_a)$.

$\underset{(19.15)}{\overset{LM}{\Longrightarrow}}$ $\exists x \in D(a)$:

$(y|v)_H = L_y v = \tilde{a}(x,v) = a(x,v) + (x|v)_H \ \forall v \in D(a)$.

$\Rightarrow \quad a(x,v) = (y - x|v)_H \quad \forall v \in D(a)$.

$\Rightarrow x \in D(A)$ and $Ax = y - x$,

i.e. $y \in rg(I + A)$ $\square$

(19.16) <u>Definition</u>

Let a be a closed, sectorial form on a Hilbert space H. A subspace W of $D(a)$ is a <u>form core</u> if W is dense in $(D(a), \|\cdot\|_a)$.

(19.17) **Proposition**

Let a be a d.d., closed, sectorial form on a Hilbert space H, $A \sim a$.

$\Rightarrow$ $D(A)$ is a form core for a.

Proof:

By Lax-Milgram we find $B \in \mathcal{L}(D(a))$ invert. with

$$a(u,v) + (u|v)_H = (Bu|v)_a \quad \forall v \in D(a).$$

It suffices to show that $B\,D(A)$ is dense in $D(a)$.

Let $w \in [B\,D(A)]^\perp$.

$$\Rightarrow \quad 0 = (Bu|w)_a = a(u,w) + (u|w)_H$$
$$= (Au + u|w)_H \quad \forall u \in D(A).$$

$\overset{I+A \text{ surj.}}{\Rightarrow}$ $w = 0$. $\square$

We now prove the converse of (19.13).

(19.18) **Theorem**

Let A be m-sectorial on a Hilbert space H.

$\Rightarrow \exists!$ d.d., closed, sectorial form a on H with $A \sim a$.

Proof:

"Existence":

Define $a: D(A) \times D(A) \longrightarrow \mathbb{C}$ by

$$a(u,v) := (Au|v)_H \quad \forall u,v \in D(A).$$

A sectorial

$\Rightarrow$ a sectorial

By (19.8)(iii) a is continuous on $(D(A), \|\cdot\|_a)$. $\Rightarrow \exists M > 0: |a(u,v)| \le M \|u\|_a \cdot \|v\|_a$ $\forall u,v \in D(A)$.

Now take the completion

$$D(\bar{a}) := (D(A), \|\cdot\|_a)^{\sim}.$$

By the universal property of the completion there is a unique $j \in \mathcal{L}(D(\bar{a}), H)$ with $j(u) = u$ $\forall u \in D(A)$:

$$\begin{tikzcd}
D(\bar{a}) \arrow[r, "j"] & H \\
D(a) \arrow[u, hook] \arrow[ur, "u \mapsto u"'] \arrow[ur, phantom, bend left, "\circlearrowleft"] &
\end{tikzcd}$$

We show: j is injective.

Let $u \in D(\bar{a})$, $j(u) = 0$.

$\Rightarrow \exists\, u_n \in D(A)$ with. $u_n \longrightarrow u$

$\Rightarrow (u_n)_{n\in\mathbb{N}}$ is Cauchy in $D(A)$ and

$u_n \longrightarrow 0$ in H.

Choose $\tilde{M} > 0$: $\|u_n\|_a \le \tilde{M}$ $\forall n \in \mathbb{N}$.

Now let $\varepsilon > 0$ and take $N \in \mathbb{N}$ with

$$\|u_n - u_m\|_a \le \frac{\varepsilon}{M \cdot \tilde{M}} \quad \forall n, m \ge N.$$

We obtain

$$\begin{aligned}
|a(u_n)| &\le |a(u_n, u_n - u_m)| + |a(u_n, u_m)| \\
&\le M \|u_n\| \cdot \|u_n - u_m\| + |(Au_n | u_m)_H| \\
&\le M \cdot \tilde{M} \cdot \frac{\varepsilon}{M \cdot \tilde{M}} + \|Au_n\|_H \cdot \underbrace{\|u_m\|_H}_{\to 0} \quad \forall n, m \ge N
\end{aligned}$$

$\overset{m\to\infty}{\Rightarrow}$ $|a(u_n)| \le \varepsilon$ $\forall n \ge N$.

$\Rightarrow a(u_n) \xrightarrow{n\to\infty} 0$

$\Rightarrow \|u_n\|_a^2 = \operatorname{Re} a(u_n) + \|u_n\|_H^2 \xrightarrow{n\to\infty} 0$

$\Rightarrow u = 0 \quad \Rightarrow j$ is injective.

We identify $D(\bar{a})$ with the subspace $j(D(\bar{a})) \subseteq H$.

There is a unique cont. extension (7)

$$\bar{a} : D(\bar{a}) \times D(\bar{a}) \longrightarrow \mathbb{C}$$

of a to $D(\bar{a}) \times D(\bar{a})$.

Choose $\theta \in [0, \frac{\pi}{2})$ with. $a(u) \in \overline{\Sigma_\theta}$ $\forall u \in D(A)$.

For $u = \lim\limits_{n\to\infty} \underset{\overset{\frown}{D(A)}}{u_n} \in D(\bar{a})$ we obtain

$$\bar{a}(u) = \lim_{n\to\infty} a(u_n) \in \overline{\Sigma_\theta}$$

$\Rightarrow$ $\bar{a}$ is sectorial.

Let $B \sim \bar{a}$. Take $u \in D(A)$.

Since $\bar{a}(u,v) = (Au|v)_H$ $\forall v \in D(A)$

we obtain by density of $D(A)$ in $D(\bar{a})$:

$$\bar{a}(u,v) = (Au|v)_H \quad \forall v \in D(\bar{a})$$

$\Rightarrow$ $u \in D(B)$ and $Bu = Au$.

$\Rightarrow$ $A \subseteq B$ $\underset{-A,\,-B \text{ generators}}{\Longrightarrow}$ $A = B$.

"Uniqueness":

Let b be a closed, d.d., sectorial form on H with $A \sim b$.

$\Rightarrow$ $a(u,v) = (Au|v)_H = b(u,v)$ $\forall u,v \in D(A)$. $(*)$

In particular we have

$$\|u\|_a = \|u\|_b \quad \forall u \in D(A).$$

By definition of $\bar{a}$, $D(A)$ is dense in $(D(\bar{a}), \|\cdot\|_a)$. By (19.17), $D(A)$ is dense in $(D(b), \|\cdot\|_b)$.

$\overset{(*)}{\Longrightarrow}$ $\bar{a} = b$ $\square$

§20 The generation theorem for general holomorphic C_0-semigroups

(20.1) Theorem

Let A be an operator on a Banach space X with

(a) $\exists\, \theta \in (0, \frac{\pi}{2}]$: $\Sigma_{\theta+\frac{\pi}{2}} \subseteq \rho(A)$

(b) $\exists\, M \geq 1$: $\|\lambda R(\lambda, A)\| \leq M$. $\forall \lambda \in \Sigma_{\theta+\frac{\pi}{2}}$

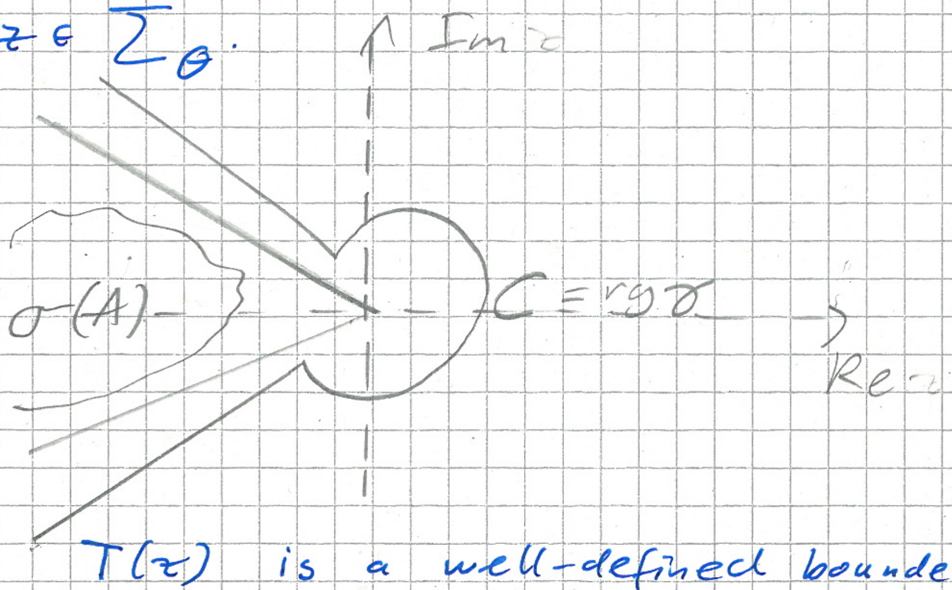
Define $T(0) := I$ and

$$T(z) := \frac{1}{2\pi i} \int_C e^{\mu z} R(\mu, A)\, d\mu$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\gamma(t) z} R(\gamma(t), A)\, \gamma'(t)\, dt$$

for a piecewise smooth curve C with parametrization $\gamma: \mathbb{R} \longrightarrow C \subseteq \Sigma_{\theta+\frac{\pi}{2}}$ satisfying

* $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$ and
* $\lim_{t\to\pm\infty} \frac{\gamma(t)}{|\gamma(t)|} = e^{\pm i(\frac{\pi}{2}+\delta)}$ for some $\delta \in (|\arg z|, \theta)$

and $z \in \Sigma_\theta$.



Then: $T(z)$ is a well-defined bounded operator $\forall z \in \Sigma_\theta$ and the definition does not depend on C.

Moreover:

(i) $\forall \tilde{\theta} \in (0,\theta)$: $\|T(z)\|$ is uniformly bounded on $\Sigma_{\tilde{\theta}}$

(2)

(ii) $z \longmapsto T(z)$ is ~~analytic~~ holomorphic on Σ_θ

(iii) $T(z_1+z_2) = T(z_1)\,T(z_2) \quad \forall z_1, z_2 \in \Sigma_\theta$.

<u>Proof (partial):</u>

Let $\tilde{\theta} \in (0,\theta)$ and set $\varepsilon := \frac{\theta - \tilde{\theta}}{2}$.

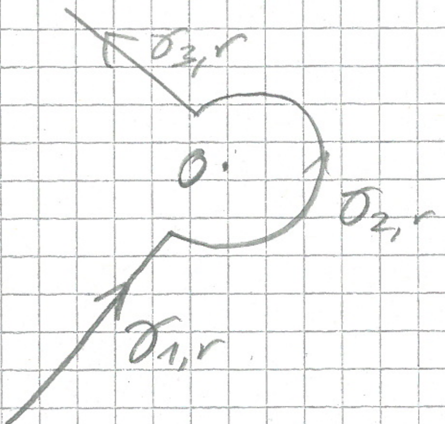
For $r>0$ we define

$$\gamma_{1,r}(t) := -t\, e^{-i(\frac{\pi}{2}+\tilde{\theta}+\varepsilon)} \quad \forall t \in (-\infty, -r),$$

$$\gamma_{2,r}(t) := r\, e^{it} \quad \forall t \in [-(\tfrac{\pi}{2}+\tilde{\theta}+\varepsilon), \tfrac{\pi}{2}+\tilde{\theta}+\varepsilon),$$

$$\gamma_{3,r}(t) := t\, e^{i(\frac{\pi}{2}+\tilde{\theta}+\varepsilon)} \quad \forall t \in (r, \infty)$$

Concatenation of the associated curves $C_{i,r}$ yields a curve C_r:



Now let $z \in \Sigma_{\tilde{\theta}}$. Then C_r is a curve as in the theorem if we set $r := \frac{1}{|z|}$.

In order to see that

$$\frac{1}{2\pi i} \int_{C_r} e^{\mu z} R(\mu, A)\, d\mu \qquad (*)$$

exists, we look at the integrals

$$\frac{1}{2\pi i} \int_{C_{i,r}} \| e^{\mu z} R(\mu, A) \|\, d\mu.$$

Take $\mu \in C_{3,r} = \operatorname{rg} \gamma_{3,r}$. $\quad \tilde\theta \le \frac{\pi}{2}$

$$\Rightarrow \frac{\pi}{2} + \varepsilon \le \underbrace{\frac{\pi}{2} + \tilde\theta + \varepsilon}_{= \arg \mu} + \underbrace{\arg z}_{\in (-\tilde\theta, \tilde\theta)} \le \frac{3\pi}{2} - \varepsilon$$

$$\Rightarrow \frac{1}{|\mu z|} \operatorname{Re}(\mu z) = \operatorname{Re}\left(e^{i(\arg\mu + \arg z)}\right)$$
$$= \cos\underbrace{(\arg\mu + \arg z)}_{\in (\frac{\pi}{2}+\varepsilon, \frac{3\pi}{2} - \varepsilon)} \le \cos\left(\frac{\pi}{2} + \varepsilon\right) = -\sin(\varepsilon)$$

$$\Rightarrow \int_{C_{3,r}} \| e^{\mu z} R(\mu, A) \| \, d\mu \overset{(b)}{\le} \int_{C_{3,r}} \frac{|e^{\mu z}|}{|\mu|} \cdot M \, d\mu$$
$$\le \int_{C_{3,r}} \frac{e^{-|\mu z| \sin(\varepsilon)}}{|\mu|} \cdot M \, d\mu = \int_r^\infty \frac{e^{-t|z|\sin(\varepsilon)}}{t} \cdot M \, dt$$

$s = t|z|$
$$\Rightarrow \int_{C_{3,r}} \| e^{\mu z} R(\mu, A) \| \, d\mu \le \int_1^\infty \frac{e^{-s \sin(\varepsilon)}}{s} \cdot M dt \qquad (**)$$

The same estimate holds for $C_{1,r}$.

Now take $\mu \in C_{2,r} = \operatorname{rg} \gamma_{2,r}$. $\Rightarrow \mu = re^{it}$ for some $t \in \mathbb{R}$.

$$\Rightarrow |e^{\mu z}| = e^{r \operatorname{Re} e^{it} z} \le e^{r \cdot |z|} = e.$$

$$\Rightarrow \int_{C_{2,r}} \| e^{\mu z} R(\mu, A) \| \, d\mu \le e \int_{-\frac{\pi}{2} - \tilde\theta - \varepsilon}^{\frac{\pi}{2} + \tilde\theta + \varepsilon} \frac{M}{r} \cdot r \, dt$$

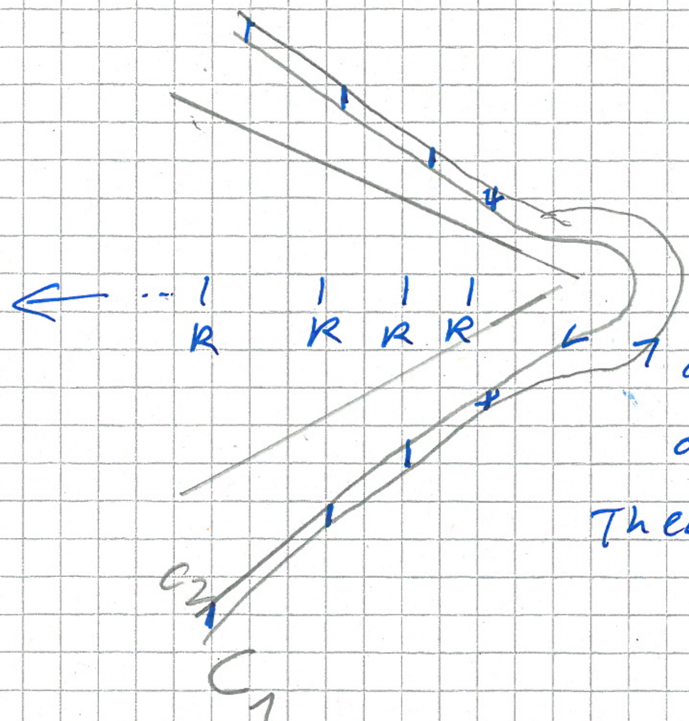
$$\Rightarrow \int_{C_{2,r}} \| e^{\mu z} R(\mu, A) \| \, d\mu \le 2\pi \cdot e \cdot M \qquad (***)$$

$(**) + (***)$ imply $T(z) \in \mathcal{L}(X)$ and uniform boundedness on $\Sigma_{\tilde\theta}$.

We omit the proof of (ii) + (iii) and

note that curve-independence follows from Cauchy's integral theorem (+ some thought). $\square$

curve independence (sketch) ④



By connecting C_1 and C_2 with vertical lines over $R \in (-\infty, 0)$ we obtain closed curves and can apply CIT. Then $R \to -\infty$.

(30.2) Proposition

Let A be an operator on a Banach space X with (a) + (b).
If A is densely defined, then the semigroup T of (30.1) is strongly continuous and A is its generator.

We omit the proof, see II.4.a of Engel-Nagel.

(30.3) Theorem

Let A be an operator on a Banach space X. The following are equivalent:

(i) $\exists\, \theta \in (0, \frac{\pi}{2}]$: $\Sigma_{\theta+\frac{\pi}{2}} \subseteq \rho(A)$, $\exists\, M \geq 1$: $\|\lambda R(\lambda, A)\| \leq M \;\forall \lambda \in \Sigma_{\theta+\frac{\pi}{2}}$ and A is densely defined.

(ii) A generates a bounded holomorphic C_0-semigroup (i.e., a strongly cont. sgr.

with a bounded holomorphic extension to some sector Σ_ϑ).

5

(iii) A generates a bounded C_0-semigroup and $\exists C \geq 1$ with

$$\|s\, R(r+is, A)\| \leq C \quad \forall s \in \mathbb{R}, r > 0.$$

Proof: (20.1) + (20.2) $\Rightarrow$ „(i) $\Rightarrow$ (ii)".

„(ii) $\Rightarrow$ (iii)": For $\tilde\theta < \vartheta$ we obtain that $t \longmapsto T(e^{\pm i\tilde\theta} t)$ are bounded C_0-semigroups with generators $e^{\pm i\tilde\theta} A$.

In particular there is $\tilde C \geq 1$ with

$$\|(\operatorname{Re} \lambda)\, R(\lambda, e^{\pm i\theta} A)\| \leq \tilde C \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

$$\Rightarrow \|R(r+is, A)\| = \|e^{-i\tilde\theta} R(e^{-i\tilde\theta}(r+is), e^{-i\tilde\theta} A)\| \leq \frac{\tilde C}{\operatorname{Re}(e^{-i\theta}(r+is))}$$

Writing $e^{-i\tilde\theta} = a - ib$, with $a, b > 0$

we obtain:

$$\operatorname{Re} e^{-i\theta}(r+is) = ar + sb \geq sb$$

and therefore

$$\|sR(r+is, A)\| \leq \frac{\tilde C}{b} \quad \forall s > 0.$$

The same estimate holds for $s < 0$.

„(iii) $\Rightarrow$ (i)":

Since A generates a bounded C_0-sgr. we have $\Sigma_{\frac{\pi}{2}} \subseteq \rho(A)$ and $\exists \tilde M \geq 1$:

(1) $$\|(\operatorname{Re} \lambda)\, R(\lambda, A)\| \leq \tilde M \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

By (3.5) we know

$$\|R(\lambda, A)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \quad \forall \lambda \in \rho(A)$$

Combining this with (iii) we obtain

$$\operatorname{dist}(r+is, \sigma(A)) \geq \frac{|s|}{C} > 0 \quad \forall r>0,\ s \in \mathbb{R}\setminus\{0\}.$$

$$\Rightarrow \quad \operatorname{dist}(is, \sigma(A)) \geq \frac{|s|}{C} > 0 \quad \forall s \in \mathbb{R}\setminus\{0\}$$

$$\Rightarrow \quad i\mathbb{R}\setminus\{0\} \subseteq \rho(A).$$

Since the resolvent map is continuous, (iii) implies

$$(2) \quad \|R(\mu, A)\| \leq \frac{C}{|\mu|} \quad \forall \mu \in i\mathbb{R}\setminus\{0\}.$$

Now let $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, $\operatorname{Re}\lambda \leq 0$ and $\left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \frac{1}{C}$. Then:

$$|\lambda - i\operatorname{Im}\lambda| \cdot \|R(i\cdot\operatorname{Im}\lambda, A)\| \underset{(2)}{\leq} |\operatorname{Re}\lambda| \cdot \frac{C}{|\operatorname{Im}\lambda|} < 1.$$

$$\underset{(3.3)}{\Rightarrow} \quad \lambda \in \rho(A) \text{ and } R(\lambda, A) = \sum_{n=0}^{\infty} (-\operatorname{Re}\lambda)^n R(i\operatorname{Im}\lambda, A)^{n+1}$$

Setting $\theta := \arctan\frac{1}{C}$ we therefore obtain:

$$\Sigma_{\frac{\pi}{2}+\theta} = \Sigma_{\frac{\pi}{2}} \cup \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0, \operatorname{Im}\lambda \neq 0, \left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \frac{1}{C}\right\} \subseteq \rho(A).$$

We now have to verify an estimate as in (b) of (20.1).

By (iii) we have

$$(3) \quad \|(\operatorname{Im}\lambda) R(\lambda, A)\| \leq C \quad \forall \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > 0.$$

Equations (1) + (3) imply

$$\|\lambda R(\lambda, A)\| \leq 2 \cdot \max\{\hat{M}, C\} \quad \forall \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > 0.$$

For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \leq 0$, $\operatorname{Im}\lambda \neq 0$ and $\left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| \leq \frac{q}{C}$, $q<1$, we obtain

$$\|R(\lambda,A)\| \leq \sum_{n=0}^{\infty} |\operatorname{Re}\lambda|^n \|R(i\operatorname{Im}\lambda, A)^{n+1}\|$$

$$\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{(\operatorname{Im}\lambda)^{n+1}} \cdot |\operatorname{Re}\lambda|^n$$

$$\leq \sum_{n=0}^{\infty} q^n \frac{C}{|\operatorname{Im}\lambda|} = \frac{1}{1-q}\frac{C}{|\operatorname{Im}\lambda|}$$

and since

$$C^2 \cdot \frac{|\lambda|^2}{|\operatorname{Im}\lambda|^2} = \frac{C^2|\operatorname{Re}\lambda|^2 + C^2|\operatorname{Im}\lambda|^2}{(\operatorname{Im}\lambda)^2}$$

$$< 1 + C^2$$

we have:

$$\|\lambda R(\lambda,A)\| \leq \frac{1}{1-q}\sqrt{1+C^2} \quad \forall \lambda \in \Sigma_{\tilde\theta}$$

with $\tilde\theta := \arctan\left(\frac{q}{C}\right) < \theta$. $\square$

By rescalling we obtain.

(20.4) <u>Corollary</u>

Let A be an operator on a Banach space X. The following are equivalent:

(i) A is d.d., $\exists\, \theta \in (0, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$, $M \geq 1$:
$\omega + \Sigma_{\theta + \frac{\pi}{2}} \subseteq \rho(A)$ and

$$\|(\lambda-\omega) R(\lambda,A)\| \leq M \quad \forall \lambda \in \omega + \Sigma_{\theta+\frac{\pi}{2}}$$

(ii) A generates a holomorphic C_0-semigroup

(iii) A generates a C_0-sgr. T with $\|T(t)\| \leq M e^{\omega t}$ $\forall t \geq 0$ and
$\exists\, C \geq 1$: $\|s R(r+\omega+is, A)\| \leq C$ $\forall s \in \mathbb{R}$, $r > 0$.

__Recall.__ A C_0-sg T on X is __holomorphic__ if $\exists\, \theta \in (0, \pi/2]$, $M \geq 0$ s.t. T has a hol. extension

$\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ such that

$$\| \tilde{T}(z) \| \leq M \quad \text{if} \quad z \in \Sigma_\theta, \ |z| \leq 1.$$

(20.5) __Theorem.__ Let A be a ~~an~~ dd operator on X. Equivalent.

(i) A generates a hol. C_0-sg.

(ii) $\exists\, w \in \mathbb{R}$, $M \geq 0$ such

$\operatorname{Re} \lambda > w \Rightarrow \lambda \in \rho(A)$ &

$$\| \lambda R(\lambda, A) \| \leq M.$$

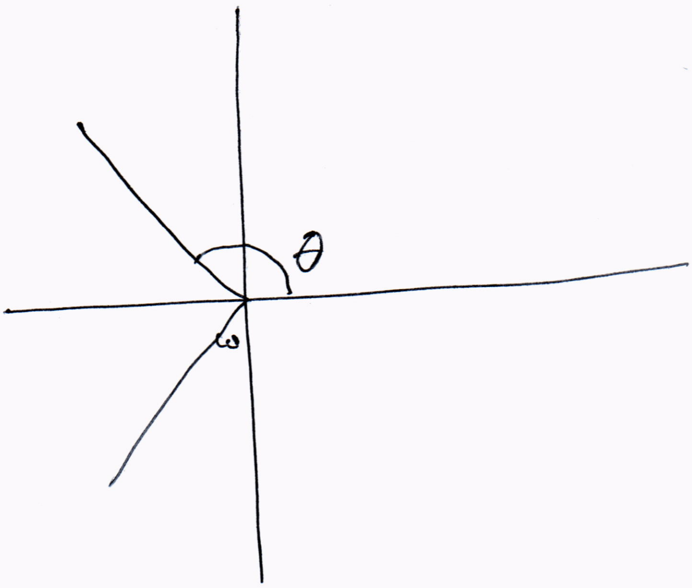
__Remarkable:__ No powers of the resolvent are needed.

Rk. (ii) $\Rightarrow$ (iii) $\exists\, \theta \in (\frac{\pi}{2}, \pi)$

$\Sigma_\theta \subset \rho(A)$ & $\| \lambda R(\lambda, A) \| \leq M'$

$\forall \lambda \in \Sigma_\theta$.

exercise.



Chapter 3 Asymptotic Behavior.

§ 21 The spectral mapping theorem.

T C_0-sg with generator A.

$$\omega(A) := \inf \{ \omega \in \mathbb{R} : \exists M_\omega \quad \|T(t)\| \leq M e^{\omega t} \}$$

growth bound or type of T.

Definition. T exponentially stable

$$:\Leftrightarrow \quad \omega(A) < 0$$

$$\Leftrightarrow \quad \exists \varepsilon > 0 \ \exists M \quad \|T(t)\| \leq M e^{-\varepsilon t}$$

(21.1) Proposition.

$$\omega(A) = \overline{\lim_{t \to \infty}} \frac{\log \|T(t)\|}{t}$$

$$= \inf_{t > 0} \frac{1}{t} \log \|T(t)\|$$

Motivation:

Theorem (Lyapunov) Let $\dim X < \infty$

$A \in \mathcal{L}(X)$. Equ:

(i) $\|e^{tA}\| \longrightarrow 0 \qquad (t \to \infty)$

(ii) $\operatorname{Re} \lambda < 0 \qquad \forall \lambda \in \sigma(A)$

(iii) $e^{tA}x \longrightarrow 0 \qquad (t \to \infty) \qquad \forall x \in X$

$u(t) = e^{tA}x$ solves

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = x \end{cases}$$

Proof. (iii) $\Rightarrow$ (ii) $\lambda \in \sigma(A) \Longrightarrow$

$\exists\, x \in X,\ x \neq 0 \quad Ax = \lambda x$

$\Rightarrow \quad e^{tA}x = e^{\lambda t}x$

$\|e^{\lambda t}x\| = e^{\operatorname{Re}\lambda t}\|x\| \longrightarrow 0$

$\Rightarrow \quad \operatorname{Re}\lambda < 0$

(ii) $\Rightarrow$ (iii) $\quad \exists\, u : X \longrightarrow \mathbb{C}^d$ isomorph.

$u A u^{-1} = B$ has Jordan normal form

$u e^{tA} u^{-1} = u e^{tB} u^{-1}$

Suffices to show $\| e^{tB} \| \longrightarrow 0$

if $\operatorname{Re} \lambda < 0 \quad \forall \lambda \in \sigma(B)$

$$B = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}$$

J_k Jordan block

$$e^{tB} = \begin{pmatrix} e^{tJ_1} & & 0 \\ & \ddots & \\ 0 & & e^{tJ_m} \end{pmatrix}$$

Suffices $\| e^{tJ_k} \| \longrightarrow 0$

$$J_k = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$$e^{tJ_k} = e^{\lambda t} \, e^{t \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}$$

$$= e^{t\lambda} \begin{pmatrix} 1 & t & \cdots & \frac{t^m}{m!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix} \longrightarrow 0 \qquad \square$$

(21.2) **Lemma.** $f : \mathbb{R}_+ \to \mathbb{R}_+$ locally bdd.

$$\omega(f) := \inf \{ w : \exists M_w \; f(t) \le M_w e^{wt} \}$$

Then
$$\omega(f) = \overline{\lim_{t\to\infty}} \; \frac{1}{t} \log f(t).$$

Pf. a) Let $\omega > \omega(f)$.

$$\log f(t) \le \log M_\omega + \omega t \Rightarrow$$

$$\frac{\log f(t)}{t} \le \frac{\log M_\omega}{t} + \omega \Rightarrow$$

$$\overline{\lim_{t\to\infty}} \; \frac{1}{t} \log f(t) \le \omega .$$

b) Let $\overline{\lim_{t\to\infty}} \; \frac{1}{t} \log f(t) < w$

$\Rightarrow \exists t_0 \; \forall t \ge t_0 \quad \frac{1}{t} \log f(t) < w$

$\Rightarrow f(t) \le e^{wt} \qquad (t \ge t_0)$

$\Rightarrow f(t) \le M e^{wt} \qquad \forall t \in \mathbb{R}_+ .$

$\Rightarrow \omega(f) = w$ $\qquad \square$

(21.3) <u>Lemma.</u> $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ loc^{ly} bded.

$$f(t+s) \leq f(t) \cdot f(s)$$

Then $\omega(f) = \inf\limits_{\tau > 0} \frac{1}{\tau} \log f(\tau)$.

<u>Proof.</u> "$\cancel{\leq}$" <u>1st case:</u> $\exists \tau > 0 \quad f(\tau) = 0$

$\Rightarrow f(\tau+s) \leq f(\tau) f(s) = 0 \quad \forall s \geq 0$.

$\Rightarrow \omega(f) = -\infty = \inf\limits_{t>0} \frac{1}{t} \cancel{t_2} \log f(t)$.

<u>2nd case:</u> $f(t) > 0 \quad \forall t > 0$.

"$\leq$" Let $\tau > 0 \quad \forall t > 0 \quad \exists!\, n \in \mathbb{N}_0,\ s \in [0, \tau)$

$t = n\tau + s \quad \Rightarrow$

$f(t) \leq f(\tau)^n f(s) \leq C_\tau f(\tau)^{t/\tau}$ if $f(\tau) \geq 1$

since $n \leq \frac{t}{\tau}$, $\quad C_\tau = \sup\limits_{s \in [0,\tau]} f(s)$.

$$C'_\tau := \begin{cases} C_\tau & \text{if} \quad f(\tau) \geq 1 \\ \frac{C_\tau}{f(\tau)} & \text{if} \quad f(\tau) < 1. \end{cases}$$

$f(\tau) < 1 \Rightarrow \quad f(t) \leq f(\tau)^n f(s) \leq C_\tau f(\tau)^n$

$$= \frac{c_\tau}{f(\tau)} \, f(\tau)^{n+1}$$

$$\leq c'_\tau \, f(\tau)^{t/\tau} \qquad \text{since } \cancel{n+1 >}$$

$$n\tau \leq t < (n+1)\tau \, , \quad \text{hence} \quad n+1 > \frac{t}{\tau}$$

In any case:

$$\log f(t) \leq \log c'_\tau + \frac{t}{\tau} \log f(\tau)$$

$$\frac{\log f(t)}{t} \leq \frac{\log c'_\tau}{t} + \frac{1}{\tau} \log f(\tau)$$

$$\Rightarrow \overline{\lim_{t\to\infty}} \frac{\log f(t)}{t} \leq \frac{1}{\tau} \log f(\tau) .$$

$$\Rightarrow \overline{\lim_{t\to\infty}} \frac{\log f(t)}{t} \leq \inf_{\tau > 0} \frac{1}{\tau} \log f(\tau) .$$

$\square$

(21.4) __Definition__. $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$

__spectral bound__

__Rk.__ $s(A) := -\infty$ if $\sigma(A) = \emptyset$.

<u>Proposition.</u> $s(A) \leq \omega(A)$.

<u>Proof.</u> Let $\mathrm{Re}\,\lambda > \omega(A)$. Choose $\omega \in (\mathrm{Re}\,\lambda, \omega(A)$. Then

$\|T(t)\| \leq M_\omega e^{\omega t} \Rightarrow$

$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x\,dt$ converges $\forall x \in X$.

$\Rightarrow \lambda \in \varrho(A)$ & $R(\lambda, A) = R(\lambda)$. $\square$

(21.5) <u>Spectral radius.</u> Let $S \in \mathcal{L}(X)$.

Then a) $\sigma(S) \neq \emptyset$

b) $r(S) := \sup\{|\mu| : \mu \in \sigma(S)\}$

$=:$ <u>spectral radius of S.</u>

<u>Theorem (Hadamard).</u>

$$r(S) = \lim_{n\to\infty} \|S^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|S^n\|^{\frac{1}{n}}$$

$$\leq \|S\|.$$

(21.6) <u>Theorem</u> $r(T(t)) = e^{\omega(A)t} \quad (t \geq 0)$

<u>Proof.</u> $r(T(t)) = \overline{\lim_{n\to\infty}} \, \|T(nt)\|^{1/n}$

$= \overline{\lim_{n\to\infty}} \, \exp\left\{ t \, \frac{1}{nt} \log \|T(nt)\| \right\}$

$\overset{(21.2)}{=} \exp\{t\,\omega(A)\}.$ □

<u>Drama:</u> $s(A) < \omega(A)$ possible!

(21.7) <u>Definition:</u> A satisfies the SMT <u>(spectral mapping theorem)</u> iff

$$\sigma(e^{tA}) \setminus \{0\} = e^{t\sigma(A)}$$

<u>Consequence:</u> $s(A) = \omega(A)$.

19.1

<u>Proof.</u> Let $\operatorname{Re}\lambda \leq \omega \quad \forall \lambda \in \sigma(A)$.

$\Rightarrow \quad |e^{\lambda t}| = e^{\operatorname{Re}\lambda\, t} \leq e^{\omega t} \quad \forall \lambda \in \sigma(A)$

SMT $\Rightarrow \quad r(T(t)) \leq e^{\omega t}$. $\quad r(T(t)) = e^{\omega(A)t}$

$\Rightarrow$ claim. □

21.6.2017

(21.8) <u>Definition</u>. a) $\sigma_p(A) := \{\lambda \in \mathbb{C} : \exists u \in D(A)$ $u \neq 0,\ Au = \lambda u\}$ =: <u>point spectrum</u> of A = set of all <u>eigenvalues</u>.

b) $\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \exists u_n \in D(A),$ $\|u_n\| = 1,\ (A-\lambda)u_n \longrightarrow 0\}$

=: <u>approximate point spectrum</u>.

(21.9) <u>Proposition</u>. A dd, closed. Then $\sigma(A) = \sigma_{ap}(A') \cup \sigma_{ap}(A)$.

~~Proof.~~

<u>Rk</u>. $\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$.

<u>Pf</u>. $\lambda \notin \sigma(A)$ $\quad u_n \in D(A),\ \|u_n\| = 1$

$\Rightarrow 1 = \|R(\lambda, A)(A-\lambda)u_n\| \leq \|R(\lambda, A)\| \, \|(A-\lambda)u_n\|$

$\square$

Pf of (24.9). Let $\lambda \notin [\sigma_p(A') \cup \sigma_{ap}(A)]$

$\Rightarrow \lambda \notin \sigma_p(A) \to (\lambda - A)$ inj.

a) $(\lambda - A)D(A)$ is closed.

In fact, $\lambda \notin \sigma_{ap}(A) \to \exists c > 0$

$\|(\lambda - A)x\| \geq c\|x\| \quad \forall x \in D(A)$

[otherwise $\forall n \quad \exists x_n \in D(A)$

$\|(\lambda - A)x_n\| < \frac{1}{n}\|x_n\|$.

$\tilde{x}_n := \frac{x_n}{\|x_n\|} \in D(A) \quad \|(\lambda - A)\tilde{x}_n\| \to 0$].

A closed $\Rightarrow (\lambda - A)D(A)$ closed.

b) Remains to show $(\lambda - A)D(A)$ dense in X. Let $\langle x', (\lambda - A)x\rangle = 0 \quad \forall x \in D(A)$.

It suffices to show that $x' = 0$

But $\langle Ax, x'\rangle = \langle x, \lambda x'\rangle \quad \forall x \in D(A)$

$\Rightarrow x' \in D(A') \quad A'x' = \lambda x' \overset{\text{hyp.}}{\Rightarrow} x' = 0$ □

(21.10) Theorem (Fourier uniqueness)

Let $u \in C([0,1]; X)$ ~~s.t.~~

$$\hat{u}(k) := \int_0^1 u(t)\, e^{-2\pi i k t}\, dt$$

k-th Fourier coefficient.

$$\hat{u}(k) = 0 \quad \forall k \in \mathbb{Z} \Rightarrow u = 0.$$

(21.11) Proposition (~~SAS~~ SMT for point spectrum).

a) $\sigma_p(\overset{T(t)}{\cancel{e^{tA}}}) \setminus \{0\} = e^{t\sigma_p(A)}$

b) $\sigma_p(\underset{T(t)'}{\cancel{e^{tA'}}}) \setminus \{0\} = e^{t\sigma_p(A')}$.

Recall:

$$(A - 2\pi i k) \int_0^1 e^{-2\pi i k s}\, T(s)x\, ds$$

$$= T(1)x - x$$

$\forall x \in X,\ k \in \mathbb{Z}$.

<u>Proof.</u> a) Let $\lambda \in \sigma_p(A)$.

$x \in D(A)$ $\quad (A-\lambda)x = 0$

$$\Rightarrow \quad e^{-\lambda t} T(t)x - x = \int_0^t (A-\lambda) e^{-\lambda s} T(s)x \, ds$$

$$= 0$$

$$\Rightarrow \quad e^{\lambda t} \in \sigma_p(T(t)).$$

~~b)~~ Conversely. 1. Let $1 \in \sigma_p(T(\cancel{t}))$

$\Rightarrow \quad \exists x \neq 0 \quad T(1)x - x = 0$

$\exists k \quad y := \int_0^1 e^{-2\pi i k s} T(s)x \, ds \neq 0$

$$(A - 2\pi i k)y = T(1)x - x = 0$$

$\Rightarrow \quad 2\pi i k \in \sigma_p(A) \quad . \quad 1 = e^{2\pi i k}.$

2. Let $\cancel{\mu \neq 0}$ $\mu = r e^{i\theta}$ $\quad \mu \in \sigma_p(T(1))$

$\cancel{\Rightarrow \quad 1 \in \sigma\left(\frac{1}{\mu} T(1)\right)}$

$$\mu = e^{(\ln r + i\theta)} = e^{\nu}$$

$$1 \in \sigma_p(e^{-\nu} T(1))$$

~~$1. \Rightarrow \quad 1 \in e^{\sigma_p(\cancel{e}A \cancel{\in} - \nu)} = e$~~

$(e^{-\nu t} T(t))_{t \geq 0}$ str.s.g. gen. $A-\nu$

1. $\Rightarrow$ $\exists x \in D(A),\ x \neq 0,\ k \in \mathbb{Z}$

$(A-\nu)x = 2\pi i k x$

$\Rightarrow$ $Ax = (\nu + 2\pi i k)x$

$\Rightarrow$ $\nu + 2\pi i k \in \sigma_p(A)$

$e^{\nu + 2\pi i k} = e^{\nu} = \mu \in \sigma_p(A)$.

3. Let $0 \neq \mu \in \sigma_p(T(t))$ $t > 0$

$S(s) = (T(ts))_{s \geq 0}$

Generator tA

2. $\Rightarrow$ $\cancel{\mu \in e^{\mu} \in \sigma_p(tA)}$

$\exists$ $\lambda_1 \in \sigma_p(tA)$ $e^{\lambda_1} = \mu$

$\lambda_1 = t\lambda$ $\lambda \in \sigma(A)$

$e^{t\lambda_1} = \mu$. a

Proof of b) similar. I omit the proof. □

(21.12) <u>Proposition.</u> $\sigma(T(t)) \supset e^{t\sigma(A)}$.

<u>Proof.</u> Wlog. $t=1$.

<u>Claim</u>: $\mu \notin \sigma(T(1)) \Rightarrow \mu \notin e^{\sigma(A)}$

Wlog. $\mu = 1$

$1 \notin \sigma(T(1)) \Rightarrow 1 \notin e^{\sigma(A)}$.

$1 \notin \sigma(T(1)) \Rightarrow 2\pi i k \notin \sigma(A)$.

$$(A - 2\pi i k) \int_0^1 e^{-2\pi i k s} T(s) x \, ds = x$$

$\forall x \in X$

$$Sx = \int_0^1 e^{-2\pi i k s} T(s) x \, ds$$

$S \in \mathcal{L}(X)$.

$SX \subset D(A)$

$(A - 2\pi i k) S = I - T(1)$

$\Rightarrow S(I - T(1))^{-1} = (A - 2\pi i k)^{-1}$

□

(21.13) <u>Proposition</u>. Let A be an operator s.t. $\rho(A) \neq \emptyset$.

Then $\partial\sigma(A) \subset \sigma_{ap}(A)$.

<u>Proof.</u> Let $\lambda \in \partial\sigma(A) \Rightarrow$

$\exists \lambda_n \in \rho(A) \quad \lambda_n \to \lambda$.

$\Rightarrow \quad \|R(\lambda_n, A)\| \to \infty$ (already shown)

$UBP \Rightarrow \quad \exists x \quad \|R(\lambda_n, A)x\| \to \infty$

$$u_n = \frac{\cancel{x}\, R(\lambda_n, A)x}{\|R(\lambda_n, A)x\|}$$

$$\lambda_n u_n - A u_n = \frac{1}{\|R(\lambda_n, A)x\|}\, x$$

$$\to 0 \quad (n \to \infty)$$

$$\lambda u_n - A u_n = (\lambda - \lambda_n) u_n + \lambda_n u_n - A u_n$$

$$\to 0 \quad \square$$

§ 22 Ljapunov's Theorem for ~~immediately~~ eventually norm continuous semigroups.

(22.1) Def. A C_0-sg T is eventually (schließlich) norm-continuous if $\exists \tau > 0$

$$\| T(\tau + t) - T(\tau) \| \longrightarrow 0 \quad \text{as } t \downarrow 0$$

Consequence: $T : [\tau, \infty) \to \mathcal{L}(X)$ is norm continuous. (exercise.

T is immediately norm-continuous if $T : (0, \infty) \to \mathcal{L}(X)$ is norm-continuous

Examples. 1. T hol. $\Rightarrow$ T immediately norm continuous.

2. $\tau > 0$, $T(\tau)$ compact $\overset{\text{Lemma}}{\Longrightarrow}$ $T : [\tau, \infty) \to \mathcal{L}(X)$ continuous.

Lemma. $S_n \to S$ strongly in $\mathcal{L}(X)$, $K \in \mathcal{L}(X)$ compact $\Rightarrow$ $S_n K \to SK$ in norm.

21.6.2017

§22 Ljapunov's Theorem for ~~immediately~~ eventually norm continuous semigroups

(22.1) Def. A C_0-sg is T **eventually** (schließlich) **norm-continuous** if $\exists \tau > 0$
$\| T(\tau + t) - T(\tau) \| \longrightarrow 0$ as $t \downarrow 0$

Consequence: $T : [\tau, \infty) \to \mathcal{L}(X)$ is norm continuous. (exercise.

T is **immediately norm-continuous** if $T : (0, \infty) \to \mathcal{L}(X)$ is norm-continuous

(22.2) **Examples**. 1. T hol. $\Rightarrow$ T immediately norm continuous.

2. $\tau > 0$, $T(\tau)$ compact $\overset{\text{Lemma}}{\Longrightarrow}$ $T : [\tau, \infty) \to \mathcal{L}(X)$ continuous.

Lemma. $S_n \to S$ strongly in $\mathcal{L}(X)$
$K \in \mathcal{L}(X)$ compact $\Rightarrow$ $S_n K \to SK$ in norm.

21.6.2017

(22.3) <u>Theorem.</u> Let T be eventually norm continuous and $1 \in \sigma_{ap}(T(1))$. Then $\exists k \in \mathbb{Z}$ s.t. $2\pi i k \in \sigma_{ap}(A)$.

<u>Proof.</u> $\exists x_n$ $\|x_n\| = 1$ $\|T(1)x_n - x_n\| \to 0$.

$u(t) := (T(t+\tau)x_n)_{n \in \mathbb{N}}$

$u : [0,1] \to \ell^\infty(X)$ continuous

[Pf. $\|u(t) - u(t_0)\|_\infty = \sup_n \|T(t+\tau)x_n - T(t_0+\tau)x_n\|$

$\leq \|T(t+\tau) - T(t_0+\tau)\| \to 0$ $t \to t_0$].

$q : \ell^\infty(X) \to \ell^\infty(X) / c_0(X) =: \hat{X}$

$q \circ u : [0,1] \to \hat{X}$ continuous. 1)

$\Rightarrow$ $\exists k \in \mathbb{Z}$

$$\int_0^1 e^{-2\pi i k t} q(u(t))\, dt \neq 0$$

$\|$

$$q\left(\left(\int_0^1 e^{-2\pi i k t} T(t+\tau)x_n \, dt\right)_{n \in \mathbb{N}}\right.$$

1) $q \circ n \neq 0$. In fact, let

$m \in [\tau, \tau+1) \cap \mathbb{N}_0$. Then

$T(m)x_n - x_n =$

$\sum_{k=0}^{m-1} T(k)\,(T(1)x_n - x_n) \longrightarrow 0$

Thus ~~q(m~~ $n(m-\tau)$ ~~t~~ $- (x_n)_{n\in\mathbb{N}} =$

$(T(m)x_n)_{n\in\mathbb{N}} - (x_n)_{n\in\mathbb{N}} \in c_0$.

Thus $q(n(m-\tau)) = q((x_n)_{n\in\mathbb{N}}) \neq 0$.

$$\Rightarrow \quad \left(\int_0^1 e^{-2\pi i k t} \, T(t+\tau) x_n \, dt \right)_{n \in \mathbb{N}} \notin c_0(X)$$

$$y_n := \int_0^1 e^{-2\pi i k t} \, T(t+\tau) x_n \, dt$$

$\Rightarrow \quad \exists \, \delta > 0 \quad \exists$ SS $(y_{n_k})_{k \in \mathbb{N}}$ s.t.

$$\| y_{n_k} \| \geq \delta > 0$$

$$(A - 2\pi i k) y_{n_k} = T(1+\tau) x_{n_k} - T(\tau) x_{n_k} \longrightarrow 0$$

$$\Rightarrow \quad 2\pi i k \in \sigma_{ap}(A) \qquad \square$$

Rk <u>Quotients.</u>

E Banach $\quad F$ closed subspace

E/F Banach for

$$\| [x] \| = \operatorname{dist}(x, F)$$

$q : E \to E/F$ contraction.

(22.3) <u>Theorem (Ljapunov).</u>

Let T be eventually norm continuous. If $\operatorname{Re}\lambda < 0$ $\forall \lambda \in \sigma(A)$ then

$$\|T(t)\| \leq Me^{-\varepsilon t}.$$

<u>Proof.</u> $r(T(1)) = e^{\omega(A)}$

claim $r(T(1)) < 1$

Assume $r(T(1)) \geq 1$

Then $\exists\, \varrho \geq 1,\ \theta \in \mathbb{R}$ s.t. $\varrho = r(T(1))$

$\Rightarrow \quad \varrho e^{i\theta} \in \sigma_{ap}(T(1))$

~~Define~~

<u>1st case:</u> $\varrho = 1,\ \theta = 0.$

$\Rightarrow 1 \in \sigma_{ap}(T(1)) \Rightarrow \exists k \quad 2\pi i k \in \sigma_{ap}(A)$

$\lightning$

2nd case $\quad S(t) = e^{(-i\theta - \log \varrho)t}\, T(t)$

is a C_0-sg. Its generator: $A - i\theta - \log \varrho$

~~r(T(1))~~ $r(S(1)) = r\left(e^{-i\theta} e^{-\log \varrho}\, T(1)\right) = 1$

1st case $\Rightarrow \quad \exists k \in \mathbb{Z} \quad 2\pi i k \in \sigma_{ap}(A - i\theta - \log \varrho)$

$\Rightarrow \quad 2\pi i k + i\theta + \log \varrho \in \sigma_{ap}(A)$

But $\quad \operatorname{Re}(2\pi i k + i\theta + \log \varrho) = \log \varrho \geq 0 \quad$ ↯

Exercise Let T be an eventually norm continuous C_0-sg with generator A.

Then

a) $\quad \sigma_{ap}(T(t)) \setminus \{0\} = e^{t\sigma_{ap}(A)}$

b) $\quad \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.

§ 23 Datko's Theorem.

(23.1) **Bochner Integral.**

(Ω, Σ) measurable space, X Banach space.

$f : \Omega \to X$ simple function $:\Leftrightarrow$

$$f = \sum_{j=1}^{n} \cancel{\alpha_j}\, x_j \mathbb{1}_{A_j} \qquad A_j \in \Sigma,\ x_j \in X.$$

f measurable $:\Longleftrightarrow$ $\exists f_n : \Omega \to X$ simple functions, $f_n(\omega) \to f(\omega)$ $\forall \omega \in \Omega$.

(23.1) **Pettis' Theorem** X separable

~~Equ.~~ $f : \Omega \to X$

(i) f ~~sepa~~ measurable

(ii) $x' \circ f$ measurable $\forall x' \in X$

(iii) $\exists W \subset X'$ separating $x' \circ f$ meas. $\forall x' \in W$.

Consequence: $\|f\|$ measurable.

(Ω, Σ, μ)

$1 \leq p < \infty$

$L^p(\Omega, X) := \{ f : \Omega \to X$ measurable :

$\int \|f(\omega)\|^p \, d\mu(\omega) < \infty \}$

Banach space for $\|f\|_p$

<u>Rk.</u> $f : \Omega \to X$ measurable $\Rightarrow$

$\exists X_0 \subset X$ separable closed subspace

$f(\omega) \in X_0$ a.e.

<u>Theorem.</u> $p = 1$ $f \in L^1(\Omega; X) \Rightarrow$

$\exists!\ y \in X$ s.t.

$$\langle x', y \rangle = \int_\Omega \langle x', f(\omega) \rangle \, d\mu(\omega)$$

$\forall x' \in X'$.

$$y =: \int_\Omega f(\omega) \, d\mu(\omega)$$

Thus $\langle \int_\Omega f(\omega) \, d\mu(\omega), x' \rangle =$

$\int \langle f(\omega), x' \rangle \, d\mu(\omega)$ $\forall x' \in X'$

Consequence:

$$\left\| \int_\Omega f(\omega)\, d\mu(\omega) \right\| \le \int_\Omega \|f(\omega)\|\, d\mu(\omega)$$

$\forall f \in L^1(\Omega, X)$.

(23.2) <u>Theorem (Datko)</u>. Let T be a C_0-sg with generator A, $1 \le p < \infty$. Equ.:

(i) $\omega(A) < 0$

(ii) $\int_0^\infty \|T(t)x\|^p \, dt < \infty \quad \forall x \in X$.

<u>Proof.</u> (ii) $\Rightarrow$ (i) $S: X \to L^p(0,\infty; X)$
$x \mapsto T(\cdot)x$

is linear & continuous.

Pf. $x_n \to x$, $T(\cdot)x_n \to f$ in L^p

$$\Rightarrow \int_0^\infty \|T(t)x_n - f(t)\|^p \, dt \longrightarrow 0 \quad (n \to \infty).$$

$\Rightarrow \exists$ ss $\|T(t)x_{n_k} - f(t)\| \longrightarrow 0$ a.e.

But $\quad T(t)x_{n_k} \longrightarrow T(t)x \quad \forall t \in (0,\infty)$

$\Rightarrow \quad f(t) = T(t)x \quad$ a.e.

i.e. $\quad f = T(\cdot)x \quad$ in $L^p(0,\infty; X)$

Consequence : $\exists c > 0$ such that

$$\int_0^\infty \|T(t)x\|^p \, dt \leq c \, \|x\|^p \quad \forall x \in X.$$

Suppose that $\omega(A) \geq 0$. $\Rightarrow$

$r(T(1)) \geq 1 \quad \Rightarrow \quad \cancel{\exists |\lambda| \geq 1, \ldots}$

$\cancel{\|T(1)x}$ $\quad \lambda \in \sigma(T(1)), \quad |\lambda| = r(T(1))$.

$\Rightarrow \quad \lambda \in \sigma_{ap}(T(1)) \quad \Rightarrow \quad \exists x_k \in X, \ \|x_k\| = 1$

$$\|T(1)x_k - \lambda x_k\| \longrightarrow 0$$

$$\Rightarrow \quad \|T(n)x_k - \lambda^n x_k\| \longrightarrow 0 \quad \forall n \in \mathbb{N}$$

[in fact $\quad T(n) - \lambda^n =$

$$\lambda^n \left[\left(\frac{T(1)}{\lambda} \right)^n - I \right] = -\lambda^n \sum_{k=1}^{n-1} \left(\frac{T(1)}{\lambda} \right)^k \left(I - \frac{T(1)}{\lambda} \right)$$

$$= \lambda^{n-1} \sum_{k=1}^{n-1} \left(\frac{T(1)}{\lambda} \right)^k \left(T(1) - \lambda I \right) \text{]}$$

$\Rightarrow \quad \exists$ ss

$$\| T(n) x_{k_\ell} - \lambda^n x_{k_\ell} \| \leq \tfrac{1}{2} \quad \forall n \in \mathbb{N} \quad \ell \geq n$$

[$n = 1$ choose x_{k_1} $\quad \| T(1) x_{k_1} - \lambda x_{k_1} \| \leq \frac{1}{2}$

$n = 2$ choose $k_2 > k_1$ $\quad \| T(2) x_{k_2} - \lambda^2 x_{k_2} \| \leq \frac{1}{2}$

$n = 3$ choose $k_3 > k_2$ $\quad \| T(3) x_{k_3} - \lambda^3 x_{k_3} \| \leq \frac{1}{2}$]

...

$$\Rightarrow \quad \| T(n) x_{k_\ell} \| = \| \lambda^n x_{k_\ell} - (\lambda^n x_{k_\ell} - T(n) x_{k_\ell}) \|$$

$$\geq 1 - \tfrac{1}{2} = \tfrac{1}{2} \qquad \forall \ell \geq n.$$

$$C_1 := \sup_{s \in [0,1]} \| T(s) \|$$

Let $m \in \mathbb{N}$, $0 \leq t \leq m$. $\exists!\, n \in \mathbb{N}_0$

$t \in (n-1, n]$ Thus $n \leq m$

For $\ell \geq m$

$$\tfrac{1}{2} \leq \| T(n) x_{k_\ell} \| = \| T(n-t) T(t) x_{k_\ell} \|$$

$$\leq C_1 \| T(t) x_{k_\ell} \|.$$

Thus $\quad \| T(t) x_{k_\ell} \| \geq \dfrac{1}{2 C_1} \quad \forall\, t \in [0, m],\ \ell \geq m.$

$$C = C\|x_{k_\ell}\|^p \geq \int_0^\infty \|T(t)x_{k_\ell}\|^p \, dt$$

$$\geq \int_0^m \|T(t)x_{k_\ell}\|^p \, dt$$

$$\geq \frac{1}{(2C_1)^p}\, m \qquad \forall \ell \geq m \qquad ⚡.$$

Chapter 4 The non-homogeneous equation

§ 24 The non-homogeneous problem: classical & mild solutions

A closed operator, $J \subset \mathbb{R}$ interval

$f \in L^1(J, X)$

$$(E) \qquad \dot{u}(t) = Au(t) + f(t) \qquad \text{on } J.$$

(24.1) Definition (classical solution)

Assume that $f \in C(J; X)$.

u classical solution $:\Longleftrightarrow$

$u \in C^1(J, X)$, $u(t) \in D(A)$ $\forall t \in J$

and (E) holds.

(24.2) Definition (mild solution)

$f \in L^1(J; X)$

u mild solution $:\Longleftrightarrow$

$u \in C(J, X)$, $\int_s^t u(r)\,dr \in D(A)$

& $\quad u(t) - u(s) = A \int_s^t u(r)\,dr + \int_s^t f(r)\,dr$

$\forall s, t \in J$.

~~Rk~~

(24.3) Proposition. Let $f \in C(J, X)$.

~~a)~~ u classical solution $\Leftrightarrow$

u mild solution & $u \in C^1(J, X)$

Proof "$\Rightarrow$" u classical solution

$\Rightarrow \quad Au = \dot{u} - f \in C(J, X)$

$\Rightarrow \quad u \in C(D(A))$ graph norm

$A \in \mathcal{L}(D(A), X) \Rightarrow$

$\int_s^t u(r)\,dr \in D(A)$ & $A\int_s^t u(r)\,dr$

$\int_s^t Au(r)\,dr = \int_s^t \dot{u}(r) - f(r)\,dr$

$= u(t) - u(s) - \int_s^t f(r)\,dr$

$\Rightarrow u$ mild solution.

"$\Leftarrow$" Let u be a mild sol. &

$u \in C^1(J; X)$. Let $t \in J$

$$u(t+h) - u(t) = \int_t^{t+h} \dot{u}(r)\,dr$$

$$= A\int_t^{t+h} u(r)\,dr + \int_t^{t+h} f(r)\,dr$$

$$\frac{1}{h}\int_t^{t+h} u(r)\,dr \to u(t)$$

$$\frac{1}{h}\int_t^{t+h} f(r)\,dr \to f(t)$$

$$\frac{1}{h}\left(u(t+h) - u(t)\right) \to \dot{u}(t)$$

A closed $\Rightarrow$ $u(t) \in D(A)$ &

$\dot{u}(t) = Au(t) + f(t)$. $\square$

Thus, if u is a mild solution, it is a classical solution as soon as $u \in C^1$.

(24.4) <u>Proposition</u>. Let $J = [0,\tau]$,

$f \in L^1(0,\tau;X)$, $x \in X$.

Let A be the generator of

a C_0-sg T. ~~$u \in C([0,\tau],X)$~~

Then $\exists!$ mild sol. u of

$$\begin{cases} \dot u = Au + f \\ u(0) = x \end{cases}$$

Namely $u(t) = T(t)x + (T * f)(t)$

$$(T * f)(t) = \int_0^t T(t-s) f(s)\, ds$$

<u>Proof</u>. 1. <u>Uniqueness</u>. u mild solution

for $x = 0$, $f = 0$ $\Rightarrow$

$\int_0^t u(r)\,dr \in D(A)$ & $A\int_0^t u(r)\,dr = u(t) - \cancel{x}$

Let $v(t) = \int_0^t u(r)\,dr$ $\Rightarrow$

$v \in C^1([0,\tau],X)$, $v(t) \in D(A)$ &, $v(0) = 0$

& $\dot v(t) = u(t) = Av(t)$. Thus

v is a classical solution of

$$\begin{cases} v'(t) = Av(t) \\ v(0) = 0 \end{cases}$$

We know that $v \equiv 0$. Hence $u \equiv 0$.

2. $f \equiv 0$ $\quad u(t) = T(t)x$

$$\int_s^t T(r)x\,dr \in D(A) \ \&$$

(AWI) $$A\int_s^t T(r)x\,dr = T(t)x - T(s)x$$

Thus $T(\cdot)x$ is the mild solution.

3. $x = 0$, $f \in L^1(0,T;X)$.

$u = T * f$.

a) $u \in C([0,\tau];X)$

$t_n \to t_0$ $\quad u(t_n) - u(t_0) =$

$$\int_0^{t_n} (T(t_n-s) f(s)\,ds - \int_0^{t_0} T(t_0-s)) f(s)\,ds =$$

$$\int_0^{t_0} \left(T(t_n-s) - T(t_0-s)\right) f(s)\, ds + \int_{t_0}^{t_n} T(t_n-s) f(s)\, ds$$

$\Rightarrow \|u(t_n) - u(t_0)\| \le$

$$\int_0^{t_0} \|(T(t_n-s) - T(t_0-s)) f(s)\|\, ds + \int_{t_0}^{t_n} M e^{\omega(t_n-s)} \|f(s)\|\, ds$$

$\longrightarrow 0$ (dominated convergence).

b) $\displaystyle\int_0^t u(r)\, dr =$

$$\int_0^t \int_0^r T(r-s) f(s)\, ds\, dr =$$

$$\int_0^t \int_s^t T(r-s) f(s)\, dr\, ds =$$

$$\int_0^t \underbrace{\int_0^{t-s} T(r) f(s)\, dr}_{\in D(A)}\, ds$$

$$A \int_0^{t-s} T(r) f(s) dr = T(t-s) f(s) - f(s)$$

$\overset{L}{\Longrightarrow} \displaystyle\int_0^t u(r)\, dr \in D(A)$ &

$$A \int_0^t u(r)\, dr = \int_0^t T(t-s) f(s)\, ds - \int_0^t f(s)\, ds$$

$$= u(t) - u(0) - \int_0^t f(s)\, ds \qquad \square$$

(24.5) <u>Lemma</u> A a.t.g., $\varrho(A) \neq \emptyset$.

Let $f \in L^1(J; X)$, $f(t) \in D(A)$ $\forall t \in J$.

Assume $Af \in L^1(J, X)$. Then

$$\int_J f(t)\,dt \in D(A) \;\&\; A \int_J f(t)\,dt = \int_J Af(t)\,dt.$$

<u>Pf</u> In the case where $\varrho(A) \neq \emptyset$.

<u>1st case</u>: $0 \in \varrho(A)$.

$Af \in L^1(J, X) \Rightarrow A^{-1}Af \in L^1(J, D(A))$

since $A^{-1} \in \mathcal{L}(X, D(A))$

(24.6) $\Rightarrow$ $\int_J f(t)\,dt \in D(A)$

$A \in \mathcal{L}(D(A), X) \Rightarrow$

$$A \int_J f(t)\,dt = \int_J Af(t)\,dt.$$

<u>2nd case</u>: $\lambda \in \varrho(A)$.

$(\lambda - A)f \in L^1(J, X) \Rightarrow$

$\int_J f(t)\,dt \in D(A)$ & $(\lambda - A)\int_J f(t)\,dt$

$= \int_J (\lambda - A)f(t)\,dt$ $\Rightarrow$ Beh.

q.e.d.

(24.6) Lemma $f \in L^1(\Omega, \Sigma, \mu; X)$

$S \in \mathcal{L}(X, Y) \Rightarrow Sf \in L^1(\Omega, \Sigma; \mu, X)$

$$S \int_\Omega f(\omega)\, d\mu(\omega) = \int_\Omega (Sf)(\omega)\, d\mu(\omega).$$

Pf. $f_n : \Omega \to X$ simple $f_n(\omega) \to f(\omega)$

$\Rightarrow Sf_n : \Omega \to Y$ " $Sf_n(\omega) \to Sf(\omega)$

$\Rightarrow Sf$ measurable.

$$\int_\Omega \| Sf(\omega) \| \, d\mu(\omega) \le \|S\| \int_\Omega \|f(\omega)\| d\mu(\omega)$$

$\Rightarrow Sf \in L^1(\Omega, \Sigma; \mu; Y)$

Let

$$\int_\Omega Sf\, d\mu \overset{!}{=} S \int_\Omega f\, d\mu$$

Let $y' \in Y'$. Then

$$\langle S \int_\Omega f\, d\mu, y' \rangle = \langle \int_\Omega f\, d\mu, S'y' \rangle$$

$$\overset{\S 23.1}{=} \int_\Omega \langle S'y', f(\omega) \rangle\, d\mu(\omega)$$

$$= \int_\Omega \langle y', Sf(\omega) \rangle\, d\mu(\omega) \overset{23.1}{=} \langle y', \int Sf(\omega)\, d\mu(\omega) \rangle$$

28.6.17

Let $-\infty < a < b < \infty$.

1

(24.7) Definition

* $f \in L^1((a,b);X)$ is weakly differentiable if $\exists f' \in L^1((a,b);X)$ with
$$-\int_a^b \varphi'(t) f(t)\, dt = \int_a^b \varphi(t) f'(t)\, dt$$
for each $\varphi \in C_c^\infty((a,b))$.

In this case f' is the weak derivative of f.

* We set
$$W^{1,1}((a,b);X) := \{ f \in L^1((a,b);X) : f \text{ weakly diff.} \}$$

(24.8) Remark

By setting
$$\|u\|_{W^{1,1}} := \int_a^b \|u(t)\|\, dt + \int_a^b \|u'(t)\|\, dt$$
for $u \in W^{1,1}((a,b);X)$ we obtain a complete norm $\|\cdot\|_{W^{1,1}}$ on $W^{1,1}((a,b);X)$.

(24.9) Theorem

(i) Each $u \in W^{1,1}((a,b);X)$ can be identified with a unique continuous representative $u \in C([a,b];X)$.

and
$$u(t) = u(s) + \int_s^t u'(r)\, dr \quad \forall a \le s < t \le b.$$

(ii) For $v \in L^1((a,b);X)$ and $x \in X$ we obtain $u \in W^{1,1}((a,b);X)$ by setting
$$u(t) := x + \int_a^t v(s)\, ds \quad \forall t \in (a,b)$$
and $u' = v$.

(iii) If $u \in W^{1,1}((a,b);X)$ then $\frac{d}{dt} u(t)$ exists almost everywhere and $u'(t) = \frac{d}{dt} u(t)$ a.e.

Now let A be generator of C_0-sgr. T on Bspace X.

(24.10) Theorem

Let $x \in D(A)$, $f \in W^{1,1}((0,T);X)$, $T \geq 0$ and u the corresponding mild solution of E with $u(0)=x$ (i.e. $u(t) = T(t)x + T * f(t)$ $\forall t \geq 0$. 2

$\Rightarrow$ $u \in C^1([0,T];X)$, i.e. u is a classical solution

(24.11) Remark We now in case of $f=0$:

$T(t)x$ is the classical solution of

$$\begin{cases} \dot{u}(t) = Au(t), \ t \geq 0, \\ u(0) = x. \end{cases}$$

if $x \in D(A)$.

Proof (of Theorem 24.10)

We may assume $x = 0$.

$$\Rightarrow u(t) = \int_0^t T(t-s) f(s)\,ds$$

$$= \underbrace{\int_0^t T(t-s) f(0)\,ds}_{1} + \overbrace{\int_0^t T(t-s) \int_0^s f'(r)\,dr\,ds}^{2} \quad \forall t \geq 0$$

with

$$1 \underset{r=t-s}{=} \underline{\int_0^t T(r) f(0)\,dr} \in D(A)$$

and

$$2 \underset{\text{Fubini}}{=} \int_0^t \int_r^t T(t-s) f'(r)\,ds\,dr$$

$$\underset{w=t-s}{=} \int_0^t \int_0^{t-r} T(w) f'(r)\,ds\,dr$$

By Lemma (24.5) we obtain

$u(t) \in D(A) \quad \forall t \in [0, T]$ and

③
$$Au(t) = T(t)f(0) - f(0) + \int_0^t A\left(\int_0^{t-r} T(w) f'(r)\,dw\right)dr$$

$$= T(t)f(0) - f(0) + \int_0^t T(t-r)f'(r)\,dr - \int_0^t f'(r)\,dr \quad \forall t \in [0,T]$$

$\Rightarrow \quad Au \in C([0,T];X).$

Lemma (24.12) now proves the claim. $\square$

(24.12) <u>Lemma</u>

Let $f \in C([0,T;X)$ and u a mild solution of (E) on $[0,T]$, $u(0)=0$.

If $u(t) \in D(A)$ and $Au(t) \in C([0,T];X)$ $\forall t \in [0,T]$, then u is classical solution.

<u>Proof</u>:

By (24.5) we obtain

$$Au(t) = A\int_0^t u(r)\,dr + \int_0^t f(r)\,dr$$

$$= \int_0^t Au(r)\,dr + \int_0^t f(r)\,dr \quad \forall t \in [0,T]$$

$\Rightarrow$ u is differentiable with

$u'(t) = Au(t) + f(t) \quad \forall t \in [0,T]$. $\square$

Exercise

Let $x \in D(A)$, $f \in L^1(0,T,X)$ and u be the corresponding mild solution of (E).

If $f(t) \in D(A)$ $\forall t \in [0,T]$ and $Af \in L^1((0,T);X)$, then u is a classical solution.

(24.13) Theorem

For a closed operator A the following are equivalent

(a) $\forall x \in X$: $\exists!$ mild solution of

$$\dot{u}(t) = Au(t) \quad \forall t \geq 0,$$
$$u(0) = x.$$

(b) A generates a C_0-sgr.

In that case: the ~~so~~ mild solution for $x \in X$ in (a) is $T(\cdot)x$ where T is the C_0-sgr. generated by A.

Proof: Exercise. 4

§25. Periodic solutions

Setting:

Let A be the generator of a C_0-sgr. T
5 on a Banach space X.
For $f \in L^1((0,1);X)$ we consider

(E) $\quad u(t) = Au(t) + f(t) \quad \forall t \in [0,1]$.

The mild solutions of (E) are given by

$$u(t) = T(t)x + T * f\,(t) \qquad \forall t \in [0,1]$$

where $x \in X$.

(25.1) Definition

A mild solution u of (E) is *periodic* if $u(0) = u(1)$.

(25.2) Remark

(E) has a periodic mild solution
$\Leftrightarrow \exists x \in X: \quad x - T(1)x = (T*f)(1)$.

(25.3) Proposition („uniqueness").

The following are equivalent:

(a) $\forall f \in L^1((0,1);X)$: $\exists$ at most one periodic mild solution of (E).

(b) The homogenous problem
$$u'(t) = Au(t) \quad \forall t \in [0,1],$$
has exactly one ~~solution~~ periodic mild solution. (namely $u = 0$).

(c) $1 \notin \sigma_p(T(1))$.

Proof:

"(a) $\Rightarrow$ (b)": obvious. ($f=0$).

"(b) $\Rightarrow$ (c)": Take $x \in X$ with $x - T(1)x = 0$.

The mild solution u with initial value x for $f=0$ (i.e., $u(t) = T(t)x \;\; \forall t \in [0,1]$) is periodic. 6

$\underset{(b)}{\Rightarrow} \quad u = 0 \Rightarrow x = 0.$

"(c) $\Rightarrow$ (a)": Take two periodic solutions u, v of (E) for some $f \in L^1((0,1);X)$.

Then $x := u(0) - v(0)$ satisfies

$$x - T(1)x = 0.$$

$\underset{(c)}{\Rightarrow} \quad u(0) = v(0) \Rightarrow u = v.$ $\square$

(25.4) **Theorem**

The following are equivalent.

(a) $\forall f \in L^1((0,1);X)$: $\exists!$ periodic mild sol. of (E).

(b) $\forall f \in C([0,1];X)$: $\exists!$ " " " " "

(c) $1 \in \rho(T(1))$.

Proof: "(a) $\Rightarrow$ (b)": ✓

"(b) $\Rightarrow$ (c)": By (25.3) we know that

$$I - T(1) \text{ is injective.}$$

Now let $x \in X$ and $f(t) := T(t)x \;\; \forall t \in [0,1]$.

$\Rightarrow \exists$ periodic mild solution for $(E)_f$

$\Rightarrow \exists y \in Y: \quad y - T(1)y = T * f(1)$

since

$$T * f(t) = \int_0^t T(t-s)T(s)x \, ds = \int_0^t T(t)x \, ds = tT(t)x$$

we obtain $\quad y - T(1)y = T(1)x.$

7

$\Rightarrow (x+y) - T(1)(x+y) = x$

$\Rightarrow (I - T(1))$ surjective

$\Rightarrow 1 \in \varrho(T(1))$.

"(c) $\Rightarrow$ (a)": Let $f \in L^1((0,1);X)$.

and $x := (I-T(1))^{-1}(T*f)(1)$

$\Rightarrow x - T(1)x = (T*f)(1)$.

$\Rightarrow$ (E) has a periodic mild sol.

$\Rightarrow$ (a). $\square$

(25.3): uniqueness

(25.5) Proposition

Let $x \in X$, with ~~T(1)x~~ $x - T(1)x = T*f(1)$

$f: \mathbb{R}_+ \longrightarrow X$ 1-periodic with

$f|_{(0,1)} \in L^1((0,1);X)$.

Set $u(t) := T(t)x + (T*f)(t) \quad \forall t \geq 0$.

~~u is periodic~~

$\Rightarrow u(t+1) = u(t) \quad \forall t \geq 0$.

Proof: Define $w(t) := u(t-n)$ for $t \in [n, n+1)$.

$\Rightarrow w \in C([0,\infty);X)$ and $w(t+1) = w(t) \ \forall t \in [0,\infty)$.

Claim: w is mild solution of

$$v'(t) = Av(t) + f(t) \quad \forall t \geq 0.$$

Let $t \geq 0$, $t \in [n, n+1)$.

$$\Rightarrow \int_0^t w(s)\,ds = \sum_{k=1}^{n} \int_{k-1}^{k} w(s)\,ds + \int_n^t w(s)\,ds$$

$$= \sum_{k=1}^{n} \int_0^1 w(s+k-1)\,ds + \int_0^{t-n} w(s+n)\,ds$$

$$= \sum_{k=1}^{n} \int_0^1 u(s)\,ds + \int_0^{t-n} u(s)\,ds \in D(A)$$

and

$$A\int_0^t w(s)\,ds = \overbrace{n\cdot(u(1)-u(0))}^{=0} - \sum_{k=1}^{n}\int_0^1 f(s)\,ds$$

$$+\, u(t-n) - u(0) - \int_0^{t-n} f(s)\,ds \qquad 8$$

$$= w(t) - w(0) - \int_0^t f(s)\,ds$$

This proves the claim.

$$(24.4) \Rightarrow \quad w(t) = T(t)\,u(0) + T * f\,(t) \quad \forall t \geq 0,$$

i.e. $w = u$.

$\square$

§ 26 Periodic solutions on Hilbert spaces

Let X be a complex Banach space.

(26.1) Proposition

1 Let T be a C_0-sgr. on X with generator A, $1 \in \varrho(T(1))$.

$\Rightarrow 2\pi i \mathbb{Z} \subseteq \varrho(A)$ and

$$\sup_{k \in \mathbb{Z}} \|R(2\pi i k, A)\| < \infty.$$

Proof:

We obtain

$$(A - 2\pi i) \int_0^1 e^{-2\pi i k s} T(s) x \, ds = T(1)x - x \quad \forall x \in X$$

and

$$\int_0^1 e^{-2\pi i k s} T(s) (A - 2\pi i) x \, ds = T(1)x - x \quad \forall x \in D(A).$$

for each $k \in \mathbb{Z}$.

$\Rightarrow$ $A - 2\pi i k$ is invertible with

$$(A - 2\pi i k)^{-1} = S_k (T(1) - I)^{-1}$$

where $S_k \in \mathcal{L}(X)$,

$$S_k x := \int_0^1 e^{-2\pi i k s} T(s) x \, ds \quad \forall x \in X, k \in \mathbb{Z}.$$

~~Moreover~~:

$\Rightarrow 2\pi i \mathbb{Z} \subseteq \varrho(A)$ and

$$\sup_{k \in \mathbb{Z}} \|R(2\pi i k, A)\| \leq \sup_{s \in [0,1]} \|T(s)\| \cdot \|R(1, T(1))\| < \infty.$$

$\square$

Let now H be a separable, complex Hilbert space.

(26.2) Reminder

$u : (0,1) \longrightarrow H$ is measurable

$\Leftrightarrow$ $(u(\cdot) | x)$ is measurable $\forall x \in H$

and this implies: $\|u(\cdot)\|$ is measurable

[$\|u(t)\| = \sup_{n\in\mathbb{N}} |(u(t)|x_n)|$ $\forall t \in (0,1)$
where $\{x_n : n\in\mathbb{N}\}$ is dense in H]

2

(26.3) <u>Remark</u>

As in the scalar-valued case $L^2((0,1);H)$ becomes a Hilbert space by setting

$$(u|v) := \int_0^1 (u(t)|v(t))_H \, dt$$

and we define the <u>Fourier coefficients</u>

$$\hat{u}(k) := \int_0^1 e^{-2\pi i k t} u(t)\, dt$$

for $k\in\mathbb{Z}$, $u\in L^2((0,1);H)$.

(26.4) <u>Theorem</u> (Plancherel)

The mapping

$$\begin{aligned} L^2((0,1);H) &\longrightarrow \ell^2(H) \\ u &\longmapsto (\hat{u}(k))_{k\in\mathbb{Z}} \end{aligned}$$

is a unitary (i.e., linear, bij., isometric) with inverse

$$\begin{aligned} \ell^2(H) &\longrightarrow L^2((0,1);H) \\ (a_k)_{k\in\mathbb{Z}} &\longmapsto \sum_{k=-\infty}^{\infty} e^{2\pi i k\cdot} a_k . \end{aligned}$$

We are now ready to prove.

(26.5) <u>Theorem</u>

For a generator A of a C_0-sgr. T on H the following are equivalent:

(a) $2\pi i\mathbb{Z} \subseteq \varrho(A)$ and $\sup_{k\in\mathbb{Z}} \|R(2\pi i k, A)\| < \infty$.

(b) $\forall f\in L^2((0,1);H)$: $\exists!$ mild periodic solution of

(E) $\quad \dot{u}(t) = Au(t) + f(t) \quad \forall t\in[0,1]$

(c) $1 \in \varrho(T(1))$.

proof:

(26.1) $\Rightarrow$ „(c) $\Rightarrow$ (a)".

3 (25.4) $\Rightarrow$ „(b) $\Rightarrow$ (c)".

Assume (a) and let $f \in L^2((0,1); H)$.

$$\overset{(26.4)}{\Rightarrow} f_N := \sum_{|k| \le N} e^{2\pi i k \cdot} \hat{f}(k) \xrightarrow[N \to \infty]{L^2} f$$

By (26.4) there is a unique $u \in L^2((0,1); H)$ with $\hat{u}(k) = R(2\pi i k, A)\hat{f}(k) \quad \forall k \in \mathbb{Z}$.

We show: u is periodic mild solution of (E).

We set $u_N := \sum_{|k| \le N} e^{2\pi i k \cdot} \hat{u}(k) \in C^1([0,1], H)$

Then $u_N(0) = u_N(1)$ and

$$\dot{u}_N(t) = \sum_{|k| \le N} 2\pi i k \, e^{2\pi i k t} \hat{u}(k)$$

$$= \sum_{|k| \le N} e^{2\pi i k t} \cdot \left(2\pi i k \cdot R(2\pi i k, A)\hat{f}(k)\right)$$

$$= \sum_{|k| \le N} e^{2\pi i k t} \hat{f}(k) + A \sum_{|k| \le N} e^{2\pi i k t} R(2\pi i k, A)\hat{f}(k)$$

$$= f_N(t) + A u_N(t) \qquad \forall t \in [0,1]$$

$\Rightarrow$ u_N is classical periodic solution of (E) with respect to f_N for each $N \in \mathbb{N}$.

$\Rightarrow (*)$: $u_N(t) = T(t) u_N(0) + T * f_N(t) \quad \forall t \in [0,1]$

We check that $u_N \longrightarrow u$ uniformly.

We obtain

$$\| T * f_N(t) - T * f(t) \| \le \int_0^t \| T(t-s)(f_N(s) - f(s)) \| \, ds$$

$$\le M e^{\omega t} \int_0^t \| f_N(s) - f(s) \| \, ds$$

$$\leq M e^{\omega t}\left(\int_0^t \|f_\nu(s) - f(s)\|^2 ds\right)^{\frac{1}{2}} t^{\frac{1}{2}}$$

$\xrightarrow[\nu\to\infty]{} 0$ uniformly for $t \in [0,1]$.

4

We also obtain with $(*)$

$$\int_0^1 T(t-s)u_\nu(s)ds = \int_0^1 T(t-s)\left(T(s)u_\nu(0) + \int_0^s T(s-r)f_\nu(r)dr\right)ds$$

$$= T(t)u_\nu(0) + \int_0^1\int_0^s T(t-r)f_\nu(r)drds \quad \forall t \in [0,1]$$

$$\Rightarrow T(t)u_\nu(0) = \int_0^1 T(t-s)u_\nu(s)ds - \int_0^1\int_r^1 T(t-r)f_\nu(r)dsdr$$

$$= \int_0^1 T(t-s)u_\nu(s)ds - \int_0^1 (1-r)T(t-r)f_\nu(r)dr$$

$$= \int_0^1 T(t-s)u_\nu(s)ds - (1-t)\int_0^1 (t-r)T(t-r)f_\nu(r)dr \quad \forall t \geq 0, \nu \in \mathbb{N}$$

As above we obtain

$$\int_0^1 T(t-s)u_\nu(s)ds \xrightarrow{\nu\to\infty} \int_0^1 T(t-s)u(s)ds$$

and

$$(1-t)\int_0^1 (t-r)T(t-r)f_\nu(r)dr \xrightarrow{\nu\to\infty} (1-t)\int_0^1 (t-r)T(t-r)f(r)dr$$

uniformly for $t \in [0,1]$.

$\overset{(*)}{\Rightarrow}$ u_ν converges uniformly on $[0,1]$ and u is necessarily the limit.

$\Rightarrow$ $u \in C([0,1];X)$ and $u(1) = u(0)$.

Since

$$u_\nu(t) - u_\nu(0) = A\int_0^t u_\nu(s)ds + \int_0^t f_\nu(s)ds$$

$$\downarrow \nu\to\infty \qquad\qquad \downarrow$$

$$u(t) - u(0) \qquad\qquad \int_0^t f(s)ds$$

and $\int_0^t u_\nu(s)ds \longrightarrow \int_0^t u(s)ds$, the closedness of A implies

$\int_0^t u(s)\,ds \in D(A)$ and

$$u(t) - u(0) = A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds \quad \forall t \geq 0. \quad (5)$$

$\Rightarrow u$ is periodic mild solution ∎

Uniqueness follows from the spectral mapping theorem for the point sp. + (25.3)

(26.6) <u>Theorem</u> (Gearhard – Prüss)

For a C_0-sgr. T on H with generator A the following are equivalent.

(a) $\omega(A) < 0$

(b) $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subseteq \varrho(A)$ and $\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| < \infty$.

(26.7) <u>Remark</u>

(a) $\Rightarrow \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \varrho(A)$ and $\sup_{\operatorname{Re}\lambda \geq 0} \|R(\lambda, A)\| < \infty$.

<u>Proof</u> (of Theorem (26.6)):

"(b) $\Rightarrow$ (a)":

"Since $r(T(1)) = e^{\omega(A)}$ it suffices to show that $r(T(1)) < 1$.

Let $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$. $\Rightarrow$ $\lambda = r e^{i\theta}$, $r \geq 1$.

We write $r = \log r_0$ with $r_0 \geq 0$.

$\Rightarrow \mu = e^{r_0 + i\theta}$ and

$$\mu - T(1) = \mu\left(I - e^{-r_0 - i\theta}T(1)\right)$$
$$= \mu\left(I - S(1)\right)$$

for $S(t) := e^{(-r_0 - i\theta)t} T(t)$, $t \geq 0$.

Now S is C_0-sgr. with generator $A - r_0 - i\theta$. By (26.7) we have

6

$2\pi i \mathbb{Z} \subseteq \varrho(A - r_0 - i\theta)$ and

the resolvent

$$R(2\pi i k, A - r_0 - i\theta) = R(2\pi i k + r_0 + i\theta, A)$$

satisfies

$$\sup_{k \in \mathbb{Z}} \| R(2\pi i k, A - r_0 - iA) \| < \infty.$$

$(26.3) \Rightarrow 1 \in \varrho(S(1)) \Rightarrow \mu \in \varrho(T(1))$

"(a) $\Rightarrow$ (b)":

Let $\omega(A) < 0 \Rightarrow \exists M \geq 1, \varepsilon > 0$:

$\| T(t) \| \leq M e^{-\varepsilon t} \quad \forall t \geq 0$.

$$\Rightarrow \| R(\lambda, A) x \| \leq \int_0^\infty e^{-\operatorname{Re}\lambda t} \| T(t) \| \, dt \cdot \| x \|$$

$$\leq \int_0^\infty e^{-(\operatorname{Re}\lambda + \varepsilon)t} \cdot M \, dt \cdot \| x \|$$

$$= \frac{M}{\operatorname{Re}\lambda + \varepsilon} \| x \| \leq \frac{M}{\varepsilon} \| x \|$$

$$\Rightarrow \| R(\lambda, A) \| \leq \frac{M}{\varepsilon} \quad \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0.$$

$\square$

Chapter 5 Invariance and Positivity.

§ 27 Invariant convex sets.

T C_0-sg on X, gen. A,

$C \subset X$ closed, convex

C <u>invariant</u> $:\Leftrightarrow$ $T(t)C \subset C$ $\quad \forall t \geq 0$

(27.1) <u>Proposition.</u> Equ. (i) C invariant

(ii) $\exists \omega$ $(\omega, \infty) \subset \rho(A)$ & $\lambda R(\lambda, A) C \subset C$ $\quad (\lambda > \omega)$.

(iii) $\exists r_0 > 0$ $\{\frac{1}{r} : 0 < r < r_0\} \subset \rho(A)$

~~Rk~~ & $(I - rA)^{-1} C \subset C$ $\quad \forall\, 0 < r < r_0$

(ii) $\Leftrightarrow$ (iii) clear.

(27.2) <u>Lemma.</u> $-\infty < a < b < \infty$

$u \in C([a,b]; X)$, $u(t) \in C$ $\quad \forall t \in [a,b]$

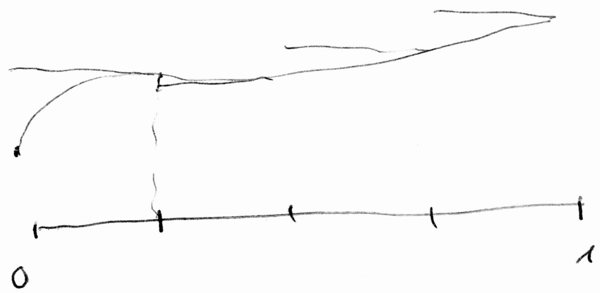
$\varphi \geq 0$, $\int_a^b \varphi(t)\, dt = 1$. $\Rightarrow$

$\int_a^b \varphi(t) u(t)\, dt \in C$

<u>Proof.</u> wlog $[0,1]$

$$u_n = u(0)\,1_{\{0\}} + \sum_{k=1}^{n} u(k/n)\,1_{((k-1)/n,\,k/n]}$$

Then $\|\varphi u_n - u_n\|_1 \to 0$



$$\Rightarrow \quad \int_0^1 \varphi u_n \longrightarrow \int_0^1 \varphi u .$$

$$\int_0^1 \varphi u_n \, dt = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{k/n} \varphi(t)\,dt \; u\left(\tfrac{k}{n}\right) \in C$$

$\square$

<u>Pf of (27.1)</u> $(i) \Rightarrow (ii)$ $\quad x \in C$

$$(1 - e^{-\lambda r})^{-1} \int_0^r \lambda e^{-\lambda t} T(t)x \, dt \in C \qquad (27.2)$$

$r \to \infty$ $\qquad \lambda R(\lambda, A)x \in C.$

$(ii) \Rightarrow (i)$ $\qquad T(t)x = \lim_{n\to\infty} \underbrace{(I - \tfrac{t}{n}A)^{-n} x}_{\in C}$

$\square$

if $x \in C$

(27.3) ~~Bem~~ Rk. a) $X = L^p(\Omega, \mu)$

$X_+ := \{ f \in L^p(\Omega, \mu) : f \geq 0 \text{ a.e.} \}$

positive cone convex closed.

$S \in \mathcal{L}(X)$.

$S \geq 0 \;:\Longleftrightarrow\; S X_+ \subset X_+$

b) $C := \{ f \in L^p(\Omega, \mu) : \cancel{0 \leq} f \leq 1 \}$

closed convex.

$SC \subset C \;\Longleftrightarrow:\;$ S ~~abs~~ sub-markovian

$\Longleftrightarrow \; \|Sf\|_\infty \leq \|f\|_\infty \quad \forall f \in L^1$ &

$S \geq 0$

$\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

If $\mathbb{K} = \mathbb{C}$ $\quad f \leq 1$ means

f is real valued & $f(\omega) \leq 1$ a.e.

In $\mathbb{K} = \mathbb{C}$: $\quad x \leq y \;:\Longleftrightarrow\; \cancel{\&}\; y - x \in \mathbb{R}_+$ ~~& $x, y \in \mathbb{R}$.~~

$x \geq 0 \;:\Longleftrightarrow\; x \in \mathbb{R}_+$

H Hilbert space, $\emptyset \neq C \subset H$ closed, convex. P = <u>minimising projection</u> to C: Thus ~~to~~

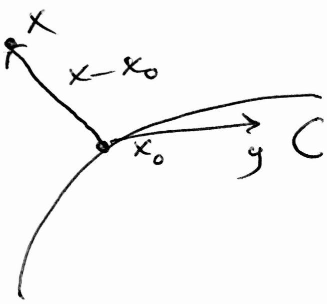
$$P : H \longrightarrow H$$

$$\| x - Px \| = \min \{ \| x - y \| : y \in C \}.$$

<u>Properties</u>: a) $\| Px - P\hat{x} \| \leq \| x - \hat{x} \|$

b) For $x \in H$, $x_0 \in C$

$$Px = x_0 \iff (x - x_0 \mid \overset{y - x_0}{\cancel{x - y}}) \leq 0 \quad \forall y \in C$$



T C_0-sg A generator

(27.4) <u>Proposition.</u> Let $w \in \mathbb{R}$.

$$\mathrm{Re}\,(Ax \mid x - Px) \leq w \| x - Px \|^2$$

$\forall x \in$ ~~dom~~ $D(A)$.

$\Rightarrow$ C invariant.

<u>Pf.</u> $(I - rA)^{-1} C \subset C \qquad 0 < r < r_0$

Let r_0 s.t. $\frac{1}{r_0} > \omega(A)$ & $r_0 \omega < 1$.

Let $(I - rA)x = y \in C$, $x \in D(A)$.

$$\mathrm{Re}\,(x - Px \mid (I - rA)x - Px) \le 0$$

$$\|x - Px\|^2 = \mathrm{Re}(x - Px \mid (I - rA)x - Px) + \overset{\mathrm{Re}\ rAx}{(\cancel{x - Px} \mid x)}$$

$$\le r\,\mathrm{Re}\,(Ax \mid x - Px) \le r\omega \|x - Px\|^2$$

$$\Rightarrow \|x - Px\| = 0 . \quad \square$$

<u>Conversely:</u>

(27.5) <u>Proposition.</u> $\|T(t)\| \le e^{\omega t}$

(quasicontractive)

C invariant $\Rightarrow$

$$\mathrm{Re}\ (Ax \mid x - Px) \le \omega \|x - Px\|^2$$

<u>Pf:</u> $x \in D(A)$ $\qquad \mathrm{Re}\,(T(t)Px - Px \mid x - Px) \le 0$

$\Uparrow$

invariance

$$\mathrm{Re}\,(T(t)x - x \mid x - Px) =$$

$$\mathrm{Re}\,(T(t)(x - Px) \mid x - Px) + \mathrm{Re}\,(T(t)Px \overset{-x}{} \mid x - Px)$$

$$\leq e^{wt}\|x - Px\|^2 + \underbrace{\mathrm{Re}\,(T(t)Px - Px \mid x - Px)}_{\leq 0} +$$

$$\mathrm{Re}\,(Px - x \mid x - Px)$$

$$\leq (e^{wt} - 1)\|x - Px\|^2 \quad \Rightarrow$$

$$\mathrm{Re}\,(Ax \mid x - Px) \leq w \|x - Px\|^2 \qquad \square$$

(27.6) <u>Corollary</u>. Let $\|T(t)\| \leq 1$. Equi

(i) $T(t)C \subset C$

(ii) $\mathrm{Re}\,(Ax \mid x - Px) \leq 0$

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(27.8) <u>Corollary.</u> $H = L^2(\Omega, \mu)$, A operator

Equ.:

(i) A generates a positive, contractive C_0-semigroup

(ii) (a) $\mathrm{Re}\,(Au \mid u^+) \le 0 \quad \forall u \in D(A)$

(b) $I - A$ surjective.

<u>Proof.</u> Let $C = L^2(\Omega)_+$. $\Rightarrow Pu = u^+$.

(i) $\Rightarrow$ (ii) $0 \ge \mathrm{Re}\,(Au \mid u - Pu) =$

$\mathrm{Re}\,(Au \mid u - u^+) = -\mathrm{Re}\,(Au \mid u^-)$

$= \mathrm{Re}\,(A(-u) \mid (-u)^+)$

(ii) $\Rightarrow$ (i) $\mathrm{Re}\,(Au \mid -u^-) = \mathrm{Re}\,(A(-u) \mid (-u)^+) \le 0$

$\Rightarrow \mathrm{Re}\,(Au \mid u) \le 0 \Rightarrow A$ diss.

Thus A is m-diss. $\square$

§ 28 Positive semigroups: spectral bound.

$X = L^p(\Omega, \Sigma, \mu) \qquad 1 \leq p < \infty.$

(28.1) Theorem. Let T be a positive C_0-sg on X with generator A. If $s(A) > -\infty$, then $s(A) \in \sigma(A)$. Moreover, for all $\operatorname{Re}\lambda > s(A)$

$$R(\lambda, A) f = \lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s) f \, ds.$$

Proof. $s := \inf \{ \mu > 0 : [\mu, \infty) \subset \rho(A)$ & $R(\lambda, A) \geq 0 \quad \forall \lambda \geq \mu \}$

a) $0 \leq R(\lambda, A) \leq R(\mu, A)$ if $s < \mu < \lambda$.

$$\frac{R(\mu, A) - R(\lambda, A)}{\lambda - \mu} = R(\mu, A)\, R(\lambda, A) \geq 0.$$

b) Assume that $s \in \varrho(A)$. Then

$R(s, A) \geq 0 \quad \Rightarrow \quad R(s, A)^m \geq 0 \quad \forall m \in \mathbb{N}$.

Let $\delta := \| R(s, A) \|^{-1}$. Then for

$\lambda \in (s - \delta, s)$, $\lambda \in \varrho(A)$ &

$$R(\lambda, A) = \sum_{n=0}^{\infty} (s-\lambda)^n R(s, A)^{n+1} \geq 0$$

$\lightning$.

10.7.2017

c) Assume that $s < 0$.

Then $R(0, A) \geq 0$

$$S(t) f := \int_0^t T(s) f \, ds \qquad (f \in X).$$

$$A S(t) f = T(t) f - f \quad \Rightarrow$$

$$S(t) f = R(0, A) f - R(0, A) T(t) f$$

$$\leq R(0, A) f \qquad \text{if } f \geq 0$$

$$\Rightarrow \quad \| S(t) \| \leq \| R(0, A) \| .$$

$\Rightarrow \quad \forall \; \mathrm{Re}\,\lambda > 0 \quad \int_0^\infty e^{-\lambda t} S(t) f \, dt$ converges.

$$\lambda \int_0^t e^{-\lambda s} S(s) f \, ds = -\int_0^t (e^{-\lambda s})' S(s) f \, ds$$

$$= -e^{-\lambda t} S(t) f + \int_0^t e^{-\lambda s} T(s) f \, ds$$

$$\Rightarrow \quad R(\lambda) f := \lim_{t \to \infty} \int_0^\infty e^{-\lambda s} T(s) f \, ds$$

exists $\forall f \in X$.

$\Rightarrow$ (see beginning of the course),

$\lambda \in \rho(A)$ & $(\lambda - A)^{-1} = R(\lambda)$.

d) Let $\mathrm{Re}\,\lambda > s$. Let $\mathrm{Re}\,\lambda > \mu > s$.

$B = A - \mu \quad \Rightarrow \quad (s-\mu, \infty) \subset \rho(B)$ &

$R(\lambda, B) \geq 0 \quad \lambda > s - \mu. \quad s - \mu < 0$,

$\Rightarrow$ & $s - \mu \in \sigma(B)$ (by b) & c).

$$\Rightarrow \quad \int_0^\infty e^{-(\lambda-\mu)t} \cdot e^{\mu t} T(t) f \, dt$$

converges for all $f \in X$, if $\mathrm{Re}\,\lambda > \mu$

i.e. $\mathrm{Re}(\lambda - \mu) > 0$. $\square$

(28.2) <u>Theorem.</u> Let $X = L^1(\Omega)$, T a positive C_0-sg with generator A. Then $s(A) = \omega(A)$.

<u>Proof.</u> Let $s(A) < 0$. Claim $\omega(A) < 0$.

(28.1) $\Rightarrow$ $\int_0^\infty T(t) f \, dt$ converges $\forall f \geq 0$.

Let $\varphi(f) = \|f^+\| - \|f^-\|$. Then $\varphi \in X'$. Thus

$$\int_0^\infty \|T(t) f\| \, dt = \lim_{t \to \infty} \int_0^t \varphi(T(s) f) \, ds$$

$$= \lim_{t \to \infty} \varphi \int_0^t T(s) f \, ds$$

$$= \varphi\left(\int_0^\infty T(t) f \, dt \right) < \infty.$$

Datko $\Rightarrow$ claim. $\square$

(28.3) <u>Theorem (Weis)</u> Let T be a positive C_0-sg on $L^p(\Omega)$ $1 \leq p < \infty$. Then $s(A) = \omega(A)$

<u>without proof</u>.

14. 7. 2017
(revolution)

29 Invariance for forms.

H Hilbert space, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

$a : D(a) \times D(a) \longrightarrow \mathbb{K}$ form

$\operatorname{Re} a(u) \geq 0 \quad \forall u \in D(a)$ (accretive)

$V := (D(a), (\ |\ )_V)$

$(u \mid v)_V := \operatorname{Re} a(u,v) + (u \mid v)_H$.

Hypothesis: V complete

(i.e., a is a closed form).

Rem: $a : V \times V \longrightarrow \mathbb{K}$ is continuous.

Rem: 1. $\|u_n\|_V \leq c \quad u_n \in V$

$\Rightarrow \exists$ ss $u_{n_k} \longrightarrow u$ in V

i.e. $(u_{n_k} \mid v)_V \longrightarrow (u \mid v)_V \quad \forall v \in V$

2. $u_n \to u$ in H, $u_n \in V$

& $\|u_n\|_V \leq c \Rightarrow$ ~~$u_n \longrightarrow u$~~ $u \in V$ & $u_n \rightharpoonup u$ in V

Rem: $X \hookrightarrow Y$ Banach spaces

$x_n \rightharpoonup x$ in X $\Rightarrow$ $x_n \rightharpoonup x$ in Y

Pf. Let $y' \in Y'$ $\Rightarrow$ $y'|_X \in X'$ $\Rightarrow$

$\langle y', x_n \rangle \to \langle y', x \rangle$. $\square$

Pf of 2. 1. $\exists w \in V$ $\exists$ ss $u_{n_k} \rightharpoonup w$

in V $\Rightarrow$ $u_{n_k} \rightharpoonup w$ in H $\Rightarrow$ $w = u$.

2. Suppose $u_n \not\rightharpoonup u$ in V.

$\Rightarrow$ $\exists v \in V$ $(u_n | v)_V \not\to (u|v)_V$

$\Rightarrow$ $\exists \varepsilon > 0$ $\exists$ ss $|(u_{n_k}|v)_V - (u|v)_V| \geq \varepsilon$

But $\exists$ ss $u_{n_{k_\ell}} \rightharpoonup u$ $\lightning$. $\square$

3. Thus $u_n \in V$, $u_n \to u$ in H

$\operatorname{Re} a(u_n) \leq c$ $\Rightarrow$ $u \in V$ & $u_n \rightharpoonup u$ in V.

(29.1) <u>Lemma.</u> $u_n \in V$ $\quad u_n \to u$ in H

$v_n \in V$, $\quad \|v_n\|_V \le C$

$$\operatorname{Re} a(u_n, u_n - v_n) \le 0$$

$$\Rightarrow \quad u \in V \quad \& \quad u_n \to u \text{ in } V.$$

<u>Pf.</u> $\quad \operatorname{Re} a(u_n) \le \operatorname{Re} a(u_n, v_n) \le M \|u_n\|_V \|v_n\|_V$

$$\|u_n\|_V^2 = \|u_n\|_H^2 + \operatorname{Re} a(u_n)$$

$$\le \cancel{\|u_n\|} C' + MC \|u_n\|_V = \alpha + 2\beta \|u_n\|_V$$

$$\Rightarrow \quad \sup_{n \in \mathbb{N}} \|u_n\|_V < \infty$$

$$\cancel{\alpha} (\|u_n\|_V - \beta)^2 = \|u_n\|_V^2 - 2\beta \|u_n\|_V + \beta^2$$

$$\le \alpha + \beta^2$$

$$\Rightarrow \quad \|u_n\|_V - \beta \le \alpha + \beta^2$$

$\overset{3.}{\Rightarrow}$ claim. $\square$

Hyp: $\overline{D(a)} = H$

$a \sim A$; i.e.

$$D(A) = \{ u \in V : \ \exists f \in H \quad a(u,v) = (f|v)_H \ \ \forall v \in V \}$$

$Au := f$.

A is m-sectorial.

$\Rightarrow$ $-A$ generates a C_0-sg T on H.

$C \subset H$ closed, convex, P the minimising projection.

(29.2) <u>Proposition.</u> C invariant $\Rightarrow$

$$PD(a) \subset D(a).$$

~~Proof.~~ $\cancel{(I + tA)^{-1}}$

(29.3) Lemma. $J_t = (I+tA)^{-1}u \longrightarrow u$ in V

as $t \downarrow 0$.

Proof. $J_t u + tAJ_t u = u$ $\qquad AJ_t u = \dfrac{u - J_t u}{t}$

Proof. Let $u \in V$.

$$R_r = (I + rA)^{-1}$$

$$R_r + rAR_r = I \qquad AR_r = \frac{I - R_r}{r}$$

~~R~~ Let $r_n \downarrow 0$

$$R_{r_n}Pu =: u_n \longrightarrow Pu \text{ in } H.$$

$$\operatorname{Re} a(u_n, u_n - u) =$$

$$\operatorname{Re} \left(A R_{r_n} Pu \mid \cancel{P} u_n - u\right) =$$

$$\operatorname{Re} \frac{1}{r_n}\left(Pu - R_{r_n}Pu \mid R_{r_n}Pu - u\right) =$$

$$\operatorname{Re} \frac{1}{r_n}\left(R_{r_n}Pu - Pu \mid u - R_{r_n}Pu\right) =$$

$$\operatorname{Re} \frac{1}{r_n} \underbrace{\left(R_{r_n}Pu - Pu \mid u - Pu\right)}_{\leq 0} + \operatorname{Re} \frac{1}{r_n}\left(R_{r_n}Pu - Pu \mid Pu - R_{r_n}Pu\right)$$

$$\leq -\frac{1}{r_n} \| R_{r_n}Pu - Pu \| \leq 0.$$

$$(29.1) \Rightarrow Pu \in V. \ \square$$

(29.3) Lemma. $R_r u \to u$ in V

$\forall u \in V$.

Proof. $r_n \downarrow 0$ $\quad u_n = R_{r_n} u$

$$\operatorname{Re} a(u_n, u_n - u) = \frac{1}{r_n} \operatorname{Re} (u - R_{r_n} u \mid R_{r_n} u - u)$$

$$\leq 0$$

$$(29.1) \Rightarrow u_n \to u \quad \text{in } V \qquad \square$$

(29.4) Theorem. Äqu.:

(i) C invariant

(ii) $PV \subset V$ &

$\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad (u \in V)$

(iii) $\exists D$ dense in V $\quad P(D) \subset V$

$\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad \forall u \in D$

(iv) $PV \subset V$ and

$\operatorname{Re} a(u, u - Pu) \geq -\omega \|u - Pu\|^2$

$\forall u \in V$ and some $\omega \in \mathbb{R}$.

Proof. (i) $\Rightarrow$ (ii) $\quad PV \subset V$ by (29.2)

Let $u \in V$.

$$\mathrm{Re}\; a(R_r Pu, u - Pu) = \frac{1}{r}(Pu - R_r Pu \mid u - Pu)$$
$$\geq 0$$

$r \downarrow 0 \Rightarrow$

$$\mathrm{Re}\; a(Pu ; u - Pu) \geq 0$$

(since $a(\cdot\,, u - Pu) \in V'$).

(iii) $\Rightarrow$ (ii) Let $u \in V$ $\exists u_n \in D$

$u_n \to u$ in V.

$$\mathrm{Re}\; a(Pu_n, Pu_n - u_n) \leq 0$$

$Pu_n \to Pu$ in H.

(29.1) $\Rightarrow$ $Pu \in V$ & $Pu_n \to Pu$ in V

$\Rightarrow$ $\cancel{\lim \|Pu_n\|}$

$$\mathrm{Re}\; a(u) = \lim \{ \mathrm{Re}\; a(Pu - Pu_n, Pu) + \mathrm{Re}\; a(Pu_n, Pu) \}$$
$$= \lim \mathrm{Re}\; a(Pu_n, Pu)$$
$$= \lim \{ \mathrm{Re}\; a(Pu_n, u - Pu_n) + \mathrm{Re}\; a(Pu_n) \}$$

$$\Rightarrow \quad \operatorname{Re} a(Pu) = \lim_{n\to\infty} \operatorname{Re} a(Pu_n, Pu)$$

$$\leq \underline{\lim}_{n\to\infty} \operatorname{Re} a(Pu_n)^{\frac{1}{2}} \operatorname{Re} a(Pu)^{\frac{1}{2}}$$

$$\Rightarrow \quad \operatorname{Re} a(Pu)^{\frac{1}{2}} \leq \underline{\lim}_{n\to\infty} \operatorname{Re} a(Pu_n)^{\frac{1}{2}} .$$

$$\Rightarrow \quad \operatorname{Re} a(Pu, u - Pu) =$$

$$\operatorname{Re} a(Pu, u) - \operatorname{Re} a(Pu) \geq$$

$$\operatorname{Re} a(Pu, u) - \underline{\lim} \operatorname{Re} a(Pu_n) =$$

$$\lim \operatorname{Re} a(Pu_n, u)$$

$$\|$$

$$\lim \left[\operatorname{Re} a(Pu_n, u - u_n) + \operatorname{Re} a(Pu_n, u_n) \right]$$

$$- \underline{\lim} \operatorname{Re} a(Pu_n, u_n)$$

Άρα $\quad \operatorname{Re} a(Pu, u - Pu) \geq$

$$\lim \operatorname{Re} a(Pu_n, u_n) - \underline{\lim} \operatorname{Re} a(Pu_n) =$$

$$\overline{\lim} \operatorname{Re} a(Pu_n, u_n - Pu_n) \geq 0 .$$

(ii) $\Rightarrow$ (~~iii~~ v)

$$\mathrm{Re}\, a(u, u-Pu) = \mathrm{Re}\, a(Pu, u-Pu) + \mathrm{Re}\, a(u-Pu, u-Pu)$$

$$\geq \mathrm{Re}\, a(Pu, u-Pu) \geq 0.$$

(iv) $\Rightarrow$ (i) $\quad u \in D(A) \Rightarrow$

$$\mathrm{Re}\,(Au \mid u - Pu) = \mathrm{Re}\, a(u, u-Pu)$$

$$\geq -\omega \|u - Pu\|^2.$$

$\overset{\S\, 27}{\Longrightarrow}$ invariance. $\square$

14.7.

17.7.2017

(Schwarzzahl)
Primzahlen

§ 30 Interpolation

$M \subset \mathbb{C}$ $\qquad C^b(M) := \{f : M \to \mathbb{C} : \text{bounded continuous}\}$

$\mathrm{Hol}(\Omega) := \{f : \Omega \to \mathbb{C} \text{ hol.}\}$

$\Omega \subset \mathbb{C}$ open.

(30.1) **Maximum Principle:** $\Omega \subset \mathbb{C}$ open bded, $f \in \mathrm{Hol}(\Omega) \cap C(\bar{\Omega})$

$$\Rightarrow \quad \|f\|_{\bar{\Omega}} = \|f\|_{\partial\Omega}$$

$$\|f\|_M = \sup_{z \in M} |f(z)| \qquad M \subset \mathbb{C}.$$

(30.2) **Theorem (maximum principle on S)**

$S := \{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 1\}$

$h \in C^b(\bar{S}) \cap \mathcal{H}(S) \Rightarrow$

$$\|h\|_{\bar{S}} = \|h\|_{\partial S}.$$

Pf: $S_k = \{ z \in S : |\operatorname{Im} z| \leq k \}$

$$\varphi_n(z) = \frac{n}{z+n} \qquad \varphi_n \in \operatorname{Hol}(S) \cap C^0(\bar{S})$$

$$\| \varphi_n h \|_{\bar{S}_k} = \| \varphi_n h \|_{\partial S_k}$$

$$\| \varphi_n \|_{\{|\operatorname{Im} z| \geq k\}} \longrightarrow 0 \qquad k \to \infty$$

Let $n \in \mathbb{N}$. $\qquad \exists k \in \mathbb{N}$

$$\| \varphi_n h \|_{\bar{S}} = \| \varphi_n h \|_{S_k}$$

$$\| \varphi_n h \|_{\partial S} = \| \varphi_n h \|_{\partial S_k}$$

$$\Rightarrow \quad \| \varphi_n h \|_{\bar{S}} = \| \varphi_n h \|_{\partial S} \leq \| h \|_{\partial S}$$

$\forall n \in \mathbb{N}$

$$n \to \infty \quad \Rightarrow \quad \| h \|_{\bar{S}} \leq \| h \|_{\partial S} . \quad \square$$

(30.3) <u>Corollary</u> (3-lines Theorem).

Let $h \in C^b(\bar{S}) \cap \mathcal{H}ol(S)$

Then $\|h\|_{\{Re z = \tau\}} \leq \|h\|_{\{Re z = 0\}}^{1-\tau} \cdot \|h\|_{\{Re z = 1\}}^{\tau}$

<u>Proof.</u> Let $b_j > \|h\|_{\{Re z = j\}}$ $\quad j = 0, 1.$

$$H(z) := \left(\frac{b_0}{b_1}\right)^z h(z)$$

$$\overset{30.2}{\Rightarrow} \quad \left(\frac{b_0}{b_1}\right)^\tau \|h\|_{\{Re z = \tau\}} \leq$$

$$\max\left\{\left(\frac{b_0}{b_1}\right)^0 b_0 \,,\, \left(\frac{b_0}{b_1}\right)^1 b_1\right\} = b_0$$

$$\Rightarrow \quad \|h\|_{\{Re z = \tau\}} \leq b_1^\tau \, b_0^{1-\tau}. \qquad \square$$

§ 31 The Stein interpolation theorem

(Ω, Σ, μ) $\quad$ σ-finite.

$\Sigma_c := \{ B \in \Sigma : \mu(B) < \infty \}$

$S(\Sigma_c) := \text{lin}\{ 1_A : A \in \Sigma_c \}$

simple functions.

$L^1_{loc}(\Omega) = \{ u \cancel{\in L^1} : \Omega \to \mathbb{C}$ meas.

$u 1_A \in L^1(\mu) \quad \forall A \in \Sigma_c \}$

Rk. $\quad u \in L^1_{loc}(\Omega), v \in S(\Sigma_c) \Rightarrow uv \in L^1(\Omega, \mu)$

$p_0, p_1, q_0, q_1 \in [1, \infty], \quad M_0, M_1 \geq 0$

$\tau \in \cancel{]}[0,1]$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \qquad \frac{1}{q_\tau} = \frac{1-\tau}{q_0} + \frac{\tau}{q_1}$$

$$M_\tau = M_0^{1-\tau} M_1^{\tau}$$

$$\Phi : \overline{S} \longrightarrow L(S(\Sigma_c), L^1_{loc})$$

$$L(E,F) = \{ T : E \to F \text{ linear} \}$$

(a) $\quad \| \Phi(j+is)u \|_{q_j} \leq M_j \|u\|_{p_j}$

$j = 0, 1, \; s \in \mathbb{R}, \; u \in S(\Sigma_c)$

(ii) $\quad A, B \in \Sigma_c \Rightarrow$

$$z \in \overline{S} \mapsto \int_\Omega (\Phi(z) 1_A) 1_B \, d\mu$$

$$\in \mathcal{H}ol(S) \cap C^b(\overline{S}).$$

(31.1) Theorem (Stein)

$$\| \Phi(\tau + is) u \|_{q_\tau} \leq M_\tau \|u\|_{p_\tau}$$

$\forall u \in S(\Sigma_c), \; s \in \mathbb{R}, \; \tau \in [0,1]$

(31.2) Corollary (Riesz - Thorin)

$B \in L(S(\Sigma_c), L^1_{loc}(\Omega))$

$\|Bu\|_{q_j} \leq M_j \|u\|_{p_j}$

$$\Rightarrow \quad \| Bu \|_{q_\tau} \le M_\tau \, \| u \|_{p_\tau} .$$

(31.3) <u>Lemma.</u> $p, p' \in [1, \infty]$,

$\frac{1}{p} + \frac{1}{p'} = 1$. $u \in L^1_{loc}$ s.t.

$$\left| \int u v \, d\mu \right| \le c \, \| v \|_{p'}$$

$\forall v \in \mathcal{S}(\Sigma_c)$

$$\Rightarrow u \in L^p(\mu), \quad \| u \|_p \le c .$$

<u>Proof of Stein.</u> Let $\tau \in (0,1)$.

$u, v \in \mathcal{S}(\Sigma_c)$ $\| u \|_{p_\tau} = \| v \|_{q'_\tau} = 1$.

<u>Claim:</u> $\left| \int \phi(\tau) u v \, d\mu \right| \le M_\tau$.

For $z \in \overline{S}$ $\quad \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$

$$\beta(z) := \frac{1-z}{q'_0} + \frac{z}{q'_1}$$

$$F(z) := \begin{cases} |u|^{\alpha(z) p_\tau} \operatorname{sgn} u & p_\tau \neq \infty \\ u & \text{if } p_\tau = \infty \end{cases}$$

$$G(z) := \begin{cases} |v|^{\beta(z) q'_\tau} \operatorname{sgn} v & q'_\tau \neq \infty \\ v & \text{if } q'_\tau = \infty \end{cases}$$

$$\Rightarrow \quad F(\tau) = |u| \operatorname{sgn} u = u$$

$$G(\tau) = v$$

$$F(z), G(z) \in \mathcal{S}(\Sigma_c)$$

[Pf. $u = \sum_{j=1}^{n} c_j 1_{A_j}$ $\quad A_j \in \Sigma_c$ pairwise disjoint

$$F(z) = \sum_{j=1}^{m} |c_j|^{\alpha(z) p_\tau} (\operatorname{sgn} c_j) 1_{A_j}$$

if $p_j \neq \infty$

similar G, a]

$$h(z) := \int \left(\phi(z) F(z) \right) G(z) \, d\mu$$

$$h \in \mathcal{H}ol(S) \cap C^b(\bar{S})$$

[Pf. $\quad u = c_1 1_A \qquad v = c_2 1_B$

$$\int \phi(z) F(z) G(z) \, d\mu =$$

$$c_1^{\alpha(z) p_t} (\operatorname{sgn} c_1) \,\&\, c_2^{\beta(z) q_t'} \operatorname{sgn} c_2 \cdot$$

$$\int \phi(z) 1_A 1_B \, d\mu \quad]$$

$$\| F(\sigma + is) \|_{p_\sigma} = 1 \qquad \forall \sigma \in [0,1], \ s \in \mathbb{R}$$

[Pf. $\quad \displaystyle\int \left| |u|^{\alpha(\sigma + is) p_t} \right|^{p_\sigma} d\mu = \|u\|^{p_t}_{L^{p_t}}$

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_0}$$

$$\alpha(\sigma + is) = \frac{1-\sigma}{p_0} + \frac{\sigma}{p_0} + \frac{-is}{p_0} + \frac{is}{p_1}$$

$$\left(\left| |u|^{\alpha(\sigma+is) p_t} \right|^{p_\sigma} = |u|^{\alpha(\sigma) p_\sigma p_t} = |u|^{p_t} \quad \right]$$

Similarly $\quad \| G(\sigma + is) \|_{q_\sigma'} = 1.$

$\Longrightarrow$

$$\| \phi(is) F(is) \|_{q_0} \leq M_0 \| F(is) \|_{p_0} = M_0$$

$$\Rightarrow |h(is)| = \left| \int \phi(is) F(is) G(is) \, d\mu \right|$$

$$\leq \| \phi(is) F(is) \|_{q_0} \| G(is) \|_{q_0'}$$

$$\leq M_0$$

Similarly $\quad |h(1+is)| \leq M_1$

3-Line Thm. $\Rightarrow$

$$\left| \int (\phi(\tau) u) v \, d\mu \right| = |h(\tau)| \leq M_\tau$$

$$\Rightarrow \left| \int (\phi(\tau) u) v \, d\mu \right| \leq M_\tau \|u\|_{p_\tau} \|v\|_{q_\tau'}$$

$(u, v \in \mathcal{S}(\Sigma_c)$

$$\Rightarrow \| \phi(\tau) u \|_{q_\tau} \leq M_\tau \|u\|_{p_\tau}. \qquad u \in \mathcal{S}(\Sigma_c)$$

17.7.2017

§ 31 Interpolation of semigroups.

(Ω, Σ, μ). $p_1 \in [1, \infty)$, $\theta \in (0, \frac{\pi}{2}]$ $\mathbb{K} = \mathbb{C}$

T bound hol C_0-sg of angle θ,

$$M_1 := \sup_{z \in \Sigma_\theta} \|T(z)\|_{\mathcal{L}(L^{p_1})} < \infty.$$

$p_0 \in [1, \infty]$, $p_0 \neq p_1$.

$$\|T(t)u\|_{p_0} \leq M_0 \|u\|_{p_0} \qquad u \in L^{p_1} \cap L^{p_0}$$

$\tau \in (0,1)$

$$M_\tau := M_0^{1-\tau} M_1^\tau, \quad \theta_\tau = \tau\theta,$$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$$

(31.1) __Theorem.__ For $z \in \Sigma_{\theta_\tau}$ $\exists!$ $T_{p_\tau}(z) \in \mathcal{L}(L^{p_\tau})$ consistent with $T(z)$. Moreover,

$$\|T_\tau(z)\| \leq M_\tau \qquad (z \in \Sigma_{\theta_\tau})$$

and

$$(T_\tau(z))_{z \in \Sigma_{\theta_\tau}}$$

is a hol. C_0-sg of angle θ_τ.

Proof. Let $0 < \theta' < \theta$

$$\psi : \bar{S} \longrightarrow \overline{\Sigma_{\theta'}} \setminus \{0\}$$

$$z \longmapsto e^{i\theta' \cdot z}$$

is bijective, holomorphic on S.

$$[\, z = is + i6 \qquad e^{i\theta' \cdot z} = e^{-\theta' s}\, e^{i\theta' 6} \,]$$

$$\phi := T \circ \psi : \bar{S} \to L(\mathcal{S}, L^1_{loc}(\Omega))$$

$$z \mapsto \int \phi(z) 1_B \, 1_C = \int_\Omega T(e^{i\theta' z}) 1_A \, 1_C \, d\mu$$

is holomorph and $\in C(\bar{S})$.

Stein $\Rightarrow$

$$\| T(\psi(\tau + is)) u \|_{p_\tau} \le M_\tau \| u \|_{p_\tau}.$$

$$\psi(\tau + is) = e^{i\theta'(\tau + is)} = e^{-\theta' s} e^{i\theta' \tau} \in \Sigma_{\theta_\tau}$$

$$w \in \Sigma_{\theta_\tau} \Rightarrow \exists \theta' \; \exists s \quad \psi(\tau + is) = w$$

$$\Rightarrow \quad \| T(w) u \|_{p_\tau} \le M_\tau \| u \|_{p_\tau} \qquad u \in \mathcal{S}$$

$$w \in \Sigma_{\theta_\tau}.$$

$$\Rightarrow \quad \exists \; T_{p_\tau}(w) \in \mathcal{L}(L^{p_\tau})$$ consistent with

$$T(w)$$

$$w \mapsto \int T(w)(1_A)\, 1_B$$ holomorphic $\Rightarrow$

$$T_{p_\tau} : \Sigma_{\theta_\tau} \to \mathcal{L}(L^{p_\tau})$$ holomorphic.

Strong continuity.

$$\| T_{p_\tau}(t) f - f \|_{L^{p_\tau}} \le \| T(t) f - f \|_{p_0}^{1-\tau} \, \| T(t) f - f \|_{p_1}^{\tau}$$

$\| T(t) f - f \|_{p_0} \downarrow 0$, $\| T(t) f - f \|_{p_1}$ bded.

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \qquad t \downarrow 0$$

$$\Rightarrow \quad T_{p_\tau}(t) f \to f \quad \text{in } L^{p_\tau} \quad \forall f \in \mathcal{S}$$

$\mathcal{S}$ dense $\Rightarrow$ $T_{p_\tau}(t) f \to f$ $\quad t \downarrow 0$

$\forall f \in L^{p_\tau}$. $\qquad \square$

§ 32 Invariance and interpolation.

$S \in \mathcal{L}(L^2(\mu))$

S substochastic $:\Longleftrightarrow$ $S \geq 0$ &
$\|Sf\|_1 \leq \|f\|_1$

S submarkovian $:\Longleftrightarrow$ $f \leq 1 \Rightarrow Sf \leq 1$

$\Longleftrightarrow$ $S \geq 0$ &
$\|Sf\|_\infty \leq \|f\|_\infty$

(32.1) Lemma. S submarkovian $\Longleftrightarrow$
S^* substochastic.

$H = L^2(\mu)$ $\quad a$ a densely defined closed
accretive form. $A \sim a$

$-A$ generates a C_0-sg. T

$\Leftrightarrow \quad a(\mathrm{Re}\, u, \mathrm{Im}\, u) \in \mathbb{R} \quad \forall u \in V$

$\Leftrightarrow \quad a(u,v) \in \mathbb{R} \quad \forall u,v \in V_{\mathbb{R}} := V \cap L^2(\Omega, \mathbb{R})$.

(b) We may assume that $\mathbb{K} = \mathbb{R}$

$C = \{ u \in L^2(\Omega; \mathbb{R}) : u \le 1 \}$

$Pu = u \wedge 1$

$u - Pu = u - u \wedge 1 = (u-1)^+$ $\square$

~~Thus ~~

(32.4) Theorem. T C_0-sg on L^2

a) Assume T is substochastic & submarkovian.

Then $\forall p \in [1, \infty)$ $\exists$ C_0-sg T_p

s.t. a) $T_2 = T$

b) $T_p(t) f = T_q(t) f \quad \forall f \in L^p \cap L^q$

consistency.

b) T_p is holomorphic for $1 < p < \infty$

$\| T_p(t) f \|_p \le \| f \|_p \qquad 1 \le p \le \infty$.

Pf. $\| T(t) f \|_\infty \le \| f \|_\infty$

$\| T(t) f \|_1 \le \| f \|_1$

Riesz Thorin $\Rightarrow$ $\| T(t) f \|_p \le \| f \|_p$.

& Continuity at 0: Let $t_n \downarrow 0$

$u \in L^p_{\mathbb{R}}(\Omega) \cap L^2(\Omega)$ Then $T(t_n) u \to u$

in L^2. Hence $T(t_n) u \to u$ a.e. after ss.

Lemma 32.5 $\Rightarrow$ $T(t_n) u \to u$ in L^p. $\square$

(32.5) Lemma. Let $1 \le p < \infty$ and

let $f_n, f \in L^p(\Omega)$ $f_n \to f$ a.e.

$\overline{\lim} \| f_n \|_p \le \| f \|_p$ $\Rightarrow$ $f_n \to f$ in L^p.

Pf. $\tilde{f}_n := (\operatorname{sgn} f_n) (|f| \wedge |f_n|)$

$\tilde{f}_n(x) \to f(x)$ a.e.

[1st. case $f(x) \neq 0$ & $f_n(x) \to f(x)$

$\Rightarrow \exists n_0 \ \forall n \ge n_0$ $|f_n(x)| \neq 0$ $\Rightarrow$

$\operatorname{sgn} f_n(x) = \dfrac{f_n(x)}{|f_n(x)|} \longrightarrow \dfrac{f(x)}{|f(x)|}$

$|f(x)| \wedge |f_n(x)| \longrightarrow |f(x)|$

hence $\tilde{f}_n(x) \longrightarrow f(x)$ ~~to~~ if ~~tt~~ $f(x) \neq 0$

$\tilde{f}_n(x) \to 0$.

2nd case $f(x) = 0 \Rightarrow$

Thus the claim follows from Lebesgue's

Theorem.

$$|f_n| = \underbrace{|\tilde{f}_n|}_{\cancel{|f| \wedge |f_n|}} + \underbrace{|f_n - \tilde{f}_n|}_{\cancel{}}$$

[Pf. a) $|f(x)| < |f_n(x)| \Rightarrow$

$$|\tilde{f}_n| + |f_n - \tilde{f}_n| = \underbrace{|f| \wedge |f_n|}_{|f|} + \left|f_n - \frac{f_n}{|f_n|}|f|\right|$$

$$= |f| + |f_n| \left|1 - \frac{|f|}{|f_n|}\right|$$

$$= |f| + \cancel{|f_n|}\,\big||f_n| - |f|\big|$$

$$= |f| + |f_n| - |f| = |f_n|.$$

b) $|f(x)| \geq |f_n(x)| \Rightarrow$

$$|\tilde{f}_n| + |f_n - \tilde{f}_n| = |f_n| + \left|f_n - \frac{f_n}{|f_n|}|f_n|\right|$$

$$= |f_n| + \cancel{|f_n| \cdot \frac{0}{|f_n|}} \quad]$$

$$\Rightarrow \quad |f_n|^p \geq |\tilde{f}_n|^p + |f_n - \tilde{f}_n|^p$$

$$\Big[\; c = a + b \quad \Rightarrow \quad c^p = (a+b)^p \geq a^p + b^p$$

$$\cancel{p(\log(a+b)) \geq \log(a^p + b^p)}$$

$$p \geq 1$$

$$\begin{aligned}(a+b)^p &= (a+b)(a+b)^{p-1} \\ &= a(a+b)^{p-1} + b(a+b)^{p-1} \\ &\geq a a^{p-1} + b b^{p-1} = a^p + b^p. \; \Big]\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \|f_n - \tilde{f}_n\|_p^p &\leq \int \left(|f_n|^p - |\tilde{f}_n|^p \right) \\ &= \|f_n\|_p^p - \|\tilde{f}_n\|_p^p \longrightarrow 0\end{aligned}$$

$$\Rightarrow \quad f_n = (f_n - \tilde{f}_n) + \tilde{f}_n \longrightarrow f \quad \text{in } L^p(\Omega). \; \square$$

§ 33 Elliptic operators.

$\Omega \subset \mathbb{R}^d$ open. , ~~$\mathbb{K} = \mathbb{R}$~~.

(33.1) Lemma. $u \in H_0^1(\Omega)$ real $\Rightarrow$

$u \wedge 1$, $(u-1)^+ \in H_0^1(\Omega)$ &

$D_j(u \wedge 1) = D_j u \, 1_{\{u<1\}}$

$D_j(u-1)^+ = D_j u \, 1_{\{u>1\}}$.

$a_{ij} \in L^\infty(\Omega)$ real valued , $0 < \alpha \leq 1$

$$\sum_{i,j=1}^d a_{ij}(x)\, \xi_i \overline{\xi_j} \geq \alpha |\xi|^2$$

$\forall \xi \in \mathbb{C}^d \quad \forall x \in \Omega$.

$H = L^2(\Omega)$

$D(a) = H_0^1(\Omega)$

$$a(u,v) = \int_\Omega \sum_{i,j=1}^d a_{ij}(x)\, \partial_i u(x)\, \overline{\partial_j v(x)}\, dx.$$

(33.2) Lemma. a is closed, accretive, dd.

Proof. $\operatorname{Re} a(u) \geq \alpha \int_\Omega |\nabla u|^2 \, dx$

$\Rightarrow$ accretive. &

$$\|u\|_a^2 = \operatorname{Re} a(u) + \|u\|_H^2 \geq \alpha \|u\|_{H^1}^2$$

Moreover,

$$\operatorname{Re} a(u) \leq c \int_\Omega \sum_{i,j=1}^d |\partial_i u| \, |\partial_j u| \, dx$$

$$= c \int_\Omega \Big(\sum_{j=1}^d |\partial_j u| \Big)^2 dx$$

$$\leq c \, 2^d \int_\Omega \sum_{j=1}^d |\partial_j u|^2 \, dx$$

Hence norms are equivalent $\Rightarrow$

closed. □

Rk $\displaystyle \Big(\sum_{j=1}^d a_j \Big)^2 \leq 2^d \sum_{j=1}^d a_j^2$

[$d \Rightarrow d+1$ $\displaystyle \Big(\sum_{j=1}^{d+1} a_j \Big)^2 \leq 2 \Big(\sum^d a_j \Big)^2 + 2 a_{d+1}^2$

$(a+b)^2 \leq 2(a^2+b^2)$ $\displaystyle \leq 2 \, 2^d \sum_{j=1}^d a_j^2 + 2 a_{d+1}^2$ □

(33.3) **Theorem.** There exist consistent C_0-sg T_p on L^p ~~consist~~ such that $T_2 = T$.

Moreover, $\|T_p(t)\| \leq 1$ $\quad 1 \leq p < \infty$,

Finally T_p is hol. for $1 < p < \infty$.

Rk. Even T_1 is holomorphic (more difficult)

ENDE