Semigroup Properties by Gaussian Estimates

by

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May 2, 1997

0   Introduction

It is a most interesting phenomenon that, in many cases, solutions of parabolic equations are given by kernels (so-called heat kernels) which can be estimated by the Gaussian kernel. Investigations in this direction go back at least to the sixties. We refer to the monographs of Davies [Da1], Robinson [R] and Varopoulos, Saloff-Coste, Coulhon [VSC] for the history and more information. Besides their intrinsic interest there are many different motivations to establish Gaussian estimates: they concern Hölder continuity of the solutions, Harnack's inequality, the construction of stochastic processes among others.

But more recently, it turned out that Gaussian estimates imply very important and interesting properties of the underlying semigroup. In this article we give an account on such implications. They concern spectral properties, regularity of the Cauchy problem and functional calculus. One of the most important results is that Gaussian estimates for a holomorphic semigroup on $L^2(\Omega)$ imply holomorphy of the semi-group in $L^1(\Omega)$. Whereas the other parts of the paper are surveys, we include a complete simple proof of this fact in Section 4 and 5 (which is somehow dispersed in the literature, otherwise).

It is striking that Gaussian estimates can be proved by quadratic form methods for elliptic operators with various boundary conditions under very mild regularity assumptions on the coefficients and the underlying domain. We give an account on such results in the second section adding some elementary special examples, as illustrations. Thus, as a concrete result, this paper contains a variety of examples

of holomorphic semigroups on $L^1(\Omega)$ generated by symmetric and non-symmetric elliptic operators with boundary conditions.

1   Gaussian estimates for semigroups.

Let $\Omega \subset I\!\!R^d$ be an open set, $1 \leq p < \infty$. We consider $L^p(\Omega)$ as a subspace of $L^p(R^d)$ identifying a function $f \in L^p(\Omega)$ with its extension to $I\!\!R^d$ by 0 . The Gaussian semigroup on $L^p(I\!\!R^d)$ is denoted by $G = (G(t))_{t \geq 0}$ where

$$(G(t)f)(x) = (4\pi t)^{-d/2} \int_{I\!\!R^d} e^{-|x-y|^2/4t} f(y)dy$$

$(x \in I\!\!R^d , t > 0 , f \in L^p(I\!\!R^d))$, $1 \leq p < \infty$. The generator of G is the Laplacian Δ on $L^p(I\!\!R^d)$ with distributional domain

$$D(\Delta) = \{f \in L^p(I\!\!R^d) : \Delta f \in L^p(I\!\!R^d)\} .$$

Definition 1.1   *A* C_0-*semigroup on* $L^2(\Omega)$ *satisfies a* **Gaussian estimate** *if there exist constants* $c > 0 , b > 0$ *such that*

(1.1) $$|T(t)f| \leq cG(bt)|f| \quad (0 < t \leq 1)$$

for all $f \in L^2(\Omega)$.

Here the inequality (1.1) is understood a.e. on Ω . If T satisfies the Gaussian estimate (1.1), then it is easy to see [Ar, Sec. 4] that

(1.2) $$|T(t)f| \leq ce^{\omega t}G(bt)|f| \quad (t \geq 0)$$

for all $f \in L^2(\Omega)$ where $\omega = \ln c$.

Next we explain how (1.2) is related to a representation of $T(t)$ by an integral kernel. Let $K \in L^\infty(\Omega \times \Omega)$. Then

(1.3) $$(B_K f)(x) = \int_\Omega K(x,y)f(y)dy$$

defines an operator $B_K \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Usually the following proposition is called the **Dunford-Pettis criterion**. We refer to [Ar Bu] for a short and elementary proof.

Proposition 1.2 *The mapping* $K \mapsto B_K$ *is an isometric isomorphism from* $L^\infty(\Omega \times \Omega)$ *onto* $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. *Moreover, this mapping is an order isomorphism; i.e.,* $B_K \geq 0$ *(i.e.,* $f \geq 0 \Rightarrow B_K f \geq 0$*) if and only if* $K(x,y) \geq 0$ *a.e.*

Let $B \in \mathcal{L}(L^p(\Omega))$, $1 \leq p \leq \infty$. We set

$$(1.4) \qquad \|B\|_{\mathcal{L}(L^1, L^\infty)} = \sup\{\|Bf\|_{L^\infty} : f \in L^1 \cap L^p , \|f\|_{L^1} \leq 1\} .$$

If $\|B\|_{\mathcal{L}(L^1, L^\infty)} < \infty$, it follows from the Dunford-Pettis criterion that there exists a unique kernel $K \in L^\infty(\Omega \times \Omega)$ such that $Bf = B_K f$ for all $f \in L^p \cap L^\infty$ and

$$(1.5) \qquad \|K\|_{L^\infty(\Omega \times \Omega)} = \|B_K\|_{\mathcal{L}(L^1, L^\infty)} .$$

We say that B is **represented by the kernel K**. If $B_0 \in \mathcal{L}(L^p(\Omega))$ is a second operator such that $|B_0 f| \leq B|f| (f \in L^p(\Omega))$, then it follows that B_0 is represented by a kernel $K_0 \in L^\infty(\Omega \times \Omega)$ such that $|K_0(x,y)| \leq K(x,y)$ a.e. As an immediate consequence we obtain an alternative more concrete description of Gaussian estimates.

Corollary 1.3 *A* C_0*-semigroup* T *on* $L^2(\Omega)$ *satisfies a Gaussian estimate if and only if* $T(t)$ *is represented by a kernel* $K(t, \cdot, \cdot) \in L^\infty(\Omega \times \Omega)$ *such that*

$$(1.6) \qquad |K(t,x,y)| \leq ce^{\omega t} t^{-d/2} \exp(-b|x-y|^2/t)$$

for all $t > 0$, *where* $c, b > 0$ *are constants.*

If T satisfies a Gaussian estimate, it can be extended to $L^p(\Omega) (1 \leq p \leq \infty)$ in the following way.

Proposition 1.4 *Let* T *be a* C_0*-semigroup on* $L^2(\Omega)$ *satisfying a Gaussian estimate. Then for* $1 \leq p < \infty$ *there exists a unique* C_0*-semigroup* T_p *on* $L^p(\Omega)$ *and a unique adjoint semigroup* T_∞ *on* $L^\infty(\Omega)$ *such that*

   a) $T_2(t) = T(t)$ $(t \geq 0)$;

   b) $T_p(t)f = T_q(t)f$ *if* $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \leq p, q \leq \infty$, $t > 0$.

Here we say that $S = (S(t))_{t \geq 0} \subset \mathcal{L}(L^\infty(\Omega))$ is an **adjoint semigroup** if there exists a C_0-semigroup $(R(t))_{t \geq 0}$ on $L^1(\Omega)$ such that $R(t)^* = S(t)$. The semigroups T_p are called **the extensions** of T to $L^p(\Omega)$, $1 \leq p \leq \infty$.

Proof. It is clear from domination that there exists $T_p(t) \in \mathcal{L}(L^p(\Omega))$ such that $T_p(t) = T(t)f$ for $f \in L^2 \cap L^p$, $1 \leq p < \infty$ and T_p satisfies the semigroup property. Strong continuity demands an additional argument (see [Ar] or [AE]). The semigroup T_∞ can be defined as follows. Also the adjoint $T^* = (T(t)^*)_{t \geq 0}$ satisfies a Gaussian estimate and so T_1^* can be defined as before. Define T_∞ as the adjoint semigroup of T_1^* . ∎

2 Examples.

We describe elliptic operators associated to sesquilinear forms. Let $\Omega \subset I\!\!R^d$ be an open set,

$$H^1(\Omega) = \{u \in L^2(\Omega) : D_j u = \frac{\partial u}{\partial x_j} \in L^2(\Omega) , j = 1 \cdots d\}$$

the first **Sobolev space** with norm

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} .$$

Let V be a closed subspace of $H^1(\Omega)$ which is dense in $L^2(\Omega)$. Let $a : V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Assume that a is **coercive**, i.e.,

$$(2.1) \qquad \operatorname{Re} a(u, u) + \lambda_0(u|u)_{L^2(\Omega)} \geq \alpha \|u\|^2_{H^1}$$

for all $u \in V$ and some $\lambda_0 \in I\!\!R, \alpha > 0$. Define the operator A associated with a by

$$D(A) = \{u \in V : \exists v \in L^2(\Omega) \text{ such that } a(u, \varphi) = (v|\varphi)_{L^2}$$
$$\text{for all } \varphi \in V\} , Au = v .$$

Then $-A$ generates a holomorphic C_0-semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(\Omega)$. We call it the **semigroup associated with a**. In the following, we give a series of examples of such semigroups which admit a Gaussian estimate.

First of all, we notice that the Gaussian semigroup itself is associated with the form a on $V = H^1(I\!\!R^d)$ given by $a(u, v) = \int_{I\!\!R^d} \nabla u \cdot \overline{\nabla v} \, dx$.

Example 2.1 (The Dirichlet Laplacian). *Let* $V = H^1_0(\Omega)$ *(the closure of the space* $C^\infty_c(\Omega)$ *of all test functions in* $H^1(\Omega)$*)* , $a(u, v) = \int \nabla u \overline{\nabla v}$. *The operator associated with* a *is denoted by* $-\Delta^D_\Omega$. *Then* $(e^{t\Delta^D_\Omega})_{t \geq 0}$ *is represented by a kernel* $k(t, \cdot, \cdot)$ *satisfying*

$$(2.2) \qquad 0 \leq k(t, x, y) \leq (4\pi t)^{-d/2} e^{-|x-y|^2/4t}$$

$(t > 0 , x, y \in I\!\!R^d)$.

This is well-known (cf. [AB 1], [AB 2]).

This estimate (2.2) is very special since it is a direct domination by the Gauss kernel with constants $c = b = 1$, $\omega = 0$ in (1.2).

Example 2.2 (Elliptic operator with constant coefficients). Let $C = (c_{ij})_{i,j=1\cdots d}$ be a strictly positive definit matrix and define the operator A on $L^2(I\!\!R^d)$ by

$$A = -\sum_{i,j=1}^{d} c_{ij} D_i D_j \ .$$

Thus A is associated with the form a defined on $V = H^1(I\!\!R^d)$ by $a(u,v) = \sum_{i,j=1}^{d} c_{ij} \int_{R^d} \nabla u \overline{\nabla v}\, dx$. Denote by $v_1, \cdots v_d > 0$ the eigenvalues of C . Then the kernel of e^{-tA} is given by

$$k(t,x,y) = (4\pi t)^{-d/2}(v_1 \cdots v_d)^{-\frac{1}{2}} \exp(-(C^{-1}(x-y)|(x-y))/4t) \ .$$

So we have the estimate

$$0 \leq k(t,x,y) \leq (4\pi t)^{-d/2}(v_1 \cdots v_d)^{-1/2} \exp(-b|x-y|^2/4t)$$

where $b = \left(\min_{k=1\cdots d} v_k\right)^{-1}$.

As an illustration we next consider the simplest example of a non-symmetric operator. Here we can also determine the kernel explicitly.

Example 2.3 Let $Af = -f'' + f'$ on $L^2(I\!\!R)$; i.e. A is associated with the form $a(u,v) = \int_R u'\overline{v}'\, dx + \int u'\overline{v}\, dx$, $V = H^1(I\!\!R)$. Then $e^{-tA} = G(t)S(t)$ where G is the Gaussian semigroup and $(S(t)f)(x) = f(x-t)$. Hence e^{-tA} is represented by the kernel

$$
\begin{aligned}
k(t,x,y) &= (4\pi t)^{-\frac{1}{2}} \exp(-(x-y-t)^2/4t) \\
&= (4\pi t)^{-\frac{1}{2}} \exp(-(x-y)^2/4t - t/4 + (x-y)/2) \ .
\end{aligned}
$$

By the old Young's trick we have

$$\frac{1}{2}(x-y) = \frac{1}{4}2\left((x-y)\cdot \varepsilon^{\frac{1}{2}}t^{-\frac{1}{2}}\right)\cdot\left(\varepsilon^{-\frac{1}{2}}t^{\frac{1}{2}}\right) \leq \frac{1}{4}\left\{(x-y)^2 t^{-1}\varepsilon + t\varepsilon^{-1}\right\} \ .$$

Choosing $\varepsilon = \frac{1}{2}$ one obtains

$$0 \leq k(t,x,y) \leq e^{t/4}(4\pi t)^{-\frac{1}{2}} \exp(-(x-y)^2/8t) \ .$$

Example 2.4 (the Neumann Laplacian). Let $V = H^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \overline{\nabla v}\, dx$. We denote the operator associated with a by $-\Delta_\Omega^N$.

a) Assume that Ω has a Lipschitz boundary. Then the semigroup $(e^{t\Delta_\Omega^N})_{t\geq 0}$ satisfies a Gaussian estimate, (see [Da1]) or Theorem 2.6 below). Here, in general, we cannot take $\omega = 0$ in the estimate (1.6). In fact, assume that Ω is bounded. Then

$$e^{t\Delta_\Omega^N} 1_\Omega = 1_\Omega \ (t \geq 0) \quad \text{whereas} \quad \lim_{t\to\infty} \|G(t)1_\Omega\|_{L^2} = 0 \ .$$

b) Some regularity property of Ω is needed in order to have Gaussian estimates. For example, if $d = 1$ and $\Omega = (0,1) \setminus \{\frac{1}{n} : n \in I\!N\}$. Then $e^{t\Delta_\Omega^N}$ is not compact since

$$e^{t\Delta_\Omega^N} 1_{(\frac{1}{n+1}, \frac{1}{n})} = 1_{(\frac{1}{n+1}, \frac{1}{n})} \text{ for all } n \in I\!N$$

so that the eigenvalue 1 has infinite multiplicity (cf. Proposition 3.1).

After these first examples we formulate some general results on elliptic operators.

Let a_{ij}, b_j, c_j, $c_0 \in L^\infty(\Omega)$, $i,j \in \{1, \cdots, d\}$. We assume throughout the ellipticity condition

$$(2.3) \qquad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \text{ for all } \xi \in I\!R^d \ x - a.e. \text{, where } \mu > 0 \text{.}$$

Moreover, we assume throughout that the second order coefficients are real-valued (cf. Remark 2.7). The first order coefficients may be complex in some cases. As before we consider a closed subspace V of $H^1(\Omega)$ which contains $H_0^1(\Omega)$. Define $a : V \times V \to \mathbb{C}$ by

$$(2.4) \qquad \begin{aligned} a(u,v) \quad &= \quad \sum_{i,j=1}^{d} \int_\Omega a_{ij}D_i u \overline{D_j v} + \sum_{j=1}^{d} \int_\Omega b_j D_j u \bar{v} + \\ &\quad \sum_{j=1}^{d} \int_\Omega c_j u \bar{v} + \int_\Omega c_0 u \bar{v} \text{ .} \end{aligned}$$

Then a is continuous, sesquilinear and coercive. We note by A_V the operator associated with V. First we consider Dirichlet boundary conditions; i.e., $V = H_0^1(\Omega)$.

Theorem 2.5 *Let $V = H_0^1(\Omega)$. Assume that the first order coefficients b_j, c_j are in $W^{1,\infty}(\Omega)$, $j = 1, \cdots, d$. Assume that one of the following conditions is satisfied*

a) $a_{ij} \in W^{1,\infty}(\Omega)$ $i,j = 1, \cdots, d$ or

b) the coefficients b_j, c_j are real-valued.

Then the semigroup $(e^{-tA_V})_{t \geq 0}$ satisfies a Gaussian estimate.

Notice that the semigroup is positive in the second case [Ou3]. For more general V the following holds.

Theorem 2.6 *Assume that all coefficients are real and $b_j, c_j \in W^{1,\infty}(\Omega)$ $(j = 1, \cdots d)$. For V we assume the following hypotheses:*

a) $v \in V$ implies $|v|, \inf\{|v|, 1\} \in V$,

b) $v \in V$, $u \in H^1(\Omega)$, $|u| \leq v$ implies $u \in V$,

c) $v \in V$ implies $b_j v$, $c_j v \in H_0^1(\Omega)$ $(j = 1 \cdots d)$.

In the case, where $V \neq H_0^1(\Omega)$ we assume in addition that Ω has Lipschitz boundary. Then $(e^{-tAv})_{t \geq 0}$ is a positive semigroup satisfying Gaussian estimates.

For example, assume that Ω has a Lipschitz boundary $\partial \Omega = \Gamma$ which can be written as $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are closed. Choose $V = \{u_\Omega : u \in C_c^\infty(I\!R^d \setminus \Gamma_1)\}^{-H^1(\Omega)}$. If b_j, $c_j = \{\varphi|_\Omega : \varphi \in C_c(I\!R^d \setminus \Gamma_2)\}^{-W^{1,\infty}(\Omega)}$ then the conditions a), b), c) are satisfied. In the case of the Laplacian and if $\Gamma_1 \cap \Gamma_2 = \emptyset$, this corresponds to Dirichlet boundary conditions on Γ_1 and Neumann boundary conditions on Γ_2.

Remark 2.7 Complex second order coefficients are a delicate matter. A counterexample on $I\!R^d$ for $d \geq 5$ is given by Auscher-Tchamitchian [AT]. However, Gaussian estimates can be proved on $I\!R^1$ and $I\!R^2$ also in the complex case, see Auscher-McIntosh-Tchamitchian [AMT].

For symmetric elliptic operators (i.e., $b_j = c_j = 0$, $a_{ij} = a_{ji}$) and $V = H_0^1(\Omega)$ or $H^1(\Omega)$, Theorem 2.5 and Theorem 2.6 are proved in [Da1]; for the general non-symmetric case presented here see [AE] where also more general boundary conditions (e.g., Robin's boundary conditions) are given. All proofs are based on Davies' trick (see [AE, Proposition 3.3] or [Da1]) which reduces the proof to an $\mathcal{L}(L^1, L^\infty)$-estimate of a semigroup obtained from e^{-tA} by a similarity transformation. However, there are various ways to prove the $\mathcal{L}(L^1, L^\infty)$-estimate, and they are the reason of the different hypotheses in Theorem 2.5 and 2.6. The proof of the first given in [AE] is the simplest one, based only on a non-symmetric Beurling-Deny criterion established by Ouhabaz ([Ou 1], [Ou 3]). It has the disadvantage to need a regularity hypothesis on the second order coefficients but allows complex first order coefficients. Davies [Da1] uses logarithmic Sobolev inequalites.

As a last example we consider Schrödinger operators.

Example 2.8 (Schrödinger semigroups). Let $A = \Delta - V$ on $L^2(I\!R^d)$ where $V : I\!R^d \to I\!R$ is measurable such that V_- is in Kato's class and $V_+ \in L_{loc}^1(I\!R^d)$. Then A (with suitable domain) is a self-adjoint operator which generates a C_0-semigroup T on $L^2(I\!R^d)$. Then by [Si, Prop. B. 6.7] T satisfies a Gaussian estimate.

Remark 2.9 Here we restricted our attention to the most common class of potentials. Concerning the L^p-theory of Schrödinger operators with more general potentials we refer to the recent article by Okazawa [Ok].

3 Consequences of Gaussian estimates.

Let $\Omega \subset I\!R^d$ be open and let T be a C_0-semigroup on $L^2(\Omega)$ satisfying a Gaussian estimate. It follows that T has many remarkable properties. Most of

them concern the solutions of the corresponding Cauchy problem, some concern spectral properties of the generator. By T_p we denote the extended semigroup on $L^p(\Omega)$ and by A_p its generator, $1 \leq p \leq \infty$. (By definition, $A_\infty = B_1^*$ where B_1 is the generator of $(T^*)_1$) .

3.1   Regularity of the semigroup .

In this subsection, we show that Gaussian estimates imply remarkable regularity properties of the semigroup, like norm continuity or holomorphy.

Proposition 3.1 *Assume that* Ω *is bounded. Then* $T_p(t)$ *is compact for all* $t > 0$, $1 \leq p \leq \infty$.

Proof. **a)** It follows from the kernel representation that $T_2(t)$ is Hilbert-Schmidt, $t > 0$. For $2 \leq p \leq \infty$ one can factorize $T_p(2t) = T_{2,\infty}(t) \circ T_2(t) \circ j$ where $j : L^\infty(\Omega) \to L^2(\Omega)$ is the injection and $T_{2,\infty}(t)$ is $T_1(t)$ seen as an operator from $L^2(\Omega)$ into $L^\infty(\Omega)$. Hence $T_p(2t)$ is compact.
b) For $1 \leq p \leq 2$, the result follows from **a)** by taking adjoints.   ∎

It follows in particular that T_p is norm continuous on $(0, \infty)$ for $1 \leq p < \infty$. But it seems to be unknown whether T_p is automatically holomorphic if Ω is bounded. If Ω is unbounded, this is not the case. Indeed, it has been shown by Voigt (oral communication) that the operator $\Delta + ix$, properly defined [LM], is the generator of a C_0-semigroup T on $L^2(I\!\!R)$ such that $|T(t)f| \leq G(t)|f|$ $(t \geq 0)$. But T is not holomorphic and not even eventually norm continuous. We are grateful to a clarifying discussion with F. Räbiger on this last point. However, if for some $p_0 \in [1, \infty)$, T_{p_0} is holomorphic, then all the semigroups T_p are holomorphic $(1 \leq p < \infty)$, and this holds for arbitrary open subsets Ω of $I\!\!R^d$. In the preceding section, we saw a large class of examples which all are holomorphic on $L^2(\Omega)$. Recall, that we assume throughout this section that T_2 satisfies a Gaussian estimate.

Theorem 3.2 If T_2 is holomorphic, then T_p is holomorphic for all $p \in [1, \infty)$.

First of all, it should be said that in the case $1 < p < \infty$, Theorem 3.2 is valid for every semigroup which extends to L_p , $1 \leq p < \infty$. This is a consequence of Stein's interpolation theorem (see [Da1, Thm. 1.4.2]). But, in general, the semi-group may fail to be holomorphic in L_1 (see [Da1, Thm. 4.3.6] for an example). The point in Theorem 3.2 is the case $p = 1$, and, for this case, applied to elliptic operators, it has several predecessors. In fact, Stewart [St1], [St2] proved holomorphy of the semigroup generated by elliptic operators on the space $C(\bar{\Omega})$ or $C_0(\Omega) = \{f \in C(\bar{\Omega}) : f_{|\partial\Omega} = 0\}$ for bounded regular Ω and for various boundary conditions. Using Stewart's result, Amann [Am] proved holomorphy on $L^1(\Omega)$ by skillful duality arguments. These results are valid also for operators which are not in divergence form. Then it was shown by Cannarsa and Vespri [CV] that a certain

class of more general elliptic operators generates an analytic semigroup in $L^1(\mathbb{R}^d)$ (see also the remark at the end of Section 4). For arbitrary open subsets of $\mathbb{R}^d$, it was shown in [AB 1], [AB 2] that the Laplacian with Dirichlet boundary conditions generates a holomorphic semigroup in $L^1(\Omega)$.

Now, Theorem 3.2 implies in particular that the elliptic operators considered in Theorem 2.5, 2.6 generate holomorphic semigroups in $L^1(\Omega)$.

In view of Example 2.8, it also follows that Schrödinger semigroups are holomorphic in all $L^p(\mathbb{R}^d)$, $(1 \leq p \leq \infty)$. This had been shown before by Kato [K2]. In the general version, Theorem 3.2 is obtained in [AE] and [H2]. We give a complete proof of Theorem 3.2 in Section 4.

3.2 Spectral properties.

In the following we denote by $\rho_\infty(A_p)$ the connected component of the resolvent set $\rho(A_p)$ which contains a right half-plan. Then the following holds.

Theorem 3.3 *One has*

$$\rho_\infty(A_p) = \rho_\infty(A_2) \quad (1 \leq p \leq \infty).$$

In particular, if A is self-adjoint then the spectrum $\sigma(A_p)$ is independent of $p \in [1, \infty]$.

Theorem 3.3 has been proved in [Ar] after a proof for Schrödinger operators had been given by Hempel and Voigt [HV]. It implies in particular, that the spectrum of the elliptic operators given in Theorem 2.5 and 2.6 is independent of $p \in [1, \infty]$. This is remarkable since the spectrum of (very simple) self-adjoint degenerate elliptic operators may strongly depend on p (see [Ar]).

It is interesting that Theorem 3.3 is also true if, instead of the Gaussian kernel, estimates by more general kernels are considered. This is done by Miyajima and Ishikawa [MI] where the Gaussian semigroup is replaced by the semigroup generated by $-(I - \Delta)^\alpha$ $(\frac{1}{2} < \alpha \leq 1)$. In Section 5 we consider higher order Gaussian estimates. For symmetric operators Davies [Da 2] obtains spectral L^p-independence as consequence of a functional calculus designed for symmetric operators satisfying kernel estimates.

However, the full independence in the non-symmetric case is still open.

Problem. Is the spectrum of A_p independent of $p \in [1, \infty]$ (also if $\rho(A_2)$ is not connected)?

We should mention, that spectral L^p-independence is trivial when Ω is bounded. Then the resolvent of A_p is compact (by Proposition 3.1), $1 \leq p \leq \infty$. Thus $\sigma(A_p)$ consists of eigenvalues only, and since $T_p(t)$ maps $L^p(\Omega)$ into $L^\infty(\Omega)$, the point spectrum is the same for all operators A_p $(1 \leq p \leq \infty)$. Some other partial answers involving conditions on the domain are given recently by Hieber and Schrohe [HS].

3.3 H^∞-functional calculus and maximal regularity.

It has been discovered recently that Gaussian estimates imply the existence of an H^∞-functional calculus (see Duong [Du], Duong-Robinson [DR], Hieber [H1]). More precisely, if A_2 has an H^∞-functional calculus then the same is valid for A_p, $1 < p < \infty$. This is a very useful result since H^∞-functional calculus is easy to obtain on a Hilbert space; e.g. contractivity of the semigroup suffices. The following result is due to Duong and Robinson who proved it for $\Omega = \mathbb{R}^d$ or more generally, for homogeneous spaces Ω (see [AE, Theorem 5.5] for the general result).

Theorem 3.4 *Assume that T is contractive on $L^2(\Omega)$ and holomorphic of angle $\theta \in (0, \frac{\pi}{2})$. Then $-A_p$ has an $H^\infty(\Sigma(\nu))$-functional calculus for all $\nu > \frac{\pi}{2} - \theta$.*

In view of the Dore-Venni theorem H^∞-functional calculus implies maximal regularity for the non-homogeneous Cauchy problem. Very recently, Hieber and Prüß [HP] gave a direct proof of maximal regularity assuming Gaussian estimates without passing by the Dore-Venni theorem.

Theorem 3.5 *Let $1 < p, q < \infty$ and assume that $\partial\Omega$ has Lebesgue measure 0. Assume further that T is holomorphic. Let $\tau > 0$ and $f \in L^p((0, \tau); L^q(\Omega))$. Consider the function $u(t) = \int_0^t T_q(t - s) f(s)\, ds$. Then $u \in W^{1,p}((0, \tau); L^q(\Omega)) \cap L^p((0, \tau); D(A_q))$, where $D(A_q)$ is considered as a Banach space with the graph norm.*

Notice that we assume that T satisfies a Gaussian estimate throughout this section. It seems to be an open problem whether Theorem 3.5 is valid without this assumption. In fact, it is if $p = 2$. Let us make this point more precise.

Remark 3.6 (maximal regularity). Let B be the generator of a C_0-semigroup T on a Banach space X. Let $1 < p < \infty$ and $f \in L^p((0, \tau); X)$ where $\tau > 0$. Then

$$u(t) = \int_0^t T(t - s) f(s)\, ds$$

is the mild solution of the Cauchy problem

$$(CP) \qquad \begin{cases} \dot{u}(t) &= Bu(t) \quad 0 \leq t < \tau \\ u(0) &= 0 . \end{cases}$$

One says that B **satisfies maximal regularity** if

$$u \in W^{1,p}((0, \tau); X) \cap L^p((0, \tau); D(B)) \quad \text{for all } f \in L^p((0, \tau); X).$$

It turns out that this property is independent of $p \in (1, \infty)$. Assume that S is holomorphic. Then deSimon [dS] showed in 1964 that B is maximal regular if X is a Hilbert space. This result does not hold for $X = L^1$ but it seems to be an open problem whether it holds for $X = L^p$, $1 < p < \infty$. Particular cases are known, though. For example, Lamberton [La] proved 1987 maximal regularity if the semigroup is contractive for the L^r-norm for all $1 \leq r \leq \infty$. Theorem 3.5 is another particular solution of this open problem.

Remark 3.7 Theorem 3.5 is proved in [HP, Theorem 3.1] for $\Omega = I\!\!R^d$ (and more generally, for topological spaces satisfying the doubling property). As in [AE, Theorem 5.5] the more general result follows from this by a direct sum argument:
Let $\Omega_1 = I\!\!R^d \setminus \bar{\Omega}$. Since $\partial\Omega$ is a null set we have $L^2(I\!\!R^d) = L^2(\Omega) \oplus L^2(\Omega_1)$. Consider on $L^2(\Omega_1)$ the operator Δ_{Ω_1} and apply the result [HP, Theorem 3.1] to the direct sum of A and Δ_{Ω_1} considering $f : (0, \infty) \to L^2(\Omega)$ as a $L^2(I\!\!R^d)$-valued function. Then the claim follows from the result on $I\!\!R^d$. ■

Concerning elliptic operators of order 2 (as defined in Theorem 2.5, 2.6) Lamberton's direct result is more suitable than the general Theorem 3.5 based on Gaussian estimates. In fact, the following theorem hold.

Theorem 3.8 *Let* $\Omega \subset I\!\!R^d$ *be an arbitrary open set. Let* $a_{ij} \in L^\infty(\Omega)$ *be real coefficients satisfying the ellipticity condition (2.9) and let* $b_j, c_j \in W^{1,\infty}(\Omega)$ *(j =* $1, \cdots d)$ *,* $c_0 \in L^\infty(\Omega)$ *complex. Let* V *be a closed subspace of* $H^1(\Omega)$ *containing* $H_0^1(\Omega)$ *. Assume that*

(3.1) $$u \in V \quad implies \quad \inf\{1, |u|\}\, \text{sign}\, u \in V$$

where

$$\text{sign}\, u \begin{cases} u(x)/\overline{u(x)} & if \quad u(x) \neq 0\,, \\ 0 & if \quad u(x) = 0\,. \end{cases}$$

Define the form a *on* $V \times V$ *by (2.10). Then* a *is continuous, sesquilinear and coercive with respect to* $L^2(\Omega)$ *. Denote by* A_V *the operator associated with* a *and by* S *the semigroup on* $L^2(\Omega)$ *generated by* $-A_V$ *. Then there exists a family of* C_0*-semigroups* S_p *on* $L^p(\Omega)$ *,* $1 < p < \infty$ *, such that* $S_2(t) = S(t)$ *(t $\geq$ 0) and* $S_p(t)f = S_q(t)f$ *(f $\in L^p \cap L^q$, t $\geq$ 0) for* $1 < p, q < \infty$ *. Denote by* A_p *the generator of* S_p *. Then* A_p *satisfies maximal regularity.*

Proof. The semigroup S is holomorphic on $L^2(\Omega)$. It follows from [Ou1] or [Ou3, Theorem 4.2 and Remark 4.3] that $\|e^{-\omega t}S(t)f\|_{L^p} \leq \|f\|_{L^p}$ for all $f \in L^2 \cap L^\infty$ and all $t \geq 0$, $1 \leq p \leq \infty$ and for some $w \in I\!\!R$. Now it follows from the above mentioned theorem by Lamberton [La, Theorem 1] that $A_p - w$ satisfies maximal regularity. Then also A_p satisfies maximal regularity. ■

We mention, however, that the above proof is restricted to second order elliptic operators. On the other hand, Theorem 3.5 is also valid if other kernel estimates are assumed and can be applied in particular to Example 5.2 below.

4 Proof of holomorphy by Gaussian estimates .

In this section we give a complete proof of Theorem 3.2. First of all we need to choose holomorphic representations of the kernel (cf. [Ar-Bu, Lemma 3.4]).

Lemma 4.1 *Let* $D \subset \mathbb{C}$ *be open and* $F : D \to L^\infty(S)$ *holomorphic where* (S, Σ, μ) *is a measure space. Then there exists* $f : D \times S \to \mathbb{C}$ *such that* $f(z, \cdot) = F(z)$ *a.e. and* $f(\cdot, x)$ *is holomorphic for all* $x \in S$.

Proof. Let $B = B(z_0, r) = \{z \in \mathbb{C} , |z - z_0| < r\}$ such that $\bar{B} \subset D$. Then there exist $a_n \in L^\infty(S)$ such that $\sum_{n=0}^\infty \|a_n\|_\infty r^n < \infty$ and $F(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$.

Define $h : B \times S \to \mathbb{C}$ by $h(z, x) = \sum_{n=0}^\infty a_n(x)(z - z_0)^n$. Then $h(\cdot, x) : B \to \mathbb{C}$ is holomorphic for all $x \in S$ and $h(z, \cdot) = F(z)$ in $L^\infty(S)$. Let B_1, B_2 be two such discs such that $\bar{B_1} \subset D$, $\bar{B_2} \subset D$ and let $h_j : B_j \times D \to \mathbb{C}$ be functions $(j = 1, 2)$ such that $h_j(z, \cdot) = F(z)$ in $L^\infty(S)$ $(z \in B_j)$ and such that $h_j(\cdot, x)$ is holomorphic on B_j for all $x \in S$. If $B_1 \cap B_2 \neq \emptyset$, then $h_1(z, x) = h_2(z, x)$ for all $z \in B_1 \cap B_2$, $x \in S$ by the identity theorem. Now it suffices to cover D by such discs. ∎

The next criterion is very convenient to prove holomorphy of vector-valued functions. A subset W of X^* is called **separating** if for all $x \in X$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$.

Theorem 4.2 *Let* $D \subset \mathbb{C}$ *be open and* $f : D \to X$ *be locally bounded such that* $\varphi \circ f$ *is holomorphic for all* $\varphi \in W$ *where* $W \subset X^*$ *is separating. Then* f *is holomorphic.*

In the case where W is a norming subspace (i.e. if $\|x\|_W = \sup\{|\varphi(x)| : \varphi \in W , \|\varphi\| \leq 1\}$ is an equivalent norm), then this theorem can be found in [K1, p. 139]. The general case, proved in [AN], is much more convenient, but also the more special hypothesis can be checked in the situation considered in the following proof. Next we recall the following well-known version of the Phragmen-Lindelöf theorem [Co, cor. 6.4.4].

Proposition 4.3 *Let* $\gamma \in (0, \frac{\pi}{2})$ *and let* $D = \{re^{i\theta} : r > 0 , 0 < \theta < \gamma\}$. *Let* $k : \bar{D} \to \mathbb{C}$ *be continuous, holomorphic on* D *such that* $|h(z)| \leq \alpha \exp(\beta|z|)$ $(z \in D)$ *where* $\alpha, \beta > 0$. *If* $|h(r)| \leq M$, $|h(re^{i\gamma})| \leq M$ *for all* $r > 0$, *then* $|h(z)| \leq M$ *for all* $z \in D$.

The following consequence is a simplified version of [Dal, Thm. 3.4.8] which suffices for our purposes.

Lemma 4.4 *Let* $\gamma \in (0, \frac{\pi}{2})$ *and let* $k : \overline{\Sigma(\gamma)} \to \mathbb{C}$ *be continuous, holomorphic in* $\Sigma(\gamma) := \{re^\alpha : r > 0, |\alpha| < \gamma\}$ *such that*

a) $|k(z)| \leq c$ $(z \in \Sigma(\gamma))$ *and*

b) $|k(t)| \leq ce^{-b/t}$ $(t > 0)$,

where $c, b > 0$. *Then for* $0 < \theta_1 < \gamma$ *one has*

$$|k(z)| \leq c \cdot \exp(-b_1/|z|) \quad (z \in \Sigma(\theta_1))$$

where $b_1 = \frac{\sin(\gamma-\theta_1)}{\sin\gamma} \cdot b$.

Proof. Let $g(z) = k(z^{-1}) \cdot \exp\{be^{i(\frac{\pi}{2}-\gamma)} \cdot z\frac{1}{\sin\gamma}\}$. Then

(i) $|g(r)| \leq ce^{-b \cdot r} \exp\{b\operatorname{Re} e^{i(\frac{\pi}{2}-\gamma)} \cdot r\frac{1}{\sin\gamma}\} = c$ $(r > 0)$;

(ii) $|g(re^{i\gamma})| \leq c \cdot \exp\{b\operatorname{Re}(e^{i(\frac{\pi}{2}-\gamma)}re^{i\gamma})\frac{1}{\sin\gamma}\} = c$,

(iii) $|g(z)| \leq c\exp\{\frac{b}{\sin\gamma}|z|\}$, $z \in \Sigma(\gamma)$.

It follows from Proposition 4.3 that $|g(z)| \leq c$ for all $z \in D$. Replacing g by $z \mapsto \overline{g(\bar{z})}$ we see that $|g(z)| \leq c$ for all $z \in \Sigma(\gamma)$. Hence

$$
\begin{aligned}
|k(z)| &= |g(z^{-1})\exp(-be^{i(\frac{\pi}{2}-\gamma)}z^{-1}\tfrac{1}{\sin\gamma})| \\
&\leq c \cdot \exp(-b/r \cdot \operatorname{Re} e^{i(\frac{\pi}{2}-\gamma-\theta)}\tfrac{1}{\sin\gamma}) \\
&= c\exp(-b/r \, \sin(\gamma+\theta)/\sin\gamma) \\
&\leq c\exp(-b \cdot \sin(\gamma-|\theta|)/(r \cdot \sin\gamma))
\end{aligned}
$$

for $z = re^{i\theta}$, $|\theta| \leq \gamma$. ∎

Lemma 4.5 *Let* $K : \Omega \times \Omega \to \mathbb{C}$ *measurable,* $h \in L^1(\mathbb{R}^d)$ *such that* $|K(x,y)| \leq h(x-y)$ $(x, y \in \Omega)$. *Then*

$$(B_K f)(x) = \int K(x,y)f(y)dy$$

defines a bounded operator B_K *on* $L^p(\Omega)$ *and* $\|B_K\|_{\mathcal{L}(L^p)} \leq \|h\|_{L^1(\mathbb{R}^d)}$.

This is immediate from Young's inequality. For the proof of Theorem 3.2 we use the following terminology. Let $S = (S(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. We say that S **is holomorphic of angle** $\theta \in (0, \frac{\pi}{2}]$ if S has a holomorphic extension (still denoted by S) from the sector $\Sigma(\theta)$ into $\mathcal{L}(X)$ which is bounded on the sets $\{re^{i\alpha} : 0 < r \leq 1, |\alpha| \leq \theta - \varepsilon\}$ for all $\varepsilon > 0$. Then the following is easy to show (cf. [P, Thm. 2.5.2 $(d) \Rightarrow (a)$]):

(4.1) $$S(z_1)S(z_2) = S(z_1 + z_2) \quad (z_1, z_2 \in \Sigma(\theta));$$

$$(4.2) \qquad \lim_{\substack{z \to 0 \\ z \in \Sigma(\theta')}} S(z)x = x \quad (x \in X) \text{ for all } \theta' \in (0, \theta)$$

$$(4.3) \qquad \begin{array}{l} \text{for all } \theta' < \theta \text{ there exist } M \geq 0, w \in I\!R \text{ such that} \\ \|S(z)\| \leq M e^{|z|w} \quad (z \in \Sigma(\theta')) \,. \end{array}$$

Proof of Theorem 3.2 Assume that T is a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ on $L^2(\Omega)$ which satisfies a Gaussian estimate. Let $0 < \theta_1 < \theta$. Choose $\theta_1 < \gamma < \theta_2 < \theta_3 < \theta$. Replacing T by $(e^{-wt}T(t))_{t \geq 0}$ we can assume that

$$(4.4) \qquad \|T(z)\|_{\mathcal{L}(L^2)} \leq \text{ const } \quad (z \in \Sigma(\theta_3)) \,;$$

$$(4.5) \qquad |T(t)f| \leq \text{ const } G(bt)|f| \quad (t \geq 0 \,, \, f \in L^2(\Omega)) \,.$$

From this follows

$$(4.6) \qquad \|T(t)\|_{\mathcal{L}(L^1, L^2)} \leq \text{ const } t^{-d/4} \quad (t > 0) \,;$$

$$(4.7) \qquad \|T(t)\|_{\mathcal{L}(L^2, L^\infty)} \leq \text{ const } t^{-d/4} \quad (t > 0) \,.$$

(In fact, one sees from the kernel and Proposition 1.2 that $\|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq$ const $t^{-d/2}$. Since T is bounded on L^1 and L^∞, (4.6) and (4.7) follow from the Riesz-Thorin theorem). Choose $\delta \in (0, 1)$ such that $\delta t + is \in \Sigma(\theta_3)$ whenever $t + is \in \Sigma(\theta_2)$. Let $z = t + is \in \Sigma(\theta_2)$. Then by (4.4), (4.6), (4.7), $\|T(z)\|_{\mathcal{L}(L^1, L^\infty)} \leq \|T((1-\delta)t/2\|_{\mathcal{L}(L^1, L^2)}\|T(\delta t + is)\|_{\mathcal{L}(L^2)} \cdot \|T((1-\delta)t/2)\|_{\mathcal{L}(L^1, L^\infty)} \leq$ const $t^{-d/2} =$ const $(\text{Re}z)^{-d/2}$.

It follows from Theorem 4.2 that the mapping $z \mapsto T(z)_{|L^1} : \Sigma(\theta_2) \to \mathcal{L}(L^1, L^\infty) = L^\infty(\Omega \times \Omega)$ is holomorphic. By Lemma 4.1 there exists $K : \Sigma(\theta_2) \times \Omega \times \Omega \to \mathbb{C}$ such that $K(\cdot, x, y)$ is holomorphic for all $x, y \in \Omega$, $K(z \cdot, \cdot) \in L^\infty(\Omega \times \Omega)$ and $(T(z)f)(x) = \int_\Omega K(z, x, y)f(y)dy \quad (f \in L^1 \cap L^2)$. Moreover,

$$(4.8) \qquad |K(z, x, y)| \leq \text{ const } (\text{Re}z)^{-d/2} \quad (z \in \Sigma(\theta_2))$$

and by (4.5),

$$(4.9) \qquad |K(t, x, y)| \leq \text{ const } t^{-d/2} \exp(-b|x-y|^2/t) \quad (t > 0) \,.$$

Applying Lemma 4.4 to the function $z^{d/2}K(z, x, y)$ we obtain a constant $c > 0$ such that for $b_1 = b \cdot \frac{\sin(\gamma - \theta_1)}{\sin \gamma}$

$$(4.10) \qquad |K(z, x, y)| \leq c \cdot |z|^{-d/2} \exp(-b_1|x-y|^2/|z|)$$

for all $z \in \bar{\Sigma}(\theta_1)$ and all $x, y \in \Omega$. It follows from Lemma 4.5 that

$$\sup_{z \in \Sigma(\theta_1)} \|T(z)\|_{\mathcal{L}(L^p)} < \infty \,, \, 1 \leq p < \infty \,.$$

Thus there exist operators $T_p(z) \in \mathcal{L}(L^p)$ such that $T_p(z)f = T(z)f$ $(f \in L^p \cap L^2)$ $(z \in \Sigma(\theta_1))$. It follows from Theorem 4.2 that $T_p(\cdot) : \Sigma(\theta_1) \to \mathcal{L}(L^p(\Omega))$

is holomorphic. This finishes the proof. ■

As application we obtain that the operators described in Theorem 2.5, 2.6 generate holomorphic semigroups in all spaces $L^p(\Omega)$, $1 \leq p < \infty$. In the case of Dirichlet boundary conditions, Vespri [V] proved holomorphy on $L^1(\Omega)$ for real L^∞-coefficients conditions. This is more general than our result based on Theorem 2.5, 2.6 where some regularity assumption on the first order coefficients is needed. However, Vespri's proof is not elementary at all, using estimates due to De Giorgi and variants of Stewart's technique. In addition some regularity assumptions on the boundary of Ω are needed for his proof.

5   Higher order Gaussian estimates and holomorphy .

So far, we considered merely Gaussian estimates of order 2. However, the proof of Theorem 3.2 carries over to higher order with small alterations. We first give the precise definition and examples for the higher-order-case, following Hieber [H2]. Let $m \in I\!N \setminus \{1\}$. We had considered $m = 2$, before. Let

$$k_t(x) = t^{-d/m} \exp(-|x|^{m/(m-1)} \cdot t^{-1/(m-1)}) \quad (t > 0 , \ x \in I\!R^d)$$
$$G(t)f = k_t * f \quad (f \in L^2(I\!R^d)) .$$

Let $\Omega \subset I\!R^d$ be open.

Definition 5.1 *A C_0-semigroup T on $L^2(\Omega)$ satisfies a Gaussian estimate of order m if there exist constants $w \in I\!R$, $b, M > 0$ such that*

$$|T(t)f| \leq M e^{wt} G(bt)|f| \quad (t > 0) .$$

As in the case of order 2, this implies that there exist C_0-semigroups T_p on $L^p(\Omega)$ such that

a) $T(t) = T_2(t) \quad (t > 0)$;

b) $T_p(t)f = T_q(t)f \quad (f \in L^p \cap L^q , \ t > 0)$.

As before, T can be represented by a bounded integral kernel.

Example 5.2 *Let $\rho \in (0,1)$, $a_\alpha \in BUC^\rho(I\!R^d, \mathbb{C})$ for $|\alpha| = m$ and $a_\alpha \in L^\infty(I\!R^d, \mathbb{C})$ for $|\alpha| \leq m$. Suppose that there exists $\delta > 0$ such that*

$$\text{Re} \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha < -\delta|\xi|^2$$

for all $x \in I\!R^d$, $\xi \in I\!R^d$. Define the elliptic operator A on $L^2(I\!R^d)$ by

$$D(A) = H^m(I\!R^d) ; \ Af = \sum_{|\alpha|\leq m} a_\alpha D^\alpha f .$$

Then A generates a holomorphic C_0-semigroup T on $L^2(\mathbb{R}^d)$ which satisfies a Gaussian estimate of order m .

See [F, Theorem 9.4.2] for a proof.

Theorem 5.3 *Let T be a holomorphic C_0-semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on $L^2(\Omega)$. Assume that T satisfies a Gaussian estimate of order $m \geq 2$. Then the extended C_0-semigroup T_p on $L^2(\Omega)$, $1 \leq p < \infty$, are holomorphic of angle θ .*

Proof. The proof of Theroem 3.2 carries over if the following modifications are made. From (4.5) one deduces here

$$(5.1) \qquad \|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq \text{ const } t^{-d/m} .$$

Since T is bounded on L^1 and L^∞ the Riesz-Thorin theorem gives

$$(5.2) \qquad \|T(t)\|_{\mathcal{L}(L^1, L^2)} \leq \text{ const } t^{-d/2m} .$$

$$(5.3) \qquad \|T(t)\|_{\mathcal{L}(L^2, L^\infty)} \leq \text{ const } t^{-d/2m} .$$

As before one deduces from this

$$(5.4) \qquad \|T(z)\|_{\mathcal{L}(L^1, L^\infty)} \leq \text{ const } (\text{Re} z)^{-d/m} \quad (z \in \Sigma(\theta_2)) .$$

Thus the kernel $K(z; x, y)$ satisfies

$$(5.5) \qquad |K(z, x, y)| \leq \text{ const } |z|^{-d/m} \quad (z \in \Sigma(\theta_2)) ,$$

$$(5.6) \qquad |K(t, x, y)| \leq \text{ const } t^{-d/m} \exp(-b|x - y|^{m\beta} \cdot t^{-\beta})$$

where $\beta = \frac{1}{m-1}$. Let $k(z) = z^{d/m\beta} k(z^{\frac{1}{\beta}}, x, y)$. Then

$$|k(z)| \leq \text{ const } (z \in \Sigma(\gamma)) \text{ and }$$
$$|k(z)| \leq \text{ const } \cdot \exp(-b|x - y|^{\beta m} \cdot t^{-1}) .$$

By Lemma 4.4 we obtain $|k(z)| \leq \text{ const } \cdot \exp(-b_1|x - y|^{\beta m}|z|^{-1})$ $(x, y \in \Omega$, $z \in \Sigma(\theta_1)$) where $b_1 = \frac{\sin(\gamma - \theta_1)}{\sin \gamma} b$. Now the conclusion follows as before. ∎

Theorem 5.3 is due to Hieber [H2]. He uses the resolvent estimate characterizing holomorphy, which is more complicated to prove, it seems.

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