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GAUSSIAN ESTIMATES AND INTERPOLATION OF THE SPECTRUM IN L^p

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In memoriam Peter Hess

Abstract. It is shown that the spectrum of a uniformly elliptic operator on $L^p(\Omega)$ with Dirichlet or Neumann boundary conditions is independent of $p \in [1, \infty)$.

0. Introduction. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $T_p = (T_p(t))_{t\geq 0}$ be consistent C_0-semigroups on $L^p(\Omega)$ with generators A_p $(1 \leq p < \infty)$. It is natural to ask whether the spectrum $\sigma(A_p)$ of A_p is independent of $p \in [1, \infty)$. This is not the case in general (see Hempel-Voigt [13], [14], Davies [9, 4.3], Jörgens [15]); here we give a particularly simple example: if

$$(A_p f)(x) = -x\, f'(x)$$

on $L^p(0, \infty)$, then $\sigma(A_p) \cap \sigma(A_q) = \emptyset$ for $p \neq q$; see Section 3.

Our main result is the following: assume that A_2 is self-adjoint and T_2 satisfies an upper Gaussian estimate. Then $\sigma(A_p)$ is independent of $p \in [1, \infty)$.

Gaussian estimates have been studied extensively; see the books of Davies [9], Robinson [21] and Varopoulos, Saloff-Coste, Coulhon [27]. In particular, if A_p is a self-adjoint second order differential operator with Dirichlet or Neumann boundary conditions such estimates are known to hold, and thus, by our result, $\sigma(A_p)$ is independent of $p \in [1, \infty)$. Also, if $A_p = \Delta - V$ is a Schrödinger operator on $L^p(\mathbb{R}^N)$ Gaussian estimates have been established (see [24]). Thus our result generalizes that of Hempel-Voigt [14] who prove p-independence in that case. In fact, we use the same strategy as Hempel and Voigt, and show that the resolvent consists of integral operators whose kernels can be estimated. But instead of regularizing by considering powers of the resolvent as in [14] we regularize by the semigroup (see (6.9)). This simplifies the proof and gives more precise results : we obtain that the resolvent consists entirely of regular integral operators (cf. [4]).

The paper is organized as follows. In Section 1 we consider the much easier case where Ω is bounded. Consistency of the resolvents is studied in Section 2. In fact,

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the counterexamples in Section 3 show not only that the spectrum may vary with p but also that $R(\lambda, A_p)$ and $R(\lambda, A_q)$ may not be consistent for $\lambda \in \rho(A_p) \cap \rho(A_q)$ (see also [6] and [12]). The main results are formulated in Section 4 and proved in Section 6. Examples are described in Section 5.

1. Subspaces. Let E, F be Banach spaces such that $F \hookrightarrow E$ (by this we mean that F is a subspace of E and the inclusion is continuous). Let A be an operator on E. By $\sigma(A)$ we denote the spectrum and by $\rho(A) = \mathbb{C}\backslash\sigma(A)$ the resolvent set of A. The resolvent of A in $\lambda \in \rho(A)$ is denoted by $R(\lambda, A) = (\lambda - A)^{-1}$. We denote by A_F the part of A in F, i.e., A_F is given by $D(A_F) = \{x \in D(A) \cap F : Ax \in F\}$, $A_F x = Ax$.

Proposition 1.1. *Assume that there exists $\mu \in \rho(A)$ such that $R(\mu, A)F \subset F$ and that there exists $k \in \mathbb{N}$ such that $D(A^k) \subset F$. Then $\sigma(A) = \sigma(A_F)$ and $R(\lambda, A_F) = R(\lambda, A)_{|F}$ for all $\lambda \in \rho(A)$.*

Proof. a) Let $\lambda \in \rho(A)$. Iteration of the resolvent equation $R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)R(\lambda, A)$ yields

$$R(\lambda, A) = \sum_{j=1}^{k} (\mu - \lambda)^{j-1} R(\mu, A)^j + (\mu - \lambda)^k R(\mu, A)^k R(\lambda, A). \tag{1.1}$$

This shows that $R(\lambda, A)F \subset F$. It is now obvious that $\lambda \in \rho(A_F)$ and $R(\lambda, A_F) = R(\lambda, A)_{|F}$.

b) Conversely, let $\lambda \in \rho(A_F)$. The space $D(A^k)$ is a Banach space for the norm $\|x\|_{D(A^k)} = \|(\mu - A)^k x\|_E$ and $D(A^k) \hookrightarrow E$. Since $F \hookrightarrow E$ it follows from the closed graph theorem that $D(A^k) \hookrightarrow F$. Note that $R(\mu, A)^k$ is an isomorphism of E onto $D(A^k)$. Thus

$$Qx := \sum_{j=1}^{k} (\mu - \lambda)^{j-1} R(\mu, A)^j x + (\mu - \lambda)^k R(\lambda, A_F)\, R(\mu, A)^k x \quad (x \in E)$$

defines a bounded operator on E. Moreover, for $x \in E$, $Qx \in D(A)$ and

$$\begin{aligned}(\lambda - A)Qx = &\sum_{j=1}^{k} \{(\mu - \lambda)^{j-1} R(\mu, A)^{j-1} x - (\mu - \lambda)^j R(\mu, A)^j x\} \\ &+ (\mu - \lambda)^k R(\mu, A)^k x = x.\end{aligned}$$

Since for $x \in D(A)$, $AQx = QAx$, it follows that $\lambda \in \rho(A)$ and $Q = R(\lambda, A)$.

Remark. The situation described in Proposition 1.1 had been considered in [5] in order to study regularity of the Cauchy problem.

Examples 1.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that $T_p = (T_p(t))_{t \geq 0}$ are consistent C_0-semigroups on $L^p(\Omega)$ with generator A_p, $1 \leq p < \infty$. Assume furthermore that for some $k \in \mathbb{N}$, $D(A_p^k) \subset L^\infty(\Omega)$. Then $\sigma(A_p)$ is independent of $p \in [1, \infty)$. A concrete case is A the Dirichlet Laplacian or Neumann Laplacian (assuming that Ω has the extension property in the latter case). Then the hypothesis is fulfilled for $k > 0$ (cf. Section 5). In that case one can also use Proposition 2.6 below since the resolvent is compact (cf. [9, Theorem 1.6.4]).

2. Consistency of the resolvent. Let E, F be two Banach spaces. We assume that there exists a topological vector space G such that $E \hookrightarrow G$ and $F \hookrightarrow G$.

Definition 2.1. Two operators $B_E \in \mathcal{L}(E)$ and $B_F \in \mathcal{L}(F)$ are consistent if

$$B_E x = B_F x \qquad (x \in E \cap F).$$

Let T_E and T_F be C_0-semigroups on E and F, respectively, with generators A_E and A_F, respectively. We assume that T_E and T_F are consistent, i.e., that $T_E(t)$ and $T_F(t)$ are consistent for all $t \geq 0$. We will see below (Section 3) that this does not imply in general that $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent for all $\lambda \in \rho(A_E) \cap \rho(A_F)$.

Proposition 2.2. *The set $\mathcal{U}$ of all $\lambda \in \rho(A_E) \cap \rho(A_F)$ such that $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent is open and closed in $\rho(A_E) \cap \rho(A_F)$.*

Note that $E + F$ is a Banach space for the norm

$$\|u\|_{E+F} = \inf\{\|x\|_E + \|y\|_F \ : \ x \in E, y \in F, u = x + y\}$$

and $E \cap F$ is a Banach space for the norm

$$\|u\|_{E\cap F} = \|u\|_E + \|v\|_F.$$

The injections $E \cap F \hookrightarrow E \hookrightarrow E + F$, $E \cap F \hookrightarrow F \hookrightarrow E + F$ are continuous. In particular, if $x_n \in E \cap F$, $x_n \to x$ in E and $x_n \to y$ in F, then $x = y$ and $x_n \to x$ in $E \cap F$.

Proof of Proposition 2.2. It follows from the remark above that $\mathcal{U}$ is closed in $\rho(A_E) \cap \rho(A_F)$. Let $\lambda_0 \in \mathcal{U}$. Let $\varepsilon > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \varepsilon\} \subset \rho(A_E) \cap \rho(A_F)$. Then for $x \in E \cap F$,

$$R(\lambda, A_E)x = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A_E)^{n+1} x$$

and

$$R(\lambda, A_F)x = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A_F)^{n+1},$$

where $|\lambda - \lambda_0| < \varepsilon$. Since $R(\lambda_0, A_E)^{n+1}$ and $R(\lambda_0, A_F)^{n+1}$ are consistent, it follows from the remark above that $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent. □

Recall that for $x, y \in E$,

$$x \in D(A_E), A_E x = y \quad \text{iff} \quad \int_0^t T_E(s) y\, ds = T_E(t)x - x \qquad (t \geq 0). \tag{2.1}$$

In the following we assume that $E \cap F$ is dense in E and in F.

Proposition 2.3. *Let $\lambda \in \rho(A_E)$. Assume that there exists $Q \in \mathcal{L}(F)$ which is consistent with $R(\lambda, A_E)$. Then $\lambda \in \rho(A_F)$ and $R(\lambda, A_F) = Q$.*

Proof. We can assume that $\lambda = 0$ (considering $A_E - \lambda$ otherwise). It follows from (2.1) that

$$\int_0^t T_E(s)y\,ds = T_E(t)A_E^{-1}y - A_E^{-1}y \qquad (y \in E, t \geq 0).$$

Hence

$$\int_0^t T_F(s)y\,ds = T_F(t)Qy - Qy \quad (t \geq 0)$$

for all $y \in E \cap F$, and by density, for all $y \in F$. It follows from (2.1) (with E replaced by F) that $Qy \in D(A_F)$ and $A_F Qy = y$ for all $y \in F$. Since $QT_F(t)y = T_F(t)Qy$ if $y \in E \cap F$, it follows that Q and $T_F(t)$ commute ($t \geq 0$). Hence $A_F Qy = QA_F y$ if $y \in D(A_F)$. □

The following is a converse of Proposition 2.3.

Proposition 2.4. *Let $\lambda \in \rho(A_E) \cap \rho(A_F)$. If $R(\lambda, A_F)(E \cap F) \subset E \cap F$, then $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent.*

Proof. We can assume that $\lambda = 0$. Let $x \in E \cap F$. By hypothesis $A_F^{-1}x \in E \cap F$. Hence

$$\int_0^t T_E(s)x\,ds = \int_0^t T_F(s)x\,ds = T_F(t)A_F^{-1}x - A_F^{-1}x = T_E(t)A_F^{-1}x - A_F^{-1}x \quad (t \geq 0).$$

It follows from (2.1) that $A_F^{-1}x \in D(A_E)$ and $A_E(A_F^{-1}x) = x$; i.e., $A_F^{-1}x = A_E^{-1}x$.

Proposition 2.5. *Assume that*

(a) $T_E(t)E \subset F$ *for some $t > 0$ or*
(b) $D(A_E^k) \subset F$ *for some $k \in \mathbb{N}$.*

Then $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent for all $\lambda \in \rho(A_E) \cap \rho(A_F)$.

Proof. (a) Let $\lambda \in \rho(A_E) \cap \rho(A_F)$. We can assume that $\lambda = 0$. Let $x \in E \cap F$. Then by (2.1),

$$A_E^{-1}x = T_E(t)A_E^{-1}x - \int_0^t T_E(s)x\,ds = T_E(t)A_E^{-1}x - \int_0^t T_F(s)x\,ds \in E \cap F.$$

It follows from Proposition 2.4 that A_E^{-1} and A_F^{-1} are consistent.

(b) If μ is larger than the type of T_E and T_F, then $R(\mu, A_E)$ and $R(\mu, A_F)$ are consistent since they are the Laplace transforms of the consistent semigroups. Let $\lambda \in \rho(A_E) \cap \rho(A_F)$. It follows from (1.1) that $R(\lambda, A_E)(E \cap F) \subset E \cap F$. Thus the claim follows from Proposition 2.4.

Proposition 2.6. *Assume that A_E and A_F have compact resolvent. Then $\sigma(A_E) = \sigma(A_F)$.*

Proof. Since $\rho(A_E) \cap \rho(A_F)$ is convex, $R(\mu, A_E)$ and $R(\mu, A_F)$ are consistent for all $\mu \in \rho(A_E) \cap \rho(A_F)$.

Let $\lambda_0 \in \rho(A_F)$. Since $\sigma(A_E)$ consists of isolated points, there exists $\varepsilon > 0$ such that $\{\lambda \in \mathbb{C} : 0 < |\lambda - \lambda_0| \leq \varepsilon\} \subset \rho(A_E) \cap \rho(A_F)$. Since $\lambda_0 \in \rho(A_F)$, one has

$$\int_{|\lambda-\lambda_0|=\varepsilon} R(\lambda, A_F) d\lambda = 0.$$

By consistency, it follows that

$$\int_{|\lambda-\lambda_0|=\varepsilon} R(\lambda, A_E) d\lambda = 0.$$

Hence, $\lambda_0 \in \rho(A_E)$.

3. Counterexamples. In [13], [14] an example of a generator on L^p is given whose spectrum depends on $p \in [1, \infty)$. However, the spectrum of the resolvent is computed, reducing the problem to bounded operators which have been investigated in detail (see Jörgens [15, p. 194, 195], Boyd [7], Auterhoff [6], Schaefer [23]). Here we give an easy direct example (see also [1] for relations with the asymptotic behavior of the semigroup).

1. Define the consistent C_0-groups T_p on $L^p(0, \infty)$ by

$$(T_p(t)f)(x) = f(e^{-t}x) \qquad (t \in \mathbb{R}),$$

$1 \leq p < \infty$ and denote by A_p the generator of T_p. Then

(a) $\sigma(A_p) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = \frac{1}{p}\}$;

(b) $R(\lambda, A_p)$ and $R(\lambda, A_q)$ are not consistent whenever $p < q$ and $\frac{1}{q} < \operatorname{Re} \lambda < \frac{1}{p}$;

(c) A_p is given by $(A_p f)(x) = -xf'(x)$,

$$D(A_p) = \{f \in L^p(0, \infty) \ : \ x \mapsto xf'(x) \in L^p(0, \infty)\}.$$

Proof. For $f \in L^p(0, \infty)$ one has

$$\|T_p(t)f\|_p = \left(\int_0^\infty |f(e^{-t}x)|^p dx\right)^{\frac{1}{p}} = e^{\frac{t}{p}} \|f\|_p.$$

Hence $(e^{-\frac{t}{p}} T_p(t))_{t\in\mathbb{R}}$ is an isometric group on $L^p(0, \infty)$. It follows that its generator $A_p - \frac{1}{p}$ has spectrum in $i\mathbb{R}$, i.e., $\sigma(A_p) \subset \frac{1}{p} + i\mathbb{R}$. Let $\frac{1}{q} < \lambda < \frac{1}{p}$. Since the type of T_q is $\frac{1}{q}$ and the type of $(T_p(-t))_{t\geq 0}$ is $-\frac{1}{p}$ we have

$$R(\lambda, A_q) = \int_0^\infty e^{-\lambda t} T_q(t) \, dt \geq 0$$

and

$$R(\lambda, A_p) = -R(-\lambda, -A_p) = -\int_0^\infty e^{-\lambda t} T_q(-t)dt \le 0.$$

Thus $R(\lambda, A_p)$ and $R(\lambda, A_q)$ are not consistent. It follows from Proposition 2.2 that $\sigma(A_p) = \frac{1}{p}+i\mathbb{R}$, $\sigma(A_q) = \frac{1}{q}+i\mathbb{R}$ and that $R(\lambda, A_p)$ and $R(\lambda, A_q)$ are not consistent either if $\frac{1}{q} < \operatorname{Re}\lambda < \frac{1}{p}$. We have shown (a) and (b). The last point (c) will become clear from 2. □

2. Let

$$(C_p f)(x) = \frac{1}{x}\int_0^x f(y)\,dy, \qquad 1 < p < \infty.$$

Then C_p is a bounded operator on $L^p(0, \infty)$, $\|C_p\| \le \frac{p}{p-1}$, and

$$\sigma(C_p) = \left\{\tfrac{1}{1-\frac{1}{p}-is} : s \in \mathbb{R}\right\} \cup \{0\},$$

so that $\sigma(C_p) \cap \sigma(C_q) = \{0\}$ if $1 < p, q < \infty$, $p \neq q$.

This has been proven by Boyd [7]. The norm estimate of C_p is known as Hardy's inequality. We obtain both as an easy consequence of 1.

Proof. Let $1 < p < \infty$. Then by 1., $1 \in \rho(A_p)$ and

$$(R(1, A_p)f)(x) = \int_0^\infty e^{-t} f(e^{-t}x)dt = \frac{1}{x}\int_0^x f(y)dy.$$

Hence $C_p = R(1, A_p)$. Since $\|T_p(t)\| = e^{\frac{t}{p}}$ we have

$$\|R(1, A_p)\| \le \int_0^\infty e^{-t} e^{\frac{t}{p}} dt = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}.$$

Since by [18, p. 67], $\sigma(R(1, A_p)) = \{\frac{1}{1-\lambda} : \lambda \in \sigma(A_p)\} \cup \{0\}$ the assertion on the spectrum of C_p follows from 1. Now 1(c) is an immediate consequence of $R(1, A_p) = C_p$. □

3. Let $B_p = (A_p - \frac{1}{2})^2$. Then B_p generates a holomorphic semigroup since $A_p - \frac{1}{2}$ generates a C_0-group on $L^p(0, \infty)$ (see e.g. [18, Corollary p. 36]). The group generated by $A_2 - \frac{1}{2}$ is isometric, thus B_2 is self-adjoint. By the spectral mapping theorem, one has

$$\sigma(B_2) = (-\infty, 0], \qquad \sigma(B_p) = \left\{(\tfrac{1}{p} - \tfrac{1}{2} + is)^2 : s \in \mathbb{R}\right\}$$

$(1 \le p < \infty)$. Hence $\sigma(B_p) \cap \sigma(B_q) = \emptyset$ whenever $1 \le p, q \le 2$, $p \neq q$.

This follows immediately from 1.

It is easy to see that B_p is given by

$$(B_p f)(x) = x^2 f'' + 2xf' + \frac{f}{4},$$

$$D(B_p) = \{f \in L^p(0,1) \ : \ x f' \in L^p(0,1),\ x^2 f'' \in L^p(0,\infty)\}.$$

Thus B_p is a degenerate elliptic operator of second order. (It is precisely this example, up to addition of a constant, which is mentioned by Hempel-Voigt [14, p. 243]).

We will see below that the spectrum of second order uniformly elliptic operators on euclidean spaces is p-independent.

4. Define $\widetilde{A}_p$ on $L^p(\mathbb{R}, e^x dx)$ by

$$\widetilde{A}_p f = f',$$

$$D(\widetilde{A}_p) = \{f \in L^p(\mathbb{R}, e^x dx) \ : \ f' \in L^p(\mathbb{R}, e^x dx).$$

Then $\widetilde{A}_p$ generates a C_0-group and $\sigma(\widetilde{A}_p) = \frac{1}{p} + i\mathbb{R}$. Thus, the Laplacian $\widetilde{A}_p^2$ on $L^p(\mathbb{R}, e^x dx)$ generates a holomorphic semigroup and

$$\sigma(\widetilde{A}_p^2) = \{(\tfrac{1}{p} + is)^2 \ : \ s \in \mathbb{R}\}; \quad 1 \le p < \infty.$$

Proof. In fact, the mapping $f \mapsto f \circ \log$ defines an isometric isomorphism U of $L^p(\mathbb{R}, e^x dx)$ onto $L^p(0,\infty)$ $(1 \le p < \infty)$. Let T_p be the C_0-group on $L^p(0,\infty)$ considered in Example 1. Then $\widetilde{T}_p(t) = U^{-1}T_p(t)U$ defines a C_0-semigroup on $L^p(\mathbb{R}, e^x dx)$ given by $(\widetilde{T}_p(t)f)(x) = (T_p(t)Uf)(e^x) = (Uf)(e^{-t}e^x) = f(x-t)$. The operator $\widetilde{A}_p$ is the generator of $\widetilde{T}_p$. The result follows now from 1. by similarity.

Remark. Davies, Simon and Taylor [10] show that the spectrum of the Laplacian on hyperbolic space and on many Kleinian groups does depend on p. On the other hand Sturm [25] has shown p-independence of the spectrum of the Laplace-Beltrami operator on certain Riemannian manifolds.

5. Let T_p on $L^p(1,\infty)$ be given by $(T_p(t)f)(x) = f(e^t x)$. Then it is not difficult to show that $\sigma(A_p) = \{\lambda \in \mathbb{C} \ : \ \operatorname{Re}\lambda \le -\frac{1}{p}\}$. Moreover, $D(A_p) \subset L^\infty \cap L^p \subset L^q(1,\infty)$, $1 \le p < \infty$ (cf. [1]). Thus in Proposition 1.1, the assumption that $F \subset E$ is essential.

4. The main results. Let $\Omega \subset \mathbb{R}^N$ be an open set and let T be a C_0-semigroup on $L^2(\Omega)$ with generator A. We identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$ by extending functions by 0.

Denote by G_p the Gaussian semigroup on $L^p(\mathbb{R}^N)$, i.e., G_p is given by $G_p f = k_t * f$ where

$$k_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-x^2/4t} \quad (x \in \mathbb{R}^N), \quad 1 \le p < \infty.$$

Definition 4.1. We say that T *satisfies an upper Gaussian estimate* if there exist $c \ge 1$, $b > 0$ such that

$$|T(t)f| \le c\, G_2(bt)\, |f| \qquad (0 \le t \le 1) \tag{4.1}$$

for all $f \in L^2(\Omega)$.

In the following we assume that (4.1) holds. Let $w = \log c$. Then

$$|T(t)f| \leq ce^{wt} G_2(bt)\,|f| \qquad (t \geq 0,\ f \in L^2). \tag{4.2}$$

Proof. It follows from (4.1) that

$$|T(n)f| = |T(1)^n f| \leq c^n G(bn)|f| \quad (n \in \mathbb{N}).$$

Let $t > 1$. Choose $n \in \mathbb{N}$ such that $s := t - n \in [0, 1)$. Then

$$\begin{aligned}|T(t)f| &= |T(n)T(s)f| \leq c^n G(bn)\, c\, G(bs)|f| \\ &= c^{n+1} G(bt)|f| = c\, e^{nw} G(bt)|f| \leq ce^{tw} G(bt)\,|f|. \qquad \square\end{aligned}$$

As a consequence there exist consistent semigroups T_p on $L^p(\Omega)$, $1 \leq p < \infty$, such that $T = T_2$ and

$$|T_p(t)f| \leq ce^{wt}\, G_p(bt)|f| \quad (f \in L^p,\ t \geq 0). \tag{4.3}$$

Proof. Since $G_2(t)$ has bounded extensions to $L^p(\mathbb{R}^N)$, it follows that there exist consistent operators $T_p(t) \in \mathcal{L}(L^p)$ such that $T_2(t) = T(t)$ $(t \geq 0)$. The semigroup property follows by density. It remains to prove strong continuity. Let $1 \leq p < \infty$. It suffices to show that $T(t)f \to f$ $(t \downarrow 0)$ in L^p for $f \in L^p \cap L^2$. Let $f \in L^p \cap L^2$ and let $t_n \to 0$. Let $f_n = T(t_n)f$, $g_n = ce^{wt_n} G(bt_n)|f|$. Since it suffices to show that every subsequence of f_n has a subsequence which converges to f, we can assume that $f_n \to f$ almost everywhere (observe that $f_n \to f$ in L^2). Taking a subsequence again we can assume that $\|g_n - g_{n-1}\|_p \leq 2^{-n}$. Let

$$h = \sum_{n \geq 2} |g_n - g_{n-1}| + |g_1|.$$

Then $h \in L^p(\mathbb{R}^N)$ and $|f_n| \leq g_n \leq h$ $(n \in \mathbb{N})$. Now it follows from the dominated convergence theorem that $f_n \to f$ in L^p.

Remark. Here strong continuity follows from domination. It is not obvious from the consistency property alone; cf. Voigt [28].

We denote by A_p the generator of T_p. By $\rho(A_p) = \mathbb{C} \backslash \sigma(A_p)$ we denote the resolvent set of A_p and by $\rho_\infty(A_p)$ the connected component of $\rho(A_p)$ which contains a right semi-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > w\}$ for some $w \in \mathbb{R}$.

Our main results are the following.

Theorem 4.2. *Assume that T admits an upper Gaussian estimate. Then $\rho_\infty(A_p)$ is independent of $p \in [1, \infty)$.*

Corollary 4.3. *Assume that A is self-adjoint and that T admits an upper Gaussian estimate. Then $\sigma(A_p)$ is independent of $p \in [1, \infty)$.*

Proof. Since $\sigma(A_2) \subset \mathbb{R}$ one has $\rho_\infty(A_2) = \rho(A_2)$. It follows from Theorem 4.2 that $\rho_\infty(A_p) = \rho(A_2)$, $p \in [1, \infty)$. Hence $\sigma(A_p) \subset \mathbb{R}$ and so $\rho_\infty(A_p) = \rho(A_p)$. □

It will be seen in the proof of Theorem 4.2 that the resolvent of A_p consists of regular integral operators. Here we use the following definition.

Let $1 \leq p, q \leq \infty$. An operator $B \in \mathcal{L}(L^p, L^q)$ is called an *integral operator* if there exists a measurable function $K : \Omega \times \Omega \to \mathbb{C}$ such that for all $f \in L^p(\Omega)$,

$$K(x, .)f(.) \in L^1(\Omega), \qquad x - \text{a.e.} \quad \text{and}$$

$$(Bf)(x) = \int_\Omega K(x, y) f(y) dy, \qquad x - \text{a.e.}$$

In that case we say that B *is represented by the kernel* K and we write $B \sim K$. If in addition also $|K|$ defines an integral operator in $\mathcal{L}(L^p(\Omega), L^q(\Omega))$ we say that B is a *regular integral operator* (see [4]). If $B \sim K$, then $B \geq 0$ if and only if $K(x, y) \geq 0$, x, y almost everywhere.

Moreover, if $B \sim K$ and if $B_0 \in \mathcal{L}(L^p, L^q)$ satisfies

$$|B_0 f| \leq B|f| \qquad (f \in L^p),$$

then B_0 is an integral operator and its kernel K_0 satisfies $|K_0(x, y)| \leq K(x, y)$ x, y almost everywhere (see [22, IV. 9]).

Now we are able to formulate the above mentioned assertion precisely.

Theorem 4.4. *Let A be self-adjoint and assume that T admits an upper Gaussian estimate. Then $R(\lambda, A_p)$ is a regular integral operator for all $\lambda \in \rho(A_p) = \rho(A)$ $(1 \leq p < \infty)$. Moreover, if $N = 1$ then the kernel of $R(\lambda, A_p)$ is bounded $(\lambda \in \rho(A_p), 1 \leq p < \infty)$.*

Remark 4.5. It has been shown in [19] that T_p is a holomorphic C_0-semigroup of angle $\pi/2$ $(1 \leq p < \infty)$ whenever A is self-adjoint and T admits an upper Gaussian estimate. Regularity results for the Cauchy problem defined by $i A_p$ are obtained in [11].

5. Examples. Here we describe the class of operators to which our results can be applied. Let $\Omega \subset \mathbb{R}^N$ be an open set.

Example 5.1 (the Laplacian). a) The *Dirichlet-Laplacian* is defined on $L^2(\Omega)$ by $D(A) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$, $Af = \Delta f$ (see [8] for the definition of the Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$).

b) The *Pseudo-Dirichlet-Laplacian* is defined on $L^2(\Omega)$ by $D(A) = \{f \in L^2(\Omega) : \tilde{f} \in H^1(\mathbb{R}^N), \Delta f \in L^2(\Omega)\}$, $Af = \Delta f$, where $\tilde{f}$ denotes the extension by 0 of f to $\mathbb{R}^N$ and Δf is defined as an element of $\mathcal{D}(\Omega)'$. See [2], [3] for this example. Note that the Pseudo-Dirichlet and the Dirichlet-Laplacian coincide if Ω is of class C^1.

c) The *Neumann-Laplacian* is the operator A on $L^2(\Omega)$ such that $-A$ is associated with the form

$$a(u, v) = \int_\Omega \nabla u \nabla v dx$$

on $L^2(\Omega)$ with $D(a) = H^1(\Omega)$.

If A is any one of these three operators, then A is self-adjoint and generates a C_0-semigroup T on $L^2(\Omega)$ which satisfies an upper Gaussian estimate. (In the case c) we assume in addition that Ω has the extension property (see [9])). In fact, in the cases a) and b) one has $0 \leq T(t) \leq G(t)$ $(t \geq 0)$; see [2], [3], [9], [20]. In the case c) we refer to [9, Theorem 3.2.9 p. 90].

Let T_p be the consistent semigroups on $L^p(\Omega)$ such that $T_2 = T$ and denote by A_p the generator of T_p. We conclude from Corollary 4.3 that $\sigma(A_p) = \sigma(A_2)$ $(p \in [1, \infty))$. Moreover, $R(\lambda, A_p)$ is a regular kernel operator for all $\lambda \in \rho(A_p)$ (with bounded kernel if $N = 1$).

Remark. 1) The p-independence of the spectrum in the case a) and b) follows also from Schreieck-Voigt [26] ; in fact, in that case the operator is of the form "$\Delta - \mu$" with μ a suitable measure.

2) It has been proved before in [4] that $R(\lambda, A_p)$ is a regular integral operator for all $\lambda \in \rho(A_p)$ if Ω is bounded.

3) If Ω is bounded one can also use Proposition 1.1 to obtain p-independence of the spectrum.

Example 5.2 (Uniformly elliptic operators of second order). Let $a_{ij} \in L^1_{\text{loc}}(\Omega)$ such that $a_{ij} = a_{ji}$ and

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \qquad x - \text{a.e.}$$

for all $\xi \in \mathbb{R}^N$, where $\alpha, \beta > 0$. Define the bilinear form

$$a(u, v) = \sum_{i,j=1}^N \int a_{ij}(x) D_i u D_j v \, dv.$$

a) Let $D(a) = C_c^\infty(\Omega)$. Then a is closable. Let $-A$ be the operator on $L^2(\Omega)$ which is associated with $\overline{a}$. We call A a *uniformly elliptic operator with Dirichlet boundary conditions.*

b) Let $D(a) = H^1(\Omega)$. Then a is a closed form. Let $-A$ be the operator on $L^2(\Omega)$ associated with a. We call A *a uniformly elliptic operator with Neumann boundary conditions.*

Let A be a uniformly elliptic operator with Dirichlet or Neumann boundary conditions. Then A is self-adjoint and generates a positive semigroup T on L^2. Assume that Ω has the extension property if Neumann boundary conditions are considered (but Ω may be arbitrary in the case of Dirichlet boundary conditions). Then T satisfies an upper Gaussian estimate (see [9, Corollary 3.2.8 p. 89 and Theorem 3.2.9 p. 90]). Let T_p be the consistent semigroup on L^p such that $T_2 = T$ and denote by A_p the generator of T_p $(1 \leq p < \infty)$. Then $\sigma(A_p)$ is independent of $p \in [1, \infty)$. Moreover, $R(\lambda, A_p)$ is a regular integral operator for all $\lambda \in \rho(A_p)$.

Example 5.3 (Schrödinger operators). Let $A = \Delta - V$ on $L^2(\mathbb{R}^N)$ where $V : \mathbb{R}^N \to \mathbb{R}$ is measurable such that V_- is in Kato's class and $V_+ \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then A (with suitable domain) is a self-adjoint operator which generates a C_0-semigroup T on $L^2(\mathbb{R}^N)$. Then by [24, Prop. B.6.7], T satisfies an upper Gaussian estimate. Let T_p be the consistent C_0-semigroup on L^p such that $T_2 = T$ and A_p the generator of T_p. Then by Corollary 4.3, $\sigma(A_p) = \sigma(A_2)$ $(1 \leq p < \infty)$. This result is due to Hempel and Voigt [14].

Example 5.4 (Non-selfadjoint examples). Let A be any of the above operators and let $M_p \in \mathcal{L}(L^p)$ be given by $M_p f = mf$ where $m : \Omega \to \mathbb{C}$ is a bounded measurable function. Then by Theorem 4.2 the set $\rho_\infty(A_p + M_p)$ is independent of $p \in [1, \infty)$.

6. Proofs of the main results. Let $\Omega \subset \mathbb{R}^N$ be open. The following criterion is well-known (see, e.g., [4] or [17] for a proof and references).

Proposition 6.1. *The formula*

$$(B_K f)(x) = \int_\Omega K(x, y) f(y) dy \quad (f \in L^1(\Omega)) \tag{6.1}$$

establishes an isometric isomorphism $K \mapsto B_K$ *of* $L^\infty(\Omega \times \Omega)$ *onto* $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$.

We also use the following fact (see [22, Chap. IV]).

Proposition 6.2. *Let* $1 \leq p, q \leq \infty$. *Let* $B \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *be an integral operator,* $B \sim K$. *Let* $B_0 \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *such that*

$$|B_0 f| \leq B|f| \qquad (f \in L^p(\Omega)).$$

Then B_0 *is a regular integral operator and* $|K_0(x, y)| \leq K(x, y)$ x, y *almost everywhere, where* $K_0 \sim B_0$.

In the following we keep the hypotheses and notations of §4. In particular, T_p, $1 \leq p < \infty$ is a family of consistent C_0-semigroups of $L^p(\Omega)$ such that the estimate (4.3) holds. The generator of T_p is denoted by A_p.

For $\varepsilon \in \mathbb{R}^N$, $x \in \mathbb{R}^N$ we let

$$\varepsilon x = \sum_{j=1}^N \varepsilon_j x_\varepsilon.$$

Let $L^p = L^p(\Omega)$, $L^p_\varepsilon = L^p(\Omega, e^{-p\varepsilon x}dx) := \{f : \Omega \to \mathbb{C} : \int_\Omega |f(x)|^p e^{-px\varepsilon}dx < \infty\}$, where $1 \le p < \infty$, $\varepsilon \in \mathbb{R}^N$. Then $(U_{\varepsilon p} f)(x) = e^{-\varepsilon x} f(x)$ defines an isometric isomorphism of L^p_ε onto L^p. Hence $\widetilde{T}_{\varepsilon,p}(t) = U^{-1}_{\varepsilon,p} T_p(t) U_{\varepsilon,p}$ defines a C_0-semigroup $\widetilde{T}_{\varepsilon,p}$ on L^p_ε.

It follows from (4.3) and Proposition 6.2 that $T_p(t)$ is an integral operator, say $T_p(t) \sim K(t, .,.)$; it is obvious that the kernel does not depend on p. Consequently, $\widetilde{T}_{\varepsilon,p}(t) \sim K_\varepsilon(t,.,.)$ where

$$K_\varepsilon(t, x, y) = e^{\varepsilon(x-y)} K(t, x, y). \tag{6.2}$$

Example 6.3. Let $b > 0$ and consider the semigroup $S_p(t) = G_p(bt)$. Let $\varepsilon \in \mathbb{R}^N$. Let $\widetilde{S}_{\varepsilon,p}(t) = U^{-1}_{\varepsilon,p} T_p(t) U_{\varepsilon,p}$. Then

$$\widetilde{S}_{\varepsilon,p}(t) = \exp(bt\varepsilon^2)\, G_p(bt)\, V(2bt\varepsilon) \quad (t \ge 0), \tag{6.3}$$

where $(V(a)f)(x) = f(x-a)$ $(x \in \mathbb{R}^N,\ a \in \mathbb{R}^N)$. In fact,

$$\begin{aligned}
(\widetilde{S}_{\varepsilon,p}(t)f)(x) &= \frac{1}{(4\pi bt)^{N/2}} \int e^{\varepsilon(x-y)} e^{-(x-y)^2/4bt} f(y)\, dy \\
&= \frac{1}{(4\pi bt)^{N/2}} \int \exp\{-\frac{1}{4bt}((x-y) - 2bt\varepsilon)^2\} \exp(bt\varepsilon^2) f(y) dy \\
&= \exp(bt\varepsilon^2) \frac{1}{(4\pi bt)^{N/2}} \int \exp\left(-\frac{1}{4bt}(x-y)^2\right) f(y - 2bt\varepsilon) dy,
\end{aligned}$$

which is (6.3).

Proposition 6.4. 1. *Let $\varepsilon \in \mathbb{R}^N$, $1 \le p < \infty$. Then*

(a) $\widetilde{T}_{\varepsilon,p}(t)(L^p_\varepsilon \cap L^p) \subset L^p_\varepsilon \cap L^p$ $(t \ge 0)$;

(b) *there exists a C_0-semigroup $T_{\varepsilon,p}$ on L^p such that $T_{\varepsilon,p}(t)f = \widetilde{T}_{\varepsilon,p}(t)f$ $(f \in L^p \cap L^p_\varepsilon)$.*

2. *There exist $M_1 \ge 0$, $w_1 \ge 0$ such that*

$$\|T_{\varepsilon,p}(t)\| \le M_1 e^{w_1 t} \quad (t \ge 0) \tag{6.4}$$

for all $|\varepsilon| \le 1$, $1 \le p < \infty$.

3. *For $\lambda > w_1$ one has*

$$\lim_{|\varepsilon|\downarrow 0} \|R(\lambda, A_{\varepsilon,p}) - R(\lambda, A_p)\| = 0, \tag{6.5}$$

where $A_{\varepsilon,p}$ denotes the generator of $T_{\varepsilon,p}$ on $L^p(\Omega)$, $1 \le p < \infty$.

Proof. It follows from (4.3) that

$$|\widetilde{T}_{\varepsilon,p}(t)f| \le ce^{wt}\widetilde{S}_{\varepsilon,p}(t)|f| \tag{6.6}$$

($t \geq 0$, $f \in L^p_\varepsilon(\Omega) \cap L^p(\Omega)$), where $\widetilde{S}_{\varepsilon,p}$ is given by (6.3). This implies 1(a). Moreover, it follows from (6.3) and (6.6) that $\widetilde{T}_{\varepsilon,p}(t)$ is bounded for the $L^p(\Omega)$-norm. Thus, there exist $T_{\varepsilon,p}(t) \in \mathcal{L}(L^p)$ such that $T_{\varepsilon,p}(t)f = \widetilde{T}_{\varepsilon,p}(t)f$ ($f \in L^p \cap L^p_\varepsilon$). It is clear that $T_{\varepsilon,p}(t+s) = T_{\varepsilon,p}(t)T_{\varepsilon,p}(s)$ ($s, t \geq 0$). In order to prove 1(b) it remains to show that $T_{\varepsilon,p}(t)f \to f$ ($t \downarrow 0$) in $L^p(\Omega)$ for all $f \in L^p(\Omega)$. Since $T_{\varepsilon,p}(t)$ is bounded for $t \in [0, 1]$, it suffices to consider functions with compact support.

Let $f \in L^p$ such that $f(x) = 0$ for $|x| \geq r$, where $r > 0$. Then

$$\limsup_{t\downarrow 0} \|T_{\varepsilon,p}(t)f - f\|_p^p$$

$$\leq \limsup_{t\downarrow 0} \int_{\substack{|x|\leq r\\ x\in\Omega}} \left(e^{\varepsilon x}|(T_p(t)(e^{-\varepsilon\cdot}f))(x) - f(x)|\right)^p dx + \lim_{t\downarrow 0} \int_{\substack{|x|>r\\ x\in\Omega}} (T_{\varepsilon p}(t)|f|)(x)^p dx$$

$$\leq \lim_{t\downarrow 0}\{e^{|\varepsilon| rp} \int_{\substack{|x|\leq r\\ x\in\Omega}} |(T_p(t)(e^{-\varepsilon\cdot}f))(x) - e^{-\varepsilon x}f(x)|^p dx$$

$$+ \int_{|x|>r} (ce^{wt}\widetilde{S}_{\varepsilon,p}(t)|f|)(x)^p dx\}$$

$$\leq \lim_{t\downarrow 0}\{e^{|\varepsilon| rp}\|T_p(t)(e^{-\varepsilon\cdot}f) - e^{-\varepsilon\cdot}f\|_p^p + \int_{|x|>r} |ce^{wt}(\widetilde{S}_{\varepsilon,p}(t)|f|)(x) - |f(x)|^p dx\}$$

$$\leq \lim_{t\downarrow 0} \|ce^{wt}\widetilde{S}_{\varepsilon,p}(t)|f| - |f|\|_p^p = 0.$$

2. Follows from (6.6) and (6.3).

3. We first show that for $1 > \delta > 0$,

$$\limsup_{|\varepsilon|\downarrow 0} \sup_{\delta\leq t\leq 1/\delta} \|T_{\varepsilon,p}(t) - T_p(t)\| = 0. \tag{6.7}$$

In fact, $T_p(t) - T_{\varepsilon,p}(t) \sim K(t, x, y)(1 - e^{\varepsilon(x-y)})$ (see (6.2)). Since

$$|K(t, x, y)(1 - e^{\varepsilon(x-y)})| \leq c\, e^{wt} \frac{1}{(4\pi bt)^{N/2}} e^{-(x-y)^2/4bt}|1 - e^{\varepsilon(x-y)}|,$$

one has for $\delta \leq t \leq 1/\delta$, $|(T_{\varepsilon,p}(t) - T_p(t))f| \leq \text{const}\, Q_\varepsilon|f|$ ($f \in L^p$), where $Q_\varepsilon g = g * q^\varepsilon$ with $q^\varepsilon(x) = e^{-\delta x^2/4b}|1 - e^{\varepsilon x}|$. But $q^\varepsilon \in L^1(\mathbb{R}^N)$ and $\|q^\varepsilon\| \to 0$ ($|\varepsilon| \downarrow 0$) by the dominated convergence theorem. Since $\|T_{\varepsilon,p}(t) - T_p(t)\| \leq \text{const}\, \|q^\varepsilon\|_1$ for $\delta \leq t \leq \frac{1}{\delta}$, (6.7) follows.

Now let $\lambda > w_1$. Then for all $\delta > 0$,

$$\limsup_{|\varepsilon|\downarrow 0} \|R(\lambda, A_{\varepsilon,p}) - R(\lambda, A_p)\| \leq \limsup_{|\varepsilon|\downarrow 0} \int_0^\infty e^{-\lambda t}\|T_p(t) - T_{\varepsilon,p}(t)\| dt$$

$$\leq 2M_1\{(\int_0^\delta + \int_{1/\delta}^\infty) e^{-\lambda t}e^{w_1 t}dt\} \leq 2M_1\{\delta + \frac{1}{\lambda - w_1} e^{-(\lambda-w_1)/\delta}\}.$$

Since $\delta > 0$ is arbitrary, (6.5) follows. □

It is clear from the construction that the semigroups $T_{\varepsilon,p}$ on L^p and $\widetilde{T}_{\varepsilon,p}$ on L^p_ε are consistent. Consequently, $R(\lambda, A_{\varepsilon,p})$ and $R(\lambda, \widetilde{A}_{\varepsilon,p})$ are consistent for $\operatorname{Re}\lambda > \max(\omega(A_{\varepsilon,p}); \omega(\widetilde{A}_{\varepsilon,p}))$, where $\omega(B)$ denotes the type of a semigroup S with generator B. Thus we obtain from Proposition 2.2 the following.

Proposition 6.5. *The operators $R(\lambda, A_{\varepsilon,p})$ and $R(\lambda, \widetilde{A}_{\varepsilon,p})$ are consistent for all $\lambda \in [\rho(A_{\varepsilon,p}) \cap \rho(\widetilde{A}_{\varepsilon,p})]_\infty$.*

Here, for $0 \subset \mathbb{C}$ open, we let 0_∞ be the connected component of 0 which contains a right halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq w\}$ for some $w \in \mathbb{R}$.

Remark. We do not know whether the resolvents $R(\lambda, A_{\varepsilon,p})$ and $R(\lambda, \widetilde{A}_{\varepsilon,p})$ are consistent for all $\lambda \in (\rho(A_{\varepsilon,p}) \cap \rho(\widetilde{A}_{\varepsilon,p}))$. Also in the proof of [14, Proposition 3.8] a connectedness argument is needed, even though it is not carried out there explicitly.

Note that by construction, $\rho(\widetilde{A}_{\varepsilon,p}) = \rho(A_p)$ and

$$R(\lambda, \widetilde{A}_{\varepsilon,p})f = e^{\varepsilon x}(R(\lambda, A_p))(e^{-\varepsilon x} f) \quad (f \in L^p_\varepsilon)$$

for all $\lambda \in \rho(A_p)$. Thus Proposition 6.5 can be reformulated by saying

$$R(\lambda, A_{\varepsilon,p})f = e^{\varepsilon x}(R(\lambda, A_p)(e^{-\varepsilon x} f)) \tag{6.8}$$

whenever $f, e^{-\varepsilon x} f \in L^p$, $\lambda \in (\rho(A_{\varepsilon,p}) \cap \rho(A_p))_\infty$.

The following proposition expresses the upper semicontinuity of the spectrum for unbounded operators (cf. [16, p. 212]).

Proposition 6.6. *Let A be an operator on X, $\lambda \in \rho(A)$. Let K be a compact subset of $\rho(A)$. Then there exist $\varepsilon > 0$ and $c \geq 0$ such that for all operators B on X with $\lambda \in \rho(B)$ and $\|R(\lambda, B) - R(\lambda, A)\| < \varepsilon$ one has $K \subset \rho(B)$ and $\sup_{\mu \in K} \|R(\mu, B)\| \leq c$.*

Proof. We can assume $\lambda = 0$. Let $M = \sup_{\mu \in K} \|\mu - \mu^2 R(\mu, A)\|$ and $\varepsilon = \frac{1}{2M}$. Assume that B is an operator such that $0 \in \rho(B)$ and $\|A^{-1} - B^{-1}\| < \varepsilon$. Let $\mu \in K\backslash\{0\}$. Then $(\frac{1}{\mu} - A^{-1})^{-1} = [\frac{1}{\mu}(A - \mu)A^{-1}]^{-1} = -\mu A R(\mu, A) = \mu - \mu^2 R(\mu, A)$. Hence $\|(\frac{1}{\mu} - A^{-1})^{-1}.(B^{-1} - A^{-1})\| \leq \frac{1}{2}$; thus $Q = (I - (\frac{1}{\mu} - A^{-1})^{-1}(B^{-1} - A^{-1}))$ is invertible and $\|Q^{-1}\| \leq 2$. Consequently,

$$\begin{aligned}(\mu - B) &= -\mu(\frac{1}{\mu} - B^{-1})B = -\mu(\frac{1}{\mu} - A^{-1} + A^{-1} - B^{-1})B \\ &= -\mu(\frac{1}{\mu} - A^{-1})\{I - (\frac{1}{\mu} - A^{-1})^{-1}(B^{-1} - A^{-1})\}B\end{aligned}$$

is invertible and $R(\mu, B) = -B^{-1}Q^{-1}(I - \mu R(\mu, A))$. Moreover, $\|R(\mu, B)\| \leq \|B^{-1}\| 2 \sup_{\lambda \in K} \|I - \lambda R(\lambda, A)\| =: c$. $\square$

We use the following notation. Let $1 \leq p, q, r \leq \infty$, $B \in \mathcal{L}(L^p)$. Then we set

$$\|B\|_{\mathcal{L}(L^q, L^r)} := \sup \{\|Bf\|_r : f \in L^p \cap L^q, \|f\|_q \leq 1\}.$$

Proof of Theorem 4.2. Let $1 \leq p, q < \infty$, $\mu \in \rho_\infty(A_p)$. We have to show that $\mu \in \rho(A_q)$. By Proposition 2.3 it suffices to show that $\|R(\mu, A_p)\|_{\mathcal{L}(L^q)} < \infty$. Since

$$R(\mu, A_p) = \int_0^1 e^{-\mu t} T_p(t) dt + e^{-\mu} T_p(1) R(\mu, A_p), \tag{6.9}$$

it suffices to show that

$$\|T_p(1)\, R(\mu, A_p)\|_{\mathcal{L}(L^q)} < \infty. \tag{6.10}$$

Let K be the image of a continuous path relating μ with a point in $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > w_1\}$. By Proposition 6.4.3 and Proposition 6.6 there exists $\varepsilon_0 > 0$ such that $K \subset \rho(A_{\varepsilon,p})$ whenever $|\varepsilon| \leq \varepsilon_0$. Consequently, $\mu \in [\rho(A_{\varepsilon,p}) \cap \rho(A_p)]_\infty$. It follows from (6.3), (6.6) that

$$\sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(\tfrac{1}{2})\|_{\mathcal{L}(L^1,L^p)} < \infty \quad \text{and} \quad \sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(\tfrac{1}{2})\|_{\mathcal{L}(L^p,L^\infty)} < \infty.$$

Since $T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p}) = T_{\varepsilon,p}(\frac{1}{2})R(\mu, A_{\varepsilon,p})T_{\varepsilon,p}(\frac{1}{2})$, it follows that

$$c_1 := \sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})\|_{\mathcal{L}(L^1,L^\infty)} < \infty. \tag{6.11}$$

It follows from Proposition 6.1 that $T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})$ is given by a kernel K_ε such that $|K_\varepsilon(x, y)| \leq c_1$ $(x, y \in \Omega)$ almost everywhere. In particular, $K_0 \sim T_p(1)R(\mu, A_p)$. Since by (6.8),

$$\begin{aligned} T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})f &= e^{\varepsilon x}T_p(1)[e^{-\varepsilon x}(e^{\varepsilon x}R(\lambda, A_p)(e^{-\varepsilon x}f))] \\ &= e^{\varepsilon x}(T_p(1)R(\lambda, A_p))(e^{-\varepsilon x}f) \end{aligned}$$

whenever $f, e^{-\varepsilon x}f \in L^p$, it follows that

$$K_\varepsilon(x, y) = K_0(x, y)e^{\varepsilon(x-y)} \qquad x, y - \text{a.e.}$$

Hence $|K_0(x, y)e^{\varepsilon(x-y)}| \leq c_1$ (x, y) almost everywhere whenever $|\varepsilon| \leq \varepsilon_0$. Consequently, $|K_0(x, y)| \leq c_1 e^{-\varepsilon_0|x-y|}$ (x, y) almost everywhere. This implies (6.10).

Proof of Theorem 4.4. We use the identity (6.9). It has been shown above that $e^{-\mu}T_p(1)R(\mu, A_p)$ is given by a kernel K_0 satisfying $|K_0(x, y)| \leq \text{const}\, e^{-\varepsilon_0|x-y|}$ $(x, y \in \Omega)$, where $\varepsilon_0 > 0$.

On the other hand, since $T_p(t)$ is a positive integral operator, it follows from [4, Theorem 2.1] that $\int_0^1 e^{-\mu t}T_p(t)dt$ is a regular integral operator. Thus, by (6.9), $R(\mu, A_p)$ is a regular integral operator. Finally, if $N = 1$, then the kernel $K(t, x, y)$ of $T_p(t)$ is dominated by const. $t^{-\frac{1}{2}}(x, y \in \Omega, t > 0)$. Thus $\|T_p(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq \text{const}\, t^{-\frac{1}{2}}$. It follows that $\| \int_0^1 e^{-\mu t}T_p(t)dt\|_{\mathcal{L}(L^1,L^\infty)} < \infty$. Thus, by Proposition 6.1 the kernel of $\int_0^1 e^{-\mu t}T_p(t)dt$ is bounded and identity (6.9) implies that $R(\mu, A_p)$ has a bounded kernel as well.

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