GENERATORS OF POSITIVE SEMIGROUPS AND RESOLVENT POSITIVE OPERATORS

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Introduction

Even though the theory of positive semigroups has progressed rapidly during the last few years, so far an intrinsic characterization of generators of positive semigroups has not been given.

The problem is obvious from the general theory: Since the infinitesimal generator determines a semigroup uniquely, one expects to find a condition on the generator which describes the positivity of the semigroup.

From a practical point of view as well there seems to be a need for such a characterization. In fact, it lies in the very nature of the theory that frequently the generator but not the semigroup is known explicitly. Since a variety of results (concerning spectral theory, asymptotics, perturbation theory etc.) for positive semigroups is available today, it is important to find conditions on the generator which enable one to verify positivity (of the associated semigroup).

Characterizations of positivity together with additional properties are known. Phillips characterized positive contraction semigroups by dispersiveness of the generator. The more general notion of p-dissipativity with respect to a half-norm p was introduced in [5] and allows one to treat contractivity in a very general sense (see also the article by Batty and Robinson [8]).

A conditon of a different kind is the following abstract version of Kato's inequality

$$\text{(K)} \qquad \langle (\text{sign } f)Af, \phi \rangle \leq \langle |f|, A'\phi \rangle$$
$$f \in D(A),\ 0 \leq \phi \in D(A').$$

Of course, this inequality is inspired by Kato's classical inequality for the Laplacian. It was R. Nagel who conjectured that some abstract version of this inequality is equivalent to positivity.

We confirm Nagel's conjecture in the following form. Let A be the generator of a semigroup on a Banach lattice E (which for simplicity is supposed to satisfy some mild restrictions). Then the semigroup is positive if and only if A satisfies (K) and the adjoint A' of A possesses a strictly positive subeigenvector ϕ (i.e., $\phi \in D(A')$ and $A'\phi \leq \lambda\phi$ for some $\lambda \in \mathbb{R}$).

So far, the discussion has focused on finding necessary and sufficient conditions for the generator of a strongly continuous semigroup to assure the positivity of the semigroup. In Chapter II, we consider things from a different point of view.

Given an operator A (without assuming that A is a generator), what conclusions can be drawn from the positivity of the resolvent? We show that A has similar properties to a generator. In analogy to the classical theorem of Bernstein, the resolvent of A is representable as the Laplace-Stieltjes transform of an operator-valued increasing function. As a consequence, the abstract Cauchy problem associated with A has unique solutions for a large class of initial values. For more information, we refer to the detailed introduction to Chapter II.

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CHAPTER I

Kato's Inequality

A Characterization of Generators of Positive Semigroups

This chapter is devoted to the characterization of positive semigroups by Kato's inequality. The main result is stated and explained in section 1; in section 2 we give the proofs. They are based on the technique developed in [5] .
The examples in section 3 are chosen in order to demonstrate that the results cannot be essentially improved. But they also illustrate how the conditions are handled for concrete operators.
A related problem is to express in terms of the generator when one semigroup is dominated by another. This can be done in a similar manner by an inequality involving the "signum operator". It is remarkable that here it is not necessary to start with a generator. The inequality and a range condition are sufficient to obtain a semigroup.
In the last section we investigate a special kind of domination. Disjointness preserving semigroups are described as those semigroups which are dominated by a lattice semigroup. This puts a new complexion on "Kato's equality", which is known to characterize generators of lattice semigroups by a result of Nagel and Uhlig [31] .

1. The characterization.

Let E be a σ-order complete real Banach lattice [42 ,II §1]. We first describe the sign operator. Let $f \in E$. There exists a unique bounded operator 'sign f' which satisfies

(1.1) $|(\text{sign } f)g| \leq |g| \qquad (g \in E)$

(1.2) $(\text{sign } f)g = 0 \qquad \text{if } f \perp g$

(1.3) $(\text{sign } f)f = |f|$.

Here we understand by $f \perp g$ that f and g are <u>disjoint</u>, i.e. $\inf\{|f|, |g|\} = 0$.
If for $u \in E_+$ the band projection onto the band $u^{\perp\perp}$ generated by u is denoted by P_u, then

(1.4) $\text{sign } f = P_{f^+} - P_{f^-}$.

<u>Example</u>. Let $E = L^p(X,\mu)$ (where (X,μ) is a measure space and $1 \leq p \leq \infty$) and $f \in E$. Let $m \in L^\infty$ be given by

$$m(x) = \begin{cases} 1 & \text{if } f(x) > 0 \\ -1 & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

Then $\quad (\text{sign } f)g = m \cdot g \quad (g \in E)$.

Now let $(T(t))_{t \geq 0}$ be a <u>semigroup</u> (by that we always mean a strongly continuous semigroup of linear operators) on E with generator A. We first consider necessary conditions for the positivity of the semigroup.

Proposition 1.1. If $T(t) \geqq 0$ $(t \geqq 0)$ then Kato's inequality holds in the weak form, i.e.

(K) $$\langle (\text{sign } f)\, Af, \phi\rangle \leqq \langle |f|, A'\phi\rangle$$
$$(f \in D(A),\ 0 \leqq \phi \in D(A')).$$

Proof. Let $f \in D(A)$, $0 \leqq \phi \in D(A')$. Then

$$\begin{aligned}
\langle (\text{sign } f)\, Af, \phi\rangle &= \lim_{t\to 0} 1/t \,\langle (\text{sign } f)(T(t)f - f), \phi\rangle \\
&= \lim_{t\to 0} 1/t \,\langle (\text{sign } f)T(t)f - |f|, \phi\rangle \\
&\leqq \lim_{t\to 0} 1/t \,\langle |T(t)f| - |f|, \phi\rangle \\
&\leqq \lim_{t\to 0} 1/t \,\langle T(t)|f| - |f|, \phi\rangle \\
&= \lim_{t\to 0} \langle |f|, 1/t(T(t)'\phi - \phi)\rangle \\
&= \langle |f|, A'\phi\rangle. \ \square
\end{aligned}$$

Let $D(A')_+ = E'_+ \cap D(A')$. Consider the condition

(1.5) $$\overline{D(A')_+}^{\sigma(E',E)} = E'_+$$

(which is satisfied if the semigroup is positive). If (K) and (1.5) hold, then Kato's inequality holds in the strong form as well, whenever it makes sense, i.e.

(1.6) $(\text{sign } f)Af \leq A|f|$ (if $f, |f| \in D(A)$).

However, it will be seen in section 3 that (K) and (1.5) are <u>not sufficient</u> for the positivity of the semigroup. So we consider another necessary condition.

<u>Definition</u> 1.2. A subset M' of E' is called <u>strictly positive</u> if for every $f \in E_+$, $\langle f, \phi \rangle = 0$ for all $\phi \in M'$ implies $f = 0$. An element ϕ of E'_+ is called <u>strictly positive</u> if the set $\{\phi\}$ is strictly positive.

<u>Example</u> 1.3. Let $E = L^p(X,\mu)$ $(1 \leq p \leq \infty)$, where (X,μ) is a σ-finite measure space. Then $\phi \in E' = L^q(X,\mu)$ (where $1/p + 1/q = 1$) is strictly positive if and only if $\phi(x) > 0$ μ-a.e. Note that strictly positive elements of E' always exist in this case.

<u>Definition</u> 1.4. Let B be an operator on a Banach lattice F and let $u \in F$. Then u is called a <u>positive subeigenvector</u> of B if

a) $0 < u \in D(B)$ and

b) $Bu \leq \lambda u$ for some $\lambda \in \mathbb{R}$

<u>Proposition</u> 1.5. If the semigroup $(T(t))_{t\geq 0}$ is positive, then there exists a strictly positive set M' of subeigenvectors of A' (the adjoint of the generator A). Moreover, if there exist strictly positive linear forms on E, then there exists a strictly positive subeigenvector of A'.

Proof. Let $\lambda > 0$ such that $R(\lambda,A) = (\lambda - A)^{-1}$ exists and $R(\lambda,A) \geq 0$. Let $N' \subset E'_+$ be strictly positive. Then $M' := \{R(\lambda,A)'\psi : \psi \in N'\} \subset D(A') \cap E'_+$. We show that M' is strictly positive. Indeed, let $f \in E_+$ such that $\langle f, \phi\rangle = 0$ for all $\phi \in M'$. Then $\langle R(\lambda,A)f, \psi\rangle = 0$ for all $\psi \in N'$. Hence $R(\lambda,A)f = 0$ since N' is strictly positive. Consequently, $f = 0$. The set M' consists of subeigenvectors of A'. In fact, let $\psi \in N'$, $\phi = R(\lambda,A)'\psi$. Then $A'\phi = \lambda\phi - \psi \leq \lambda\phi$. □

The following is our characterization.

Theorem 1.6. The semigroup $(T(t))_{t\geq 0}$ is positive if and only if its generator A satisfies the following condition.
There exists a strictly positive set M' of subeigenvectors of A' such that

(K) $$\langle(\text{sign } f)\, Af, \phi\rangle \leq \langle|f|, A'\phi\rangle$$
$$\text{for all } f \in D(A),\ \phi \in M'.$$

Corollary 1.7. Assume that E' contains strictly positive functionals. Then the semigroup is positive if and only if there exists a strictly positive subeigenvector ϕ of A' such that

(K) $$\langle(\text{sign } f)\, Af, \phi\rangle \leq \langle|f|, A'\phi\rangle \quad \text{for all } f \in D(A).$$

Remark 1.8. For the application of our criterion the following improvement (of one direction of the characterization) is important. If condition (K) is merely satisfied for all $f \in D_o$, where D_o is a core of A, then the semigroup is positive. This will be obvious from the proofs.

Remark 1.9. In Theorem 1.6 and Corollary 1.7 one can replace inequality (K) by the inequality

$$(1.7) \qquad \langle (P_f+)Af, \phi\rangle \leqq \langle f^+, A'\phi\rangle .$$

Indeed, (1.7) for -f gives $\langle (-P_f-)Af, \phi\rangle \leqq \langle f^-, A'\phi\rangle$. Adding up both inequalities one obtains $\langle (\text{sign } f)Af, \phi\rangle \leqq \langle |f|, A'\phi\rangle$.
On the other hand, if A generates a positive semigroup, one sees by the obvious alterations in the proof of Proposition 1.1 that (1.7) holds for all $f \in D(A)$, $\phi \in D(A')_+$.

We conclude this section by formulating our result for the space $C_o(X)$. If the Banach lattice E is not σ-order complete there are some difficulties to defining the signum operator. One still can define sign f as an operator from E into E'' (cf. [3]). Here we consider merely the case $E = C_o(X)$ in which this can be done in a natural way.
Let X be a locally compact space and $E = C_o(X)$ the space of all real valued continuous functions on X which vanish at infinity. Note that E is not σ-order complete unless X is σ-Stonian. For $f \in C_o(X)$ we define the function sign f by

$$(\text{sign } f)(x) = \begin{cases} 1 & \text{if } f(x) > 0 \\ -1 & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

Then (sign f) is a bounded Borel function.
If $\mu \in M(X) = C_0(X)'$ we set $\langle g, \mu\rangle := \int g(x)\, d\mu(x)$ for every bounded Borel function g on X.

Theorem 1.10. Let A be the generator of a semigroup on $C_0(X)$. The semigroup is positive if and only if there exists a strictly positive set M' of subeigenvectors of A' such that

$$\text{(K)} \qquad \langle (\text{sign } f)Af, \mu\rangle \leq \langle |f|, A'\mu\rangle \quad \text{for all } f \in D(A),\ \mu \in M'.$$

Remark. We point out that for compact X a simpler condition is equivalent to positivity, namely a minimum principle (see [5]). For a comparison of Kato's inequality and the minimum principle we refer to [4]. Due to the non-empty interior of the positive cone the space C(X) (X compact) plays an exceptional role in our context (see also Chapter II, sec. 2).

2. The proofs

Our arguments are based on the results of [5] on p-contraction semigroups and p-dissipative operators (see also [8]).
Let F be a Banach space. A mapping $p : F \to \mathbb{R}$ is called a sublinear functional if

$$\text{(2.1)} \qquad p(f+g) \leq p(f) + p(g) \qquad (f,g \in F)$$

$$\text{(2.2)} \qquad p(\lambda f) = \lambda p(f) \qquad (f \in F,\ \lambda \in \mathbb{R}_+).$$

p is called a <u>half-norm</u> if in addition

(2.3) $p(f) + p(-f) > 0$ for all $0 \neq f \in F$.

Then $\|f\|_p := p(f) + p(-f)$ defines a norm on F. (This is the motivation for the terminology.)

<u>Examples</u> 2.1. a) $p(f) = \|f\|$ defines a continuous half-norm on F.

b) Let E be a real Banach lattice. $N(f) = \|f^+\|$ defines a continuous half-norm on E (the <u>canonical</u> <u>half-norm</u>).

c) Let E be a real Banach lattice and $\phi \in E'$. Let $p(f) = \langle f^+, \phi\rangle$ $(f \in E)$. Then p is a continuous sublinear functional. Moreover, p is a half-norm if and only if ϕ is strictly positive.

<u>Remark</u> 2.2. To every continuous half-norm p on F there corresponds a closed proper cone $F_p := \{f \in E : p(-f) \leq 0\}$. In Example 2.1. a), we have $F_p = \{0\}$; in b), $E_p = E_+$ and in c), $E_p = E_+$ if ϕ is strictly positive.

Let p be a continuous sublinear functional on F. The <u>subdifferential</u> dp of p is defined as follows. Let $f \in F$; then

$$(2.4) \qquad dp(f) = \{\phi \in F' : \langle g,\phi\rangle \leq p(g) \text{ for all } g \in F \text{ and } \langle f,\phi\rangle = p(f)\}.$$

It follows from the Hahn-Banach theorem that $dp(f) \neq \emptyset$ for all $f \in F$.

An operator A on F is called p-<u>dissipative</u> if for every $f \in D(A)$ there exists $\phi \in dp(f)$ such that $<Af, \phi> \leq 0$.

<u>Proposition</u> 2.3. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then the following are equivalent.

(i) T(t) is p-<u>contractive</u> for all $t \geq 0$;
i.e. $p(T(t)f) \leq p(f) \qquad (f \in E)$.

(ii) A is p-dissipative.

(iii) There exists a core D_o of A such that $A|_{D_o}$ is p-dissipative.

<u>Remark</u>. Suppose that p is a continuous half-norm. If A satisfies the equivalent conditions of the proposition, then the semigroup is positive for the ordering induced by p (see Remark 2.2).

For the proof of Proposition 2.3 see [5, Theorem 4.1] or [8, 2.1.1].

<u>Proposition</u> 2.4. Let A be a densely defined operator on E and $\phi \in D(A')_+$ such that $A'\phi \leq 0$. Denote by p the sublinear functional given by $p(f) = <f^+, \phi>$. If

(K) $$<(\text{sign } f)\, Af, \phi> \leq <|f|, A'\phi> \qquad (f \in D(A)),$$

then A is p-dissipative.

Proof. Let $P = I - P_{f^+} - P_{f^-}$, $Q = P_{f^+} + 1/2\ P$ and $\psi = Q'\phi$. We show that

(2.5) $\psi \in dp(f)$.

Let $g \in E$. Since $0 \leqq Q \leqq I$ we have $\langle g, \psi\rangle = \langle Qg, \phi\rangle \leqq \langle Qg^+, \phi\rangle \leqq \langle g^+, \phi\rangle = p(g)$. Moreover, $\langle f, \psi\rangle = \langle Qf, \phi\rangle = \langle P_{f^+} f + 1/2\ Pf, \phi\rangle = \langle f^+, \phi\rangle = p(f^+)$. So (2.5) follows by the definition of dp(f).

The proof will be finished when we have shown that

(2.6) $\langle Af, \psi\rangle \leqq 0$.

One has trivially

(2.7) $\langle (P_{f^+} + P_{f^-} + P)\ Af, \phi\rangle = \langle f, A'\phi\rangle$.

Addition of (2.7) and (K) gives

$\langle (2P_{f^+} + P)\ Af, \phi\rangle \leqq \langle 2f^+, A'\phi\rangle \leqq 0$.

Hence $\langle Af, \psi\rangle = \langle QAf, \phi\rangle \leqq 0$. □

Proof of Theorem 1.6. Proposition 1.1 and 1.5 give one implication. In order to show the other assume that the condition in Theorem 1.6 is satisfied. We have to show that $T(t) \geqq 0$ for all $t \geqq 0$.

Let $\phi \in M'$. Consider the half-norm $p(f) = \langle f^+, \phi\rangle$ and the operator $B = A - \lambda$, where $\lambda \in \mathbb{R}$ is such that $A'\phi \leqq \lambda\phi$. Then B satisfies $B'\phi \leqq 0$ and (K) as well. So it follows from Propo-

sition 2.4 that B is p-dissipative.
Since B generates the semigroup $(e^{-\lambda t}T(t))_{t\geq 0}$ we obtain from Proposition 2.3 that $p(e^{-\lambda t}T(t)f) \leq p(T(t)f)$ $(f \in E,\ t \geq 0)$. Hence,

$$(2.8)\quad \langle (T(t)f)^+, \phi\rangle \leq e^{\lambda t} \langle f^+, \phi\rangle \qquad (f \in E,\ t \geq 0)\ .$$

Now let $t > 0$ and $f \leq 0$. Then $f^+ = 0$, so it follows from (2.8) that $\langle (T(t)f)^+, \phi\rangle \leq 0$. Since $\phi \in M'$ is arbitrary and M' is strictly positive, it follows that $(T(t)f)^+ = 0$; i.e., $T(t)f \leq 0$. This implies that $T(t) \geq 0$. □

The proof of Theorem 1.10 is identical to the proof given above if the symboles (sign f), P_{f^+}, etc. are interpreted as Borel functions.

<u>Remark</u> 2.5. a) Proposition 1.1, which gives one implication of Theorem 1.6, had been proved (in a different way) in [3, Remark 3.9]. The other implication of Theorem 1.1 has been obtained independently by A.R. Schep [46] with a different method of proof. In particular, Schep's argument seems not to apply for the case where condition (K) is only known to hold on a core of A (cf. Remark 1.8).
b) Using Proposition 2.4 one can show with the help of the proof of [5, Theorem 2.4] that a densely defined operator which satisfies the conditions of Theorem 1.6 is closable (cf. Theorem 4.4).

Remark 2.6. The proof of Theorem 1.6 shows the following. If A is the generator of a positive semigroup and E' contains strictly positive linear forms, then there exist a continuous half-norm p on E and $w \in \mathbb{R}$ such that $A-w$ is p-dissipative. We stress that p cannot be replaced by the norm, since in general none of the semigroups $(e^{-wt}T(t))_{t\geq 0}$ $(w \in \mathbb{R})$ is contractive for the norm (cf. [7] and [17]).

3. Examples and discussion

As a first example we consider the first derivative with boundary conditions on $E = L^p[0,1]$ $(1 \leq p < \infty)$. By $AC[0,1]$ we denote the space of all absolutely continuous functions on $[0,1]$. Let A_{max} be given by

$$D(A_{max}) = \{f \in AC[0,1] : f' \in L^p[0,1]\}$$
$$A_{max}f = f' \quad (f \in D(A_{max})).$$

Lemma 3.1. Let $f \in AC[0,1]$. Then $|f| \in AC[0,1]$ and

$$|f|' = (\text{sign } f)\cdot f' \quad \text{(a.e.)} .$$

This is easy to prove.

As a consequence of the lemma, $D(A_{max})$ is a sublattice of E and

$$(3.1) \qquad (\text{sign } f)A_{max}f = A_{max}|f| \quad (f \in D(A_{max})).$$

For $\lambda > 0$ one has

(3.2) $\quad \ker(\lambda - A_{max}) = \mathbb{R}\cdot e_\lambda \quad$ where $e_\lambda(x) = e^{\lambda x}$.

Hence A_{max} is not a generator. We impose the following boundary conditions.

Let $d \in \mathbb{R}$. Consider the restriction A_d of A_{max} with the domain

$$D(A_d) = \{f \in D(A_{max}) : f(1) = df(0)\} .$$

Then A_d is the generator of the semigroup $(T_d(t))_{t\geq 0}$ given by

(3.3) $\quad T_d(t)f(x) = d^n f(x+t-n) \quad$ if $\quad x+t \in [n, n+1) \quad (n \in \mathbb{N})$.

This is not difficult to prove. Actually (3.3) defines a group if $d \neq 0$ and if we let $t \in \mathbb{R}$, $n \in \mathbb{Z}$. For $d = 0$ one obtains the nilpotent shift semigroup on E. One sees from (3.3) that the semigroup $(T_d(t))_{t\geq 0}$ is positive if and only if $d \geq 0$.

Let us fix $d < 0$. Let $A = A_d$ and $T(t) = T_d(t)$ for $t \geq 0$. Then $(T(t))_{t\geq 0}$ is a semigroup which is <u>not positive</u> . Nevertheless its generator A satisfies Kato's inequality. Even the equality is valid; i.e.

(3.4) $\quad \langle(\text{sign } f)\, Af, \phi\rangle = \langle|f|, A'\phi\rangle$
for all $f \in D(A)$, $0 \leq \phi \in D(A')$.

<u>Proof</u>. It is not difficult to see that

$$(3.5)\quad D(A') = \{\phi \in AC[0,1] : \phi' \in L^q[0,1],\ \phi(0) = d\phi(1)\}$$
$$A'\phi = -\phi \quad \text{for all } \phi \in D(A').$$

where $1/p + 1/q = 1$. Let $\phi \in D(A')_+$. Since $d < 0$, it follows that $\phi(0) = \phi(1) = 0$. Hence for $f \in D(A)$,

$$\begin{aligned} <(\text{sign } f)\, Af,\ \phi> &= <(\text{sign } f)\ f',\ \phi> = <|f|',\ \phi> \\ &= \int_0^1 |f|'(x)\ \phi(x)\ dx \\ &= |f|\phi\ \Big|_0^1 - \int_0^1 |f(x)|\ \phi'(x)\ dx \\ &= |f(1)|\phi(1) - |f(0)|\phi(0) + <|f|,\ A'\phi> \\ &= <|f|,\ A'\phi> \quad \square \end{aligned}$$

Remark 3.2. The equality (3.4) does not hold for all $\phi \in D(A')$, however. In fact, this would imply that $|f| \in D(A)$ and $(\text{sign } f)Af = A|f|$ for all $f \in D(A)$. Thus by [31,3.5] (or Corollary 5.6) the semigroup would consist of lattice homomorphisms. The reason why in this example the equality holds will be explained from a more general point of view in section 5 (see Proposition 5.9).

Even though the semigroup $(T(t))_{t\geq 0}$ is not positive its generator A has other surprising properties besides (3.4). For instance, the positive cones $D(A)_+ := D(A) \cap E_+$ and $D(A')_+ := D(A') \cap E'_+$ satisfy

$$(3.6) \qquad \overline{D(A)_+} = E_+ \quad \text{and} \quad \overline{D(A')_+}^{\sigma(E',E)} = E'_+ .$$

Thus the question following Remark 3.10 in [3] (resp. Problem 1.5 in [4]) has a negative answer.

Moreover, (3.1) shows that A satisfies Kato's inequality (in the strong sense) formally. In order to formulate this more precisely, observe that it follows from (3.2) that $D(A_{max}) = D(A) + \mathbb{R}\cdot e_\lambda$ (where $0 < \lambda \in \rho(A)$). Thus the extension A_{max} of A satisfies the following.

(3.7) A_{max} is closed.

(3.8) $D(A_{max})$ is a sublattice of E.

(3.9) D(A) has codimension one in $D(A_{max})$.

(3.10) $(\text{sign } f)\, Af = A_{max}|f|$ for all $f \in D(A)$.

It is also remarkable that there exists a dense sublattice $D_o := \{f \in D(A) : f(0) = f(1) = 0\}$ of E which is included in D(A). But D_o is not a core of A (this would imply the positivity of the semigroup by [4, Theorem 3.4] if $|d| \leq 1$).

Since $(T(t))_{t\geq 0}$ is not positive but (3.4) holds, it follows from Theorem 1.6 that there exists no strictly positive subeigenvector of A'. In fact, more is true.

(3.11) $0 \leq \phi \in D(A')$, $A'\phi \leq \mu\phi$ for some $\mu \in \mathbb{R}$ implies $\phi = 0$.

Proof. Suppose that $0 \leqq \phi \in D(A')$ such that $-\phi' = A'\phi \leqq \mu\phi$. We can assume that $0 \leqq \mu$. Let $\psi(x) = \phi(1-x)$. Then $\psi'(x) = -\phi'(1-x) \leqq \mu\phi(1-x) = \mu\psi(x)$. Since $\psi(0) = 0$, we get

$$\psi(x) = \int_0^x \psi'(y)\, dy \leqq \mu \int_0^x \psi(y)\, dy \qquad (x \in [0,1]).$$

It follows from Gronwall's Lemma that $\psi \leqq 0$. Hence $\phi = \psi = 0$. □

In view of the preceding example one might presume that the existence of a strictly positive set of subeigenvectors of the adjoint of the generator actually implies the positivity of the semigroup. This is not the case.
To give an example consider $E = L^2(\mathbb{R})$ and the operator B given by

$$Bf = f^{(3)} \quad \text{with domain}$$
$$D(B) = \{f \in C^2(\mathbb{R}) : f', f'' \in L^2(\mathbb{R}),\ f'' \in AC(\mathbb{R}),\ f^{(3)} \in L^2(\mathbb{R})\}$$

Then B is the generator of an unitary group $(U(t))_{t\in\mathbb{R}}$. In particular, B is skew-adjoint, i.e. $B' = -B$.

(3.12) B' has a strictly positive subeigenvector ϕ

Proof. Let $\lambda > 0$ and

$$\phi(x) = \begin{cases} e^{-\lambda x} & \text{for } x \geqq 1 \\ g(x) & -1 < x < 1 \\ e^{\lambda x} & \text{for } x \leqq -1 \end{cases}$$

where $g \in C^3[-1,1]$ such that $g(x) > 0$ for all $x \in [-1,1]$ and such that $\phi \in C^3(\mathbb{R})$. Moreover, choose g such that $g(0) = 1$ and $g'(0) = g''(0) = 0$. Since g, $g^{(3)} \in C^3(\mathbb{R})$ and $\inf\{g(x) : x \in [-1,1]\} > 0$ there exists $\mu \geq \lambda^3$ such that $-g^{(3)}(x) \leq \mu\, g(x)$ for all $x \in [-1,1]$. Consequently,

$$-\phi^{(3)}(x) = \begin{cases} \lambda^3 e^{-\lambda x} & (x \geq 1) \\ -g^{(3)}(x) & (x \in [-1,1]) \\ -\lambda^3 e^{\lambda x} & (x < -1) \end{cases} \leq \mu\, \phi(x).$$

Hence $B'\phi = -\phi^{(3)} \leq \mu\, \phi$. □

But the semigroup $(U(t))_{t\geq 0}$ is <u>not</u> positive. In fact, we show that there exists $f \in D(B)$ such that

$$\text{(3.13)} \qquad \langle (\text{sign } f)\, Bf, \phi\rangle > \langle |f|, B'\phi\rangle.$$

<u>Proof</u>. Let $f \in D(B)$ be such that $f(x) = e^{-x} \sin x$ in a neighborhood of 0 and $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Then

$$\langle (\text{sign } f)\, Bf, \phi\rangle = - \int_{-\infty}^{0} f^{(3)}(x)\phi(x)\, dx + \int_{0}^{\infty} f^{(3)}(x)\phi(x)\, dx.$$

Hence,

$$\langle |f|, B'\phi\rangle = \int_{-\infty}^{0} (-f(x))(-\phi^{(3)}(x))\, dx + \int_{0}^{\infty} f(x)(-\phi^{(3)}(x))\, dx$$

$$= - \int_{-\infty}^{0} f^{(3)}(x)\phi(x)\, dx + \int_{0}^{\infty} f^{(3)}(x)\phi(x)\, dx$$

$$+ [f''\phi]|_{-\infty}^{0} - [f''\phi]|_{0}^{\infty} \qquad \text{(since } \phi''(0)=\phi'(0)=0)$$

$$= <(\text{sign } f)\, Bf, \phi> + 2f''(0)\phi(0)$$

$$< <(\text{sign } f)\, Bf, \phi>$$

since $f''(0)\phi(0) = f''(0) = -2$. □

We now show that B satisfies Kato's inequality for positive elements, however; i.e.

(3.14) $\qquad P_f\, Bf \leq Bf \quad$ for all $f \in D(B)_+$.

In fact, more is true. B is <u>local</u>, i.e.

(3.15) $\qquad f \perp g$ implies $Af \perp g$ for all $f \in D(B)$, $g \in L^2(\mathbb{R})$.

<u>Proof</u>. Let A be the generator of the translation group. Then A is local by [31, 3.3]. Hence $B = A^3$ is local as well. □

So this example shows that even if there exists a strictly positive subeigenvector of the adjoint of the generator, Kato's inequality for positive elements alone does not suffice for the positivity of the semigroup.

Next we make some observations concerning positive subeigenvectors. Assume that A is the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E. Let $\phi \in D(A')_+$ and

$\lambda \in \mathbb{R}$. Then

(3.16) $A'\phi \leqq \lambda\,\phi$ if and only if $T(t)'\phi \leqq e^{\lambda t}\phi$ $(t \geqq 0)$.

Proof. If $T(t)'\phi \leqq e^{\lambda t}\,\phi$ for all $t \geqq 0$, then

$$A'\phi = \sigma(E',E)\text{-}\lim_{t\to 0} 1/t\ (T(t)'\phi - \phi) \leqq \lim_{t\to 0} 1/t\ (e^{\lambda t}\phi - \phi) = \lambda\phi.$$

For the converse let $f \in E_+$. Then

$$\langle f, T(t)'\phi\rangle = \langle f, \phi\rangle + \int_0^t \langle f, T(s)'A'\phi\rangle\, ds$$

$$\leqq \langle f, \phi\rangle + \lambda \int_0^t \langle f, T(s)'\phi\rangle\, ds.$$

It follows from Gronwall's lemma that $\langle f, T(t)'\phi\rangle \leqq e^{\lambda t} \langle f, \phi\rangle$. □

Assume now that ϕ is a subeigenvector of A'. Then it follows from (3.16) that the ideal $J := \{f \in E : \langle |f|, \phi\rangle = 0\}$ is invariant under the semigroup. From this we conclude

Proposition 3.3. If the semigroup is irreducible (see [45]), then every positive subeigenvector of A' is strictly positive.

Example. For $d > 0$ the semigroup $(T_d(t))_{t\geqq 0}$ considered at the beginning of this section is irreducible. Thus every positive subeigenvector of A'_d is strictly positive.

The existence of positive subeigenvectors is related to the Krein-Rutman theorem. If A has a compact resolvent and

$\sigma(A) \neq \emptyset$, then the Krein-Rutman theorem asserts that there exists a positive eigenvector of A' (and A) for the eigenvalue
$s(A) := \sup \{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$.
It is easy to see that A_d has compact resolvent and $\sigma(A_d) \neq \emptyset$ for $d \neq 0$. Thus A_d' has a positive eigenvector if and only if $d > 0$.

4. Domination

Frequently it is useful to be able to compare two semigroups on a Banach lattice with respect to the ordering.
In this section we assume that E is a σ-order complete complex Banach lattice [42, II §11]. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A and $(S(t))_{t\geq 0}$ a semigroup with generator B. We say, $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$ if

(4.1) $|S(t)f| \leq T(t)|f|$ for all $f \in E$, $t > 0$.

We first observe that domination of the semigroup is equivalent to domination of the resolvents. More precisely, (4.1) holds if and only if

(4.2) $|R(\lambda,B)f| \leq R(\lambda,A)|f|$ $(f \in E)$ for large real λ.

Proof. (4.2) follows from (4.1) since the resolvent is given by the Laplace transform of the semigroup. Conversely, if (4.2) holds, then

$$\begin{aligned}|S(t)f| &= \lim_{n\to\infty} |((n/t)\ R(n/t,B))^n f| \\ &\leq \lim_{n\to\infty} ((n/t)\ R(n/t,A))^n |f| \\ &= T(t)|f| \qquad (t \geq 0,\ f \in E). \quad \square\end{aligned}$$

One can describe domination by an inequality for the generators in a manner analoguous to the characterization of positive semigroups in section 1, however, no positive subeigenvectors are needed here.

We briefly want to explain the sign operator in a complex Banach lattice. Let $f \in E$. There exists a unique operator $S \in \mathcal{L}(E)$ satisfying

(4.3) $Sf = |f|$

(4.4) $|Sg| \leq |g| \qquad (g \in E)$

(4.5) $Sg = 0 \qquad$ if $g \perp f$

(see [31 ,2.1]).

<u>Example</u> 4.1. Let $E = L^p(X,\mu)$ $(1 \leq p < \infty)$ and $f \in E$. Then

$$(\text{sign } f)(x) = \begin{cases} f(x)/|f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

defines a function in L^∞. The operator S is given by
$Sg = (\text{sign } f)\cdot g \quad (g \in E)$.

We define $\text{sign } f := S \in \mathcal{L}(E)$. Thus in the case $E = L^p$ we identify the function sign f and the multiplication operator it defines.

Remark 4.2. If $(T(t))_{t\geq 0}$ is a positive semigroup on a σ-order complete complex Banach lattice, then its generator satisfies Kato's inequality in the form (K) if 'sign f' is interpreted as above (see also [4]). However, for the characterization of positive semigroups one can restrict oneself to the real case by making use of the following observation.

Let E be a complex Banach lattice. Denote by $E_{\mathbb{R}}$ the real Banach lattice associated with E. Then $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$; i.e. for $f \in E$ there exist unique elements Ref, Imf of $E_{\mathbb{R}}$ such that $f = \text{Re}f + i\text{Im}f$. Let $\overline{f} = \text{Re}f - i\text{Im}f$.

Let $(S(t))_{t\geq 0}$ be a semigroup on E with generator A. We say that $(S(t))_{t\geq 0}$ is real if $S(t)E_{\mathbb{R}} \subset E_{\mathbb{R}}$ for all $t \geq 0$. It is easy to describe this in terms of the generator. We say that A is real if $f \in D(A)$ implies $\overline{f} \in D(A)$ and $A\overline{f} = \overline{Af}$. Then

(4.6) $(S(t)_{t\geq 0}$ is real if and only A is real.

Theorem 4.3. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A and $(S(t))_{t\geq 0}$ a semigroup with generator B . The following assertions are equivalent.

(i) $|S(t)f| \leq T(t)|f|$ for all $f \in E$, $t > 0$

(ii) $\text{Re} \langle(\text{sign } f)\, Bf, \phi\rangle \leq \langle|f|, A'\phi\rangle$ for all $f \in D(B)$, $\phi \in D(A')_+$

The author learnt Theorem 4.3 from T. Kato. There are similar results due to B. Simon [47], [48] and Hess, Schrader and

Uhlenbrock [24]. Our aim is to generalize Theorem 4.3 by replacing the condition that B is a generator by a range condition. The precise formulation is the following.

<u>Theorem</u> 4.4. Let $(T(t))_{t\geq 0}$ be a <u>positive</u> semigroup with generator A. Let B be a densely defined operator such that

$$(4.7)\qquad \begin{array}{l} \mathrm{Re} \langle (\mathrm{sign}\ f)\ Bf, \phi\rangle \leqq \langle |f|, A'\phi\rangle \\ \text{for all } f \in D(B),\ \phi \in D(A')_+ \end{array}$$

Then B is closable. Moreover, if $(\lambda - B)D(B)$ is dense in E for some $\lambda > \max\{0, s(A)\}$, then $B^{\#}$ (the closure of B) generates a semigroup which is dominated by $(T(t))_{t\geq 0}$.

We will use the following notion. Let A be the generator of a positive semigroup. The <u>spectral bound</u> s(A) is defined by $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$. Note that $R(\lambda,A) \geqq 0$ for all $\lambda \geqq s(A)$ (see section II 1 for more details).

<u>Proof of Theorem</u> 4.4. 1. We show that B is closable.
Let $D(B)\ u_n \to 0$ such that $Bu_n \to v$. We have to show that $v = 0$. Considering $A - \mu$ and $B - \mu$ for some $\mu > s(A)$ instead of A and B we may assume that $s(A) < 0$. Then there exists a strictly positive set $M' \subset E'$ such that

$$(4.8)\qquad \phi \in D(A') \text{ and } A'\phi \leqq 0 \quad \text{for all } \phi \in M'$$

(see the proof of Proposition 1.5).

Let $\phi \in M'$ and p be the sublinear functional given by $p(f) = \langle |f|, \phi\rangle$. We show that B is p-dissipative.
Let $f \in D(B)$, $\psi = (\text{sign } f)'\phi$. Then it is easy to see that
$\psi \in dp(f) := \{\psi \in E' : \text{Re}\langle g,\psi\rangle \leqq p(g) \quad (g \in E) \; ; \langle f,\psi\rangle = p(f)\}$.
Moreover, by (4.7) and (4.8) one obtains that
$\text{Re}\langle Bf,\psi\rangle = \text{Re}\langle(\text{sign } f)\, Bf, \phi\rangle \leqq \langle |f|, A'\phi\rangle \leqq 0$.
Thus B is p-dissipative; i.e.
$p((\lambda - B)f) \geqq \lambda p(f) \quad$ for all $f \in E$, $\lambda > 0$.
By the proof of [5, Theorem 2.4] one sees that $p(v) = 0$; i.e. $\langle |v|,\phi\rangle = 0$. Since $\phi \in M'$ was arbitrary we conclude that $v = 0$.

2. Let $\lambda > \lambda_o := \max\{s(A),0\}$. We show that for $f \in D(B)$,

(4.9) $\quad g = (\lambda - B)f$ implies $|f| \leqq R(\lambda,A)|g|$.

Let $\psi \in E'_+$. We have to show that $\langle |f|,\psi\rangle \leqq \langle R(\lambda,A)|g|,\psi\rangle$.
Let $\phi = R(\lambda,A)'\psi \in D(A')_+$. Then by (4.7)

$$\begin{aligned}\langle |f|,\psi\rangle &= \langle |f|, (\lambda - A')\phi\rangle = \text{Re}\langle(\text{sign } f)\ (\lambda f), \phi\rangle - \langle |f|, A'\phi\rangle \\ &= \text{Re}\langle(\text{sign } f)(\lambda - B)f, \phi\rangle + \text{Re}\langle(\text{sign } f)Bf,\phi\rangle - \langle |f|,A'\phi\rangle \\ &\leqq \text{Re}\langle(\text{sign } f)(\lambda - B)f, \phi\rangle = \text{Re}\langle(\text{sign } f)\ g, \phi\rangle \\ &\leqq \langle |g|, \phi\rangle = \langle |g|, R(\lambda,A)'\psi\rangle = \langle R(\lambda,A)|g|, \psi\rangle.\end{aligned}$$

It follows from (4.9) that for $\lambda > \lambda_o$ and $f \in D(B^\#)$

(4.10) $\qquad g = (\lambda - B^\#)f$ implies $|f| \leqq R(\lambda,A)|g|$.

In particular, $(\lambda - B^\#)$ is injective for $\lambda > \lambda_o$. Moreover, (4.11) $\quad |R(\lambda,B^{\#})g| \leq R(\lambda,A)|g|$ for all $g \in E$ whenever $\lambda_o < \lambda \in \rho(B^{\#})$.

Assume now that there exists $\mu > \lambda_o$ such that $(\mu - B)D(B)$ is dense in E. Then $(\mu - B^{\#})D(B^{\#}) = E$. (Indeed, let $h \in E$. There exists $f_n \in D(B)$ such that $g_n := (\mu - B)f_n \to h$. By (4.9) it follows that $|f_n - f_m| \leq R(\lambda,A)|g_n - g_m|$. Thus (f_n) is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $f \in D(B^{\#})$ and $(\mu - B^{\#})f = h$.) Thus $\mu \in \rho(B^{\#})$.

Let $\lambda_o < \lambda \in \rho(B^{\#})$. Then it follows from (4.11) that
$||R(\lambda,B^{\#})|| \leq ||R(\lambda,A)|| \leq ||R(\lambda_o,A)|| := c$. Hence,
$\mathrm{dist}(\lambda,\sigma(B^{\#})) = r(R(\lambda,B^{\#}))^{-1} \geq ||R(\lambda,B^{\#})||^{-1} \geq 1/c$.
This implies that $[\lambda_o, \infty) \subset \rho(B^{\#})$. Moreover, it follows from (4.11) that

(4.12) $$|R(\lambda,B^{\#})^n f| \leq R(\lambda,A)^n|f| \qquad (f \in E,\ n \in \mathbb{N}).$$

Let $w > \omega(A)$ (the type of $(T(t))_{t\geq 0}$). Then it follows from (4.12) that
$||(\lambda - w)^n R(\lambda,B^{\#})^n|| \leq ||(\lambda - w)^n R(\lambda,A)^n||$ for all $\lambda > w$, $n \in \mathbb{N}$.
So by the Hille-Yosida theorem, $B^{\#}$ is the generator of a semigroup $(S(t))_{t\geq 0}$. Finally, the domination follows from (4.11). □

<u>Proof of Theorem 4.3</u> One direction follows from Theorem 4.4. The other can be proved in a way similar to Proposition 1.1. □

Example 4.5. As an illustration of Theorem 4.3 we consider the complex version of the first example of section 1.
Let $E = L^p[0,1]$. For $d \in \mathbb{C}$ let $A_d f = f'$ with domain $D(A_d) = \{f \in AC[0,1] : f(1) = df(0)\}$. Then A_d generates a semigroup $(T_d(t))_{t\geq 0}$. Let $|d| \leq c$. Then $(T_d(t))_{t\geq 0}$ is dominated by $(T_c(t))_{t\geq 0}$. This can be seen by Theorem 4.3 as follows. Let $f \in D(A_d)$, $0 \leq \phi \in D(A_c')$. Then $\phi(o) = c\phi(1)$. Hence

$$\begin{aligned}
\mathrm{Re} <(\mathrm{sign}\ f)A_d f, \phi> &= \mathrm{Re} <(\mathrm{sign}\ f)f', \phi> = <|f|', \phi> \\
&= <|f|, -\phi'> + (|f(x)|\phi(x))|_o^1 \\
&= <|f|, (A_c)'\phi> + |f(1)|\phi(1) - |f(0)|\phi(0) \\
&= <|f|, (A_c)'\phi> + |f(0)|\phi(1)(|d| - c) \\
&\leq <|f|, (A_c)'\phi>.
\end{aligned}$$

Of course, in this example domination can also be verified by inspection of the semigroups.

Example 4.6. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A. Let $M \in Z(E)$ (the center of E (see [53 , chapter 20]). For example, if $E = L^p(X,\mu)$ (where (X,μ) is a σ-finite measure space and $1 \leq p \leq \infty$) then M is the multiplication operator defined by a function in $L^\infty(X,\mu)$.
Let $B = A + M$. Then B generates a semigroup $(S(t))_{t\geq 0}$.
Assume that $\mathrm{Re}M \leq 0$. Let $f \in D(B)$ and $\phi \in D(A')_+$. Then

$$\begin{aligned}
\mathrm{Re}<(\mathrm{sign}\ f)\ Bf, \phi> &= \mathrm{Re}<(\mathrm{sign}\ f)\ Af, \phi> + \mathrm{Re}<(\mathrm{sign}\ f)Mf, \phi> \\
&= \mathrm{Re}<(\mathrm{sign}\ f)\ Af, \phi> + \mathrm{Re}<M|f|, \phi> \\
&\leq <|f|, A'\phi> .
\end{aligned}$$

Thus, by Theorem 4.3, $(S(t))_{t\geq 0}$ is dominated by $(T(t))_{t\geq 0}$.

Domination and positivity are characterized simultaneously as follows.

<u>Proposition</u> 4.7. Let E be a σ-order complete real Banach lattice. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A and let $(S(t))_{t\geq 0}$ be a semigroup with generator B. The following are equivalent.

(i) $0 \leq S(t) \leq T(t)$ for all $t \geq 0$.

(ii) $< P_f + Bf, \phi> \leq <f^+, A'\phi>$ for all $f \in D(B)$, $\phi \in D(A')_+$.

(iii) $<P_f + Bf, \phi> \leq <f^+, A'\phi>$ for all $f \in D_o$, $\phi \in D(A')_+$,
where D_o is a core of B.

<u>Remark</u> 4.8. Condition (ii) implies (4.7) (cf. Remark 1.9).

<u>Proof</u>. One proves as in Proposition 1.1 that (i) implies (ii). It is trivial that (ii) implies (iii). Assume that (iii) holds. Let $\lambda > \lambda_o = \max \{s(A), s(B), 0\}$. In a similar way as (4.10) one shows that for all $f \in D_o$

(4.13) $\lambda f - Bf = g$ implies $f^+ \leq R(\lambda, A)g^+$.

Since D_o is a core it follows that (4.13) also holds for all $f \in D(B)$. This implies that $(R(\lambda,B)g)^+ \leq R(\lambda,A)g^+$ for all

$g \in E$, $\lambda > \lambda_o$. Consequently, $0 \leqq R(\lambda,B) \leqq R(\lambda,A)$ for all $\lambda > \lambda_o$. Hence (i) holds. □

Finally, if it is known that the semigroup $(S(t))_{t\geqq 0}$ also is positive, domination can be characterized as follows.

<u>Proposition</u> 4.9. Let E be a real Banach lattice, $(T(t))_{t\geqq 0}$ a positive semigroup with generator A and $(S(t))_{t\geqq 0}$ a positive semigroup with generator B. Consider the following conditions.

(i) $S(t) \leqq T(t) \qquad (t \geqq 0)$.

(ii) $\langle Bf,\phi\rangle \leqq \langle f,A'\phi\rangle$ for all $f \in D(B)_+$, $\phi \in D(A')_+$.

(iii) $Bf \leqq Af$ for $0 \leqq f \in D(A) \cap D(B)$.

Then (i) and (ii) are equivalent and imply (iii).

Moreover, if $D(A) \subset D(B)$ or $D(B) \subset D(A)$, then (iii) implies (i).

<u>Proof</u>. Assume that (i) holds. Then for $f \in D(B)_+$, $\phi \in D(A')_+$,

$$\langle Bf,\phi\rangle = \lim_{t\to o} 1/t \langle S(t)f - f, \phi\rangle \leqq \lim_{t\to o} 1/t \langle T(t)f - f, \phi\rangle$$
$$= \langle f, A'\phi\rangle.$$

So (ii) holds. (iii) is proved similarly.

Now assume (ii). Let $\lambda > \max \{s(A), s(B)\}$. Let $g \in E_+$, $\psi \in E'_+$.

Then $\langle R(\lambda,B)g - R(\lambda,A)g, \psi\rangle =$

$\langle R(\lambda,A)g, \lambda R(\lambda,B)'\psi - \psi\rangle - \langle \lambda R(\lambda,A)g - g, R(\lambda,B)'\psi\rangle =$

$\langle f, B'\phi\rangle - \langle Af, \phi\rangle \leqq 0$ where $f = R(\lambda,A)g \in D(A)_+$ and

$\phi = R(\lambda,B)'\psi \in D(B')_+$. Hence $R(\lambda,B) \leqq R(\lambda,A)$ and (i) follows.

Finally, we prove that (iii) implies (i) if $D(B) \subset D(A)$, say.

Let $\lambda > \max \{s(A), s(B)\}$. Then $(A - B)R(\lambda,B)$ is a positive operator. Hence $R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A - B)R(\lambda,B) \geq 0$. This implies (i). □

Example 4.10. Let B be the generator of a positive semigroup $(S(t))_{t\geq 0}$, C a bounded positive operator. Then $A = B + C$ with $D(A) = D(B)$ is the generator of a semigroup $(T(t))_{t\geq 0}$. It can be seen from the product formula (see e.g. [13]) that $(T(t))_{t\geq 0}$ is positive. Since $Bf \leq Af$ for all $f \in D(B)_+$, it follows from Proposition 4.9 that $S(t) \leq T(t)$ for all $t \geq 0$.

The preceding results can be applied to the perturbation by multiplication operators. Let (X,μ) be a σ-finite measure space and $E = L^p(X,\mu)$ $(1 \leq p < \infty)$. Consider a positive semigroup $(T(t))_{t\geq 0}$ with generator A. Let $m : X \to \mathbb{R}$ be a measurable function such that $m(x) \leq 0$ for all $x \in X$. Let
$D(m) = \{f \in E : f\cdot m \in E\}$. Define the operator B with domain
$D(B) = D(A) \cap D(m)$ by $Bf = Af + m\cdot f$ $(f \in D(B))$.

Theorem 4.11. If there exists a quasi-interior subeigenvector u of A such that $u \in D(m)$, then B is closable and the closure $B^{\#}$ of B is the generator of a positive semigroup $(S(t))_{t\geq 0}$ which is dominated by $(T(t))_{t\geq 0}$.

For the proof of the theorem we need the following lemma.

Lemma 4.12. Let A and B be generators of positive semigroups $(T(t))_{t\geq 0}$ (resp., $S(t)_{t\geq 0}$). If $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$, then $s(B) \leq s(A)$.

Proof of Lemma 4.12. Let $\lambda > s(A)$. Then for all $\mu \geq \max\{\lambda, s(B)\}$ one has $0 \leq R(\mu,B) \leq R(\lambda,A)$, and so $\operatorname{dist}(\mu,\sigma(B)) \geq \|R(\mu,B)\|^{-1} \geq \|R(\lambda,A)\|^{-1}$. This implies that $[\lambda,\infty) \subset \rho(B)$. □

Proof of Theorem 4.11. There exists $\mu > 0$ such that $Au \leq \mu u$. Let $\lambda > \max\{s(A), \mu\}$. Then $\lambda R(\lambda,A)u = AR(\lambda,A)u+u \leq \mu R(\lambda,A)u + u$. Hence $R(\lambda,A)u \leq c\,u$ where $c > 0$. It follows that $R(\lambda,A)E_u \subset E_u \cap D(A) \subset D(B)$. Hence $D(B)$ is dense.
Let $f \in D(B)$, $\phi \in D(A')_+$. Then

$$(4.14)\quad \langle P_f + Bf, \phi\rangle \leq \langle f^+, A'\phi\rangle.$$

In fact,
$$\begin{aligned}\langle P_f + Bf, \phi\rangle &= \langle P_f + Af, \phi\rangle + \langle P_f + m\cdot f, \phi\rangle \\ &= \langle P_f + Af, \phi\rangle + \langle m\cdot f^+, \phi\rangle \\ &\leq \langle P_f + Af, \phi\rangle \\ &\leq \langle f^+, A'\phi\rangle \quad \text{(by (1.7))}.\end{aligned}$$

But (4.14) implies (4.7). So it follows from Theorem 4.4 that B is closable. Moreover, if we can show that $(\lambda - B^\#)D(B^\#)$ is dense in E, it follows that $B^\#$ is the generator of a semigroup $(S(t))_{t\geq 0}$. In that case (4.14) implies by Proposition 4.7 that $(S(t))_{t\geq 0}$ is dominated by $(T(t))_{t\geq 0}$.
We show now that $(\lambda - B^\#)D(B^\#)$ is dense in E.
Let $m_n = \sup\{m, -n1_X\}$ $(n \in \mathbb{N})$ and $B_n = A + m_n$. Then B_n is the generator of a positive semigroup and it follows from

Proposition 4.9 that $0 \leq R(\lambda,B_{n+1}) \leq R(\lambda,B_n) \leq R(\lambda,A)$ for all $n \in \mathbb{N}$, $\lambda > s(A)$. (Note that $s(B_n) \leq s(A)$ by Lemma 4.12). Let $0 \leq f \in E_u$. Let $g_n = R(\lambda,B_n)f$. Then $g = \inf_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} g_n$ exists. Moreover $g_n \in D(B)$ and $\lim_{n\to\infty} (\lambda - B)g_n = f + \lim_{n\to\infty} (B_n - B)g_n = f$, since $|(B_n - B)g_n| \leq (m_n - m)|g_n| = (m_n - m)|R(\lambda,B_n)f| \leq (m_n - m)R(\lambda,A)|f| \leq c'(m_n - m)u$. But $\lim_{n\to\infty} (m_n - m)u = 0$ since $u \in D(m)$. Thus $g \in D(B^\#)$ and $(\lambda - B^\#)g = f$. We have shown that $E_u \subset (\lambda - B^\#)D(B^\#)$. Hence $(\lambda - B^\#)D(B^\#)$ is dense in E. □

<u>Example</u> 4.13. If $D(A) \subset L^\infty(X,\mu)$ and $m \in L^p(X,\mu)$, then the hypotheses of Theorem 4.11 are satisfied.

5. <u>Semigroups of disjointness preserving operators</u>

In this section we consider a special case of domination. Let E be a complex Banach lattice. A bounded operator S on E is called <u>disjointness preserving</u> if

(5.1) $f \perp g$ implies $Sf \perp Sg$ $(f,g \in E)$.

Note that an operator S is a lattice homomorphism [42 , II 2.4] if and only if S is positive and disjointness preserving.

In the following we will consider <u>disjointness preserving semigroups</u> (by this we mean semigroups of disjointness preserving operators). An example is the semigroup $(T_d(t))_{t \geq 0}$ defined in section 3.

Remark 5.1. In [2] we called order bounded disjointness preserving operators Lamperti operators, and it was shown that on a σ-order complete Banach lattice every disjointness preserving operator is automatically order bounded. More recently Abramovich [1] showed that the assumption of σ-order continuity can be omitted and de Pagter [35] gave a simplified proof of this fact.

If $S \in \mathcal{L}(E)$ is disjointness preserving, then the modulus $|S|$ of S exists. $|S|$ is a lattice homorphism and is related to S by

(5.2) $|Sf| = |S|\,|f| \quad (f \in E)$.

Proposition 5.2. Let $(S(t))_{t \geq 0}$ be a disjointness preserving semigroup. Let $T(t) = |S(t)|$ $(t \geq 0)$. Then $(T(t))_{t \geq 0}$ is a strongly continuous semigroup.

Proof. Let $0 \leq s,t$ and $f \in E_+$. Then by (5.2),
$T(s)T(t)f = T(s)|S(t)f| = |S(s)S(t)f| = |S(s+t)f| = T(s+t)f$.
Since span $E_+ = E$, it follows that $(T(t))_{t \geq 0}$ is a semigroup. Moreover, for $f \in E_+$, $\lim_{t \to 0} T(t)f = \lim_{t \to 0} |S(t)f| = |f| = f$. This implies that $(T(t))_{t \geq 0}$ is strongly continuous. □

Remark. R. Derndinger [16] investigates the modulus of a semigroup in other cases.

Example 5.3. Let $d \in \mathbb{C}$ and $S(t) = T_d(t)$ be given by (3.3). Then $T(t) = T_{|d|}(t) \quad (t \geq 0)$.

Proposition 5.4. Let B be the generator of a disjointness preserving semigroup $(S(t))_{t\geq 0}$ on a Banach lattice E. Then B is local; i.e.

(5.3) $Bf \perp g$ if $f \in D(B)$, $g \in E$ such that $f \perp g$.

The proof of [31, 3.3] can be adapted in an obvious way.

We now describe the relation between the generator of a disjointness preserving semigroup and the generator of the modulus semigroup.

Theorem 5.5. Assume that E is a complex Banach lattice with order continuous norm. Let $(S(t))_{t\geq 0}$ be a semigroup with generator B . The following assertions are equivalent.

(i) $(S(t))_{t\geq 0}$ is disjointness preserving.

(ii) There exists a semigroup $(T(t))_{t\geq 0}$ with generator A such that

(5.4) $f \in D(B)$ implies $|f| \in D(A)$ and $\mathrm{Re}\,((\mathrm{sign}\, f)\, Bf) = A|f|$

Moreover, if these equivalent conditions are satisfied, then $T(t) = |S(t)| \quad (t \geq 0)$.

Remark. The relation (5.4) is equivalent to

$<\mathrm{Re}((\mathrm{sign}\ f)Bf), \phi> = <|f|, A'\phi>$ $(f \in D(B),\ \phi \in D(A'))$.

In the case where A generates a positive semigroup, this is condition (4.7) in Theorem 4.4 with the inequality replaced by the equality. It is remarkable that , in contrast to the situation considered in Theorem 4.4, here condition (ii) implies the positivity of $(T(t))_{t\geq 0}$.

Proof. This is an adaption of the proof of [31, Theorem 3.4] given by Nagel and Uhlig. Assume that (i) holds. Let $f \in D(B)$. Then $S(t)f$ is differentiable in t. By the chain rule [31 ,3.1] $T(t)|f| = |S(t)f|$ is also differentiable and $d/dt_{|t=0}\ T(t)|f| = \mathrm{Re}(\mathrm{sign}\ f)Bf$ (by [31 ,2.2] and Proposition 5.4). Hence $|f| \in D(A)$ and $A|f| = \mathrm{Re}(\mathrm{sign}\ f)Bf$. Conversely, assume that (ii) holds. Let $s > 0$, $f \in E$. We show that $|S(s)f| = T(t)|f|$. This implies that $S(s)$ is disjointness preserving and $|S(s)| = T(s)$ (by [2,Theorem 2.4]). Since $D(B)$ is dense we can assume that $f \in D(B)$. Let $\xi(t) = T(s-t)|S(t)f|$ $(t \in [0,s])$. Then using again [31 , 3.1], [31 ,2.2] and Proposition 5.4 one obtains

$d/dt\ \xi(t) = -AT(s-t)|S(t)f| + T(s-t)(\mathrm{Re}(\mathrm{sign}\ S(t)f)BS(t)f) = 0$ by the assumption (ii). Hence $\xi(0) = \xi(s)$, i.e.

$|S(s)f| = T(s)|f|$. □

For the case where $S(t) = T(t)$ $(t\geq 0)$ we obtain

Corollary 5.6 (Nagel, Uhlig [31 ,3.4]). Let $(T(t))_{t\geq 0}$ be a semigroup with generator A . The following assertions are equivalent.

(i) $T(t)$ is a lattice homomorphism for all $t \geqq 0$.

(ii) $f \in D(A)$ implies $|f| \in D(A)$ and $\mathrm{Re}((\mathrm{sign}\, f)Af) = A|f|$.

Example 5.7. Let $E = L^p(X,\mu)$ (where (X,μ) is a σ-finite measure space and $1 \leqq p < \infty$) and A_o be the generator of a semigroup of lattice homomorphisms. Let $h \in L^\infty$ and $B = A_o + h$ (i.e. B is given by $Bf = A_o f + h \cdot f$ for $f \in D(B) = D(A_o)$). Let $A = A_o + \mathrm{Re} h$. Since A_o generates a semigroup of lattice homomorphisms, we have $|f| \in D(A_o)$ whenever $f \in D(A_o)$ and $\mathrm{Re}((\mathrm{sign}\, f)A_o f) = A_o|f|$. Hence $\mathrm{Re}((\mathrm{sign}\, f)Bf) = \mathrm{Re}((\mathrm{sign}\, f)A_o f) + (\mathrm{Re} h)\cdot|f|) = A_o|f| + (\mathrm{Re} h)|f| = A|f|$ for all $f \in D(B)$. Thus it follows from Theorem 5.5 that B generates a disjointness preserving semigroup whose modulus semigroup is generated by A.

Next we describe in terms of the domain of the generator when a disjointness preserving semigroup is positive.

Proposition 5.8. Let E be a complex Banach lattice with order continuous norm and B be the generator of a disjointness preserving semigroup $(S(t))_{t \geqq 0}$. The semigroup is positive if and only if B is real and $\mathrm{span}\, D(B)_+ = D(B)$.

Proof. The conditions are clearly necessary. In order to prove sufficiency, we can assume that E is real. Denote by A the generator of $(T(t))_{t \geqq 0}$, where $T(t) = |S(t)|$. Let $f \in D(B)_+$. Since B is local we have $Bf = P_f Bf = (\mathrm{sign}\, f)\, Bf = A|f| = Af$. By assumption, $\mathrm{span}\, D(B)_+ = D(B)$. Thus it follows that $B \subset A$. This implies that $B = A$ since $\rho(B) \cap \rho(A) \neq \emptyset$. $\square$

Finally, we show that for generators of disjointness preserving semigroups Kato's inequality holds in the reverse sense.

Proposition 5.9. Let B be the generator of a disjointness preserving semigroup $(S(t))_{t\geq 0}$ on a real Banach lattice E with order continuous norm. Then

(5.5) $$\langle (\operatorname{sign} f)Bf, \phi\rangle \geq \langle |f|, B'\phi\rangle$$

for all $f \in D(B)$, $\phi \in D(B')_+$.

Proof. Let $T(t) = |S(t)|$ and denote by A the generator of $(T(t))_{t\geq 0}$. Let $f \in D(B)$, $\phi \in D(B')_+$. Then $\langle (\operatorname{sign} f)Bf, \phi\rangle = \langle A|f|, \phi\rangle = \lim_{t\to 0} (1/t) \langle T(t)|f| - |f|, \phi\rangle \geq \lim_{t\to 0} 1/t \langle S(t)|f| - |f|, \phi\rangle = \langle |f|, B'\phi\rangle$. □

Chapter II

Resolvent Positive Operators

The Hille-Yosida theorem yields the following characterization of generators of positive semigroups in terms of the resolvent.

Theorem. Let A be a densely defined operator on an ordered Banach space E. Then A generates a positive strongly continuous semigroup if and only if the following two conditions are satisfied.

a) There exists $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $R(\lambda,A) := (\lambda-A)^{-1} \geqq 0$ for all $\lambda \in (w,\infty)$ (where $\rho(A)$ denotes the resolvent set of A).

b) $\sup\{||(\lambda - w)^n R(\lambda,A)^n|| : \lambda > w,\ n \in \mathbb{N}\} < \infty$.

Given a concrete operator, condition b) is frequently difficult to verify since the powers of $R(\lambda,A)$ are involved. So we take condition a) as a definition.

Definition. An operator A on an ordered Banach space E is called resolvent positive if there exists $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $R(\lambda,A) \geqq 0$ for all $\lambda > 0$.

The purpose of this chapter is to investigate systematically resolvent positive operators. We first show that in some exceptional cases (for example, if $E = C(K)$, K compact) a resolvent positive operator is automatically a generator. On L^p-spaces and $C_o(X)$ (X locally compact) this is not true, however. In fact, there are many natural examples of such operators which are not generators. Nevertheless these operators have remarkable properties. We will prove that if A is a resolvent positive operator and if either the domain $D(A)$ of A is dense or E is reflexive, then the Cauchy problem

$$u'(t) = Au(t)$$
$$u(0) = f$$

has a unique solution $u \in C^1([0,\infty),E)$ for every initial value $f \in D(A^2)$.

But resolvent positive operators are also interesting from a structural point of view. In fact, it is natural and corresponds to the historical development to consider semigroups from the point of view of Laplace transforms. Then the Hille-Yosida theorem characterizes those operators whose resolvent is a Laplace transformation. The corresponding classical theorem is the following.

Theorem [51, 6.8]. Let $w \in \mathbb{R}$, $M > 0$ and $g \in C^\infty(w,\infty)$. There exists a measurable function f on $[0,\infty)$ satisfying $|f(t)| \leq Me^{wt}$ $(t \geq 0)$ such that

$$g(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt \qquad (\lambda > w)$$

if and only if

$$|\lambda^{n+1} g^{(n)}(\lambda)/n!| \leq M \qquad (\lambda > w,\ n = 0,1,2,\ldots).$$

To see the analogy, observe that the derivatives of the resolvent of an operator A are given by

$$R^{(n)}(\lambda,A)/n! = (-1)^n R(\lambda,A)^{n+1} \qquad (n \in \mathbb{N}).$$

There is also a classical theorem characterizing those functions which are Laplace-Stieltjes transforms of increasing functions, namely,

Theorem (Bernstein) [51, 6.7]. Let $a \in \mathbb{R}$ and $g \in C^\infty(a,\infty)$. There exists an increasing function $\alpha : [0,\infty) \to \mathbb{R}$ such that

$$g(\lambda) = \int_0^\infty e^{-\lambda t}\, d\alpha(t) \qquad (\lambda > a)$$

if and only if g is completely monotonic; i.e.,

$$(-1)^n g^{(n)}(\lambda) \geq 0 \quad \text{for all } \lambda > a,\ n = 0,1,2,\ldots$$

Now let A be a resolvent positive operator. Then

$$(-1)^n R^{(n)}(\lambda,A) = n!\, R(\lambda,A)^{n+1} \geq 0$$

for all $n \in \mathbb{N}$ and all sufficiently large λ. Thus $R(\lambda,A)$ is completely monotonic. In fact, we will show that

$$R(\lambda,A) = \int_0^\infty e^{-\lambda t} \, dS(t) \qquad (\lambda \text{ large})$$

for a strongly continuous increasing family $(S(t))_{t\geq 0}$ of positive operators. This theorem will be proved by two different approaches. One uses the Hille-Yosida theorem and can be applied when A is densely defined. The second approach is based on a vector-valued version of Bernstein's theorem which we prove in section 5. Here we have to restrict the class of spaces (allowing reflexive spaces, L^p-spaces for $1 \leq p < \infty$ and c_0), but it is no longer necessary to assume that A is densely defined.

The relations between the operators $(S(t))_{t\geq 0}$ and A are similar to those of a semigroup to its generator. For instance, the operators $(S(t))_{t\geq 0}$ induce the solutions of the Cauchy problem mentioned above.

In the last section we give a characterization of resolvent positive operators by means of Kato's inequality.

<u>General assumption</u>. Throughout this chapter E denotes an ordered Banach space with <u>generating and normal positive cone</u> E_+. Moreover, we assume that the norm $\|\ \|$ on E is chosen in such a way that

$$\pm f \leq g \text{ implies } \|f\| \leq \|g\| \qquad (f,g \in E)$$

(which can always be done). For further properties of E which will frequently be used we refer to Appendix A.

1. Basic Properties.

Even though in the definition of resolvent positive operators it is merely required that a half-line lies in the resolvent set we show that this entails much stronger consequences.

Definition 1.1. Let A be a resolvent positive operator (see the introduction to this chapter for the definition). Then

$$s(A) = \inf \{w \in \mathbb{R} : (w,\infty) \subset \rho(A) \text{ and } R(\lambda,A) \geq 0 \text{ for all } \lambda > w\}$$

is called the spectral bound of A.

We will eventually show that $s(A) = \sup \{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ (Theorem 1.4), which justifies the terminology.

Lemma 1.2. Let A be an operator on E and $\lambda_o \in \mathbb{F} \cap \rho(A)$ such that $R(\lambda_o, A) \geq 0$.

(i) If $\lambda_1 < \lambda_o$ such that $\lambda_1 \in \rho(A)$ and $R(\lambda_1, A) \geq 0$, then $[\lambda_1, \lambda_o] \subset \rho(A)$ and $0 \leq R(\lambda_o, A) \leq R(\lambda, A) \leq R(\lambda_1, A)$ for all $\lambda \in [\lambda_1, \lambda_0]$.

(ii) If $s := \inf \{\lambda_1 \in \rho(A) : R(\lambda_1, A) \geq 0\} > -\infty$, then $s \in \sigma(A)$.

Proof. a) Let $\lambda, \mu \in \rho(A) \cap \mathbb{R}$, $\lambda < \mu$, $R(\lambda, A) \geq 0$, $R(\mu, A) \geq 0$. Then $R(\lambda, A) \geq R(\mu, A)$. In fact by the resolvent equation, $R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \geq 0$.

b) If $\lambda \in \rho(A)$ and $R(\lambda, A) \geq 0$, then there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda] \subset \rho(A)$ and $R(\mu, A) \geq 0$ for all $\mu \in (\lambda - \varepsilon, \lambda]$.

In fact, let $\varepsilon > 0$ such that $|\mu - \lambda| < \varepsilon$ implies $\mu \in \rho(A)$. Then for $\mu \in (\lambda - \varepsilon, \lambda]$,

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1} \geq 0.$$

This proves b). Moreover, b) implies (ii). [In fact, if $s > -\infty$, then there exist $s < \lambda_n \in \rho(A)$ $(n \in \mathbb{N})$ such that $R(\lambda_n, A) \geq 0$ and $\lim_{n \to \infty} \lambda_n = s$. Hence if $s \in \rho(A)$, then $R(s, A) = \lim_{n \to \infty} R(\lambda_n, A) \geq 0$ and b) leads to a contradiction to the definition of s.]

We prove (i). Let $s = \inf \{r \in \mathbb{R} : r \geq \lambda_1, (r, \lambda_o] \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda \in (r, \lambda_o]\}$. We have to show that $s = \lambda_1$. It follows from b) that $s < \lambda_o$. Assume that $s > \lambda_1$. Then it follows from b) that $s \in \sigma(A)$ (cf. the proof of (ii) above). For $\lambda \in (s, \lambda_o]$, $R(\lambda, A) \leq R(\lambda_1, A)$ by a). Hence $M := \sup \{\|R(\lambda, A)\| : \lambda \in (s, \lambda_c]\} < \infty$. Consequently, for $\lambda \in (s, \lambda_o]$,

$\mathrm{dist}(\sigma(A), \lambda) \geq \|R(\lambda,A)\|^{-1} \geq M^{-1} > 0$, contradicting $s \in \sigma(A)$. This proves that $s = \lambda_1$. □

<u>Remark</u> 1.3. By Lemma 1.2, an operator A is resolvent positive whenever there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(A) \cap \mathbb{R}$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and $R(\lambda_n,A) \geq 0$ for all $n \in \mathbb{N}$.

<u>Theorem</u> 1.4. Let A be a resolvent positive operator. Then for $\mathrm{Re}\lambda > s(A)$ we have $\lambda \in \rho(A)$ and

$$(1.1) \qquad |\langle R(\lambda,A)f, \phi\rangle| \leq \langle R(\mathrm{Re}\lambda,A)f, \phi\rangle \quad \text{for all } f \in E_+,\ \phi \in E'_+.$$

Moreover, if $s(A) > -\infty$, then $s(A) \in \sigma(A)$.

<u>Proof</u>. Let $H = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > s(A)\}$ and denote by K the connected component in $H \cap \rho(A)$ containing $(s(A), \infty)$. We claim that (1.1) holds for every $\lambda \in K$. In fact, let $f \in E_+$, $\phi \in E'_+$. Then the function $\lambda \to \langle R(\lambda,A)f,\phi\rangle$ $(\lambda \in (s(A),\infty))$ is completely monotonic (see the introduction to this chapter). Hence by Bernstein's theorem, there exists an increasing function $\alpha : (0,\infty) \to \mathbb{R}$ such that

$$\langle R(\lambda,A)f, \phi\rangle = \int_0^\infty e^{-\lambda t}\, d\alpha(t) \qquad (\lambda > s(A)).$$

The functions $\lambda \to \int_0^\infty e^{-\lambda t}\, d\alpha(t)$ and $\lambda \to \langle R(\lambda,A)f,\phi\rangle$ are both holomorphic on K and coincide on $(0,\infty)$. So they are identical. Hence for $\lambda \in K$, $|\langle R(\lambda,A)f,\phi\rangle| = |\int_0^\infty e^{-\lambda t}\, d\alpha(t)| \leq \int_0^\infty e^{-\mathrm{Re}\lambda t}\, d\alpha(t) = \langle R(\mathrm{Re}\lambda,A)f,\phi\rangle$. This proves the claim.

For $r > s(A)$ let $K_r = \{\lambda \in K : \mathrm{Re}\lambda \geq r\}$. Then for $\lambda \in K_r$, $|\langle R(\lambda,A)f,\phi\rangle| \leq \langle R(\mathrm{Re}\lambda,A)f,\phi\rangle \leq \langle R(r,A)f,\phi\rangle$ (by Lemma 1.2). Hence $\sup\{\|R(\lambda,A)\| : \lambda \in K_r\} < \infty$. Since $\mathrm{dist}(\lambda,\sigma(A)) \geq \|R(\lambda,A)\|^{-1}$, it follows that $K_r = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > r\}$. Since $r > s(A)$ was arbitrary, we conclude that $H \subset \rho(A)$ and (1.1) holds for all $\lambda \in H$. Finally, it follows from Lemma 1.2 that $s(A) \in \sigma(A)$, whenever $s(A) > -\infty$. □

Now let A be a resolvent positive operator. Then by Lemma 1.2,

(1.2) $\quad R(\mu,A) \geq R(\lambda,A) \geq 0 \quad$ whenever $s(A) < \mu \leq \lambda$.

In particular, for every $\lambda_0 > s(A)$,

(1.3) $\quad \sup\{\|R(\lambda,A)\| : \lambda \geq \lambda_0\} < \infty$.

If A is densely defined, then

(1.4) $\quad \lim_{\lambda\to\infty} R(\lambda,A)f = 0 \quad$ for all $f \in E$ and

(1.5) $\quad \lim_{\lambda\to\infty} \lambda R(\lambda,A)f = f \quad$ for all $f \in D(A)$.

Proof. a) Let $f \in D(A)$. Let $\mu > s(A)$ and $g = (\mu-A)f$. Then $R(\lambda,A)f = R(\lambda,A)R(\mu,A)g = 1/(\lambda-\mu)\ (R(\mu,A)g - R(\lambda,A)g) \to 0$ for $\lambda \to \infty$ (because of (1.3)).

b) (1.4) follows from a) and (1.3) by a 3ε-argument since $D(A)$ is dense.

c) Let $f \in D(A)$. Let $g = (\mu-A)f$ where $\mu > s(A)$. Then

$\lambda R(\lambda,A)f = \lambda R(\lambda,A)R(\mu,A)g = \lambda/(\lambda-\mu) \ (R(\mu,A)g - R(\lambda,A)g) \to$ $R(\mu,A)g = f$ for $\lambda \to \infty$ (by (1.4)). This proves (1.5). □

Remark. Note that in general $\underline{\lim}_{\lambda\to\infty} \|\lambda R(\lambda,A)\| = \infty$ even if A is densely defined (see Example 7.3 b)).

We will use the following definitions.

(1.6) $D(A)_+ = E_+ \cap D(A)$ and $D(A')_+ = E'_+ \cap D(A')$

(where we assume A to be densely defined in the second definition). Since $D(A) = R(\lambda,A)E$ and $D(A') = R(\lambda,A)'E'$ and $R(\lambda,A) \geq 0$ for $\lambda > s(A)$ one obtains

(1.7) $D(A) = D(A)_+ - D(A)_+$ and $D(A') = D(A')_+ - D(A')_+$.

In particular, D(A) is an ordered Banach space with respect to the graph norm and the positive cone $D(A)_+$. However, this cone is not normal in general.

Remark. Operators with positive resolvent have been considered by Kato [29] and Nussbaum [34]. Theorem 1.4 is proved by Nussbaum (l.c.) by reduction to the corresponding result for bounded positive operators [41, App. 2.2].

2. Resolvent Positive Operators which are Automatically Generators.

Resolvent positive operators admit norm estimates for the resolvent. On $C(K)$ (K compact), they are sufficient to yield the norm condition required by the Hille-Yosida theorem. In other special cases, an additional mild norm condition or order condition suffices to obtain a semigroup.

Lemma 2.1. Let A be a resolvent positive operator such that $s(A) < 0$. Then

(2.1) $$R(0,A) = R(\lambda,A) + \lambda R(\lambda,A)^2 + \lambda^2 R(\lambda,A)^3 + \ldots + \lambda^{n-1} R(\lambda,A)^n + \lambda^n R(\lambda,A)^n R(0,A)$$

for all $n \in \mathbb{N}$, $\lambda \geq 0$. Consequently,

(2.2) $$\sup \{ \|\lambda^n R(\lambda,A)^n R(0,A)\| : n \in \mathbb{N}, \lambda \geq 0 \} < \infty.$$

Proof. By the resolvent equation,

$$R(0,A) = R(\lambda,A) + \lambda R(\lambda,A) R(0,A)$$

($\lambda > 0$). This is (2.1) for $n = 1$. Iterating this equation yields (2.1) for all $n \in \mathbb{N}$. □

Remark 2.2. Relation (2.1) implies that $\lambda^{n-1} R(\lambda,A)^n \leq R(0,A)$ for all $n \in \mathbb{N}$, $\lambda \geq 0$. Hence $\sup \{ \|\lambda^{n-1} R(\lambda,A)^n\| : \lambda \geq 0, n \in \mathbb{N} \} < \infty$. On the other hand, if A is densely defined, then A generates a bounded strongly continuous semigroup if and only if $\sup \{ ||\lambda^n R(\lambda,A)^n|| : \lambda \geq 0, n \in \mathbb{N}\} < \infty$.

A subset C of E_+ is called <u>cofinal</u> in E_+ if for every $f \in E_+$ there exists $g \in C$ such that $f \leq g$.

If $(T(t))_{t\geq 0}$ is a positive strongly continuous semigroup with generator B, then the <u>type</u> (or <u>growth bound</u>) $\omega(B)$ is defined by

$$\omega(B) = \inf \{w \in \mathbb{R} : \text{there exists } M \geq 1 \text{ such that } ||T(t)|| \leq Me^{wt} \text{ for all } t \geq 0\}.$$

One always has $s(B) \leq \omega(B) < \infty$, but it can happen that $s(B) \neq \omega(B)$ even if B generates a positive group ([23] and [52]).

<u>Theorem</u> 2.3. Let A be a densely defined resolvent positive operator. If $D(A)_+$ is cofinal in E_+ or if $D(A')_+$ is cofinal in E'_+, then A is the generator of a strongly continuous positive semigroup. Moreover, $s(A) = \omega(A)$.

<u>Proof</u>. a) Assume that $s(A) < 0$. We claim that A generates a bounded strongly continuous semigroup, if one of the conditions in the theorem is satisfied.
We first assume that D(A) is cofinal. Let $f \in E_+$. Then there exists $g \in D(A)_+$ such that $f \leq g$. Let $h = -Ag$ and $k \in E_+$ such that $h \leq k$. Then $f \leq g = R(0,A)h \leq R(0,A)k$. It follows from (2.1) that $\lambda^n R(\lambda,A)^n f \leq \lambda^n R(\lambda,A)^n R(0,A)k \leq R(0,A)k$. Hence

$\sup \{||\lambda^n R(\lambda,A)^n f|| : \lambda \geq 0, n \in \mathbb{N}\} < \infty$. Since $E = E_+ - E_+$, it follows that $\{\lambda^n R(\lambda,A)^n : \lambda \geq 0, n \in \mathbb{N}\}$ is strongly bounded; thus it is norm-bounded by the uniform boundedness principle. The Hille-Yosida theorem implies that A generates a strongly continuous bounded semigroup.

If $D(A')_+$ is cofinal in E'_+ consider $f \in E_+$, $\phi \in E'_+$. Then there exists $\psi \in E'_+$ such that $\phi \leq R(0,A)'\psi$. Hence by (2.1), $\langle\lambda^n R(\lambda,A)^n f, \phi\rangle \leq \langle\lambda^n R(\lambda,A)^n f, R(0,A)'\psi\rangle = \langle\lambda^n R(\lambda,A)^n R(0,A)f, \psi\rangle \leq \langle R(0,A)f, \psi\rangle$. Since E_+ and E'_+ are generating this implies that $\{\lambda^n R(\lambda,A)^n : \lambda \geq, n \in \mathbb{N}\}$ is weakly bounded, and so norm-bounded. Again the Hille-Yosida theorem implies the claim.

b) If $s(A)$ is arbitrary consider $B = A - w$ for some $w > s(A)$. Then $s(B) < 0$, and so by a), B is the generator of a bounded semigroup $(T(t))_{t\geq 0}$. Hence A generates the semigroup $(e^{wt}T(t))_{t\geq 0}$. Moreover $\omega(A) \leq w$. □

Corollary 2.4. Assume that $\text{int } E_+ \neq \emptyset$. If A is a densely defined resolvent positive operator, then A is the generator of a strongly continuous positive semigroup and $s(A) = \omega(A)$.

Proof. Since $\text{int } E_+ \neq \emptyset$ and $D(A)$ is dense, there exists $u \in \text{int } E_+ \cap D(A)$. The set $\{u\}$ is clearly cofinal in E_+. □

Corollary 2.5. Let A be a densely defined operator with positive resolvent on $L^1(X,\mu)$ (where (X,μ) is a σ-finite measure space). If there exists $\phi \in D(A') \cap L^\infty(X,\mu)$ such that $\phi(x) \geq \varepsilon > 0$ for almost all $x \in X$, then A is the generator of a strongly continuous positive semigroup.

Remark. Corollary 2.4 has been proved in [5] and Corollary 2.5 by Batty and Robinson [8] with a different approach using half-norms.

Theorem 2.6. Let A be a densely defined resolvent positive operator. If there exist $\lambda_o > s(A)$ and $c > 0$ such that

$$(2.4) \qquad \|R(\lambda_o,A)f\| \geq c\,\|f\| \qquad (f \in E_+),$$

then A is the generator of a strongly continuous semigroup and $s(A) = \omega(A)$.

Proof. Let $s(A) < w \leq \lambda_o$. Let $B = A - w$. Then $s(B) < 0$. Since $R(0,B) = R(w,A) \geq R(\lambda_o,A)$ (by (1.2)), it follows from (2.4) that $\|R(0,B)f\| \geq \|R(\lambda_o,A)f\| \geq c\,\|f\|$ for all $f \in E_+$. Using (2.2) one obtains a constant $M>0$ such that $\|(\lambda R(\lambda,B))^n f\| \leq c^{-1}\|R(0,B)(\lambda R(\lambda,B))^n f\| \leq M\,\|f\|$ for all $f \in E_+$, $\lambda \geq 0$, $n \in \mathbb{N}$. Since $E = E_+ - E_+$ it follows that the set $\{\lambda^n R(\lambda,B)^n : n \in \mathbb{N}, \lambda \geq 0\}$ is strongly bounded and so norm-bounded. Thus by the Hille-Yosida theorem, $B = A - w$ generates a bounded strongly continuous positive semigroup. Hence A is a generator and $\omega(A) \leq w$. □

Remark. Theorem 2.6 (except the assertion concerning the spectral bound) is due to Batty and Robinson [8] (with a different proof) who analyse condition (2.4) in more detail.

Theorem 2.7. Suppose that the norm is additive on the positive cone, i.e. $\|f+g\| = \|f\| + \|g\|$ for all $f,g \in E_+$ (e.g. $E = L^1(X,\mu)$). Let A be a densely defined operator. Then the following assertions are equivalent.

(i) A generates a strongly continuous positive group.

(ii) A and -A are resolvent positive and there exist $\lambda > \max\{s(A), s(-A)\}$ and $c > 0$ such that

(2.5) $||R(\lambda,\pm A)f|| \geq c\,||f||$ for all $f \in E_+$.

Proof. Assume that A generates a positive group $(T(t))_{t\in\mathbb{R}}$. Then there exist $w > 0$, $M \geq 1$ such that $||(T(-t)|| \leq M e^{wt}$ for all $t \geq 0$. This implies that $||T(t)f|| \geq M^{-1} e^{-wt}||f||$ $(f \in E)$. Hence for $\lambda > \omega(A)$, $f \in E_+$, $||R(\lambda,A)f|| = || \int_0^\infty e^{-\lambda t} T(t)f\,dt || = \int_0^\infty e^{-\lambda t} ||T(t)f||\,dt \geq M^{-1} \int_0^\infty e^{-\lambda t} e^{-wt}||f||\,dt = ((\lambda+w)M)^{-1} ||f||$. Similarly for $R(\lambda,-A)$ where $\lambda > \omega(-A)$. Thus (ii) holds. The converse follows from Theorem 2.6. □

Remark. Condition ((2.4) does not hold for generators of positive groups on every Banach lattice. For example, it fails for the generator of the rotation group on $C(T)$, where T is the 1-dimensional torus [8, Example 2.2.13].

Example 2.8. We show by an example that condition (2.5) cannot be omitted in Theorem 2.7.

Let B be the generator of the group $(T(t))_{t\in\mathbb{R}}$ on $L^1(\mathbb{R})$ given by $T(t)f(x) = f(x+t)$. Then $D(B) = \{f\in AC(\mathbb{R}) : f' \in L^1(\mathbb{R})\}$ and

$R(\lambda,B)f(x) = e^{\lambda x}\int_x^\infty e^{-\lambda y} f(y)\,dy$ and $-R(-\lambda,B)f(x) = R(\lambda,-B)f(x) = e^{-\lambda x}\int_{-\infty}^x e^{\lambda y} f(y)\,dy$ for $\lambda > 0$, $f \in E$. For $n \in \mathbb{N}$ let

$$p_n(f) = (3/2)^n \int_{2^{-n}}^{2^{-n+1}} |f(x)|\,dx$$

and $E_o = \{f \in L^1(\mathbb{R}) : \sum_{n=1}^\infty p_n(f) < \infty\}$. Then E_o is a Banach lattice with the norm $||f||_o := ||f||_1 + \sum_{n=1}^\infty p_n(f)$. Of course, E_o is isomorphic to $L^1(\mathbb{R},\mu)$ for a suitable measure μ. We show that $R(\lambda,B)E_o \subset E_o$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. In fact, let $\lambda > 0$, $f \in E_o$. Then

$$p_n(R(\lambda,B)f) \leq (3/2)^n \int_{2^{-n}}^{2^{-n+1}} e^{\lambda x} \int_x^\infty e^{-\lambda y} |f(y)|\,dy\,dx \leq$$

$$||f||_1\ (3/2)^n \int_{2^{-n}}^{2^{-n+1}} dx = (3/4)^n ||f||_1.$$ Hence $\sum_{n=1}^\infty p_n(R(\lambda,B)f) < \infty$.

Similarly for $\lambda < 0$. Let A be the operator on E_o defined on $D(A) = \{f \in E_o \cap D(B) : Bf \in E_o\}$ by $Af = Bf$. Then it is easy to see that $\mathbb{R} \setminus \{0\} \subset \rho(A)$ and $R(\lambda,A) = R(\lambda,B)_{|E_o}$ for $\lambda \in \mathbb{R} \setminus \{0\}$. Hence $R(\lambda,A) \geq 0$ and $R(\lambda,-A) = -R(-\lambda,A) \geq 0$ for $\lambda > 0$. We show that $D(A)$ is dense in E. Let $f \in E_o$. For $n \in \mathbb{N}$ let $f_n = f\cdot 1_{\mathbb{R}\setminus[0,\,2^{-n}]}$. Then $f-f_n = f\cdot 1_{[0,2^{-n}]}$. Hence $||f - f_n||_1 \to 0$ for $n \to \infty$. Moreover, $p_m(f-f_n) = 0$ for $m \leq n$ and $p_m(f-f_n) = p_m(f)$ for $m > n$. Hence $\sum_{m=1}^\infty p_m(f-f_n) = \sum_{m=n+1}^\infty p_m(f) \to 0$ for $n \to \infty$. We have shown that $E_{oo} := \{f \in E_o :$ there exists $\varepsilon > 0$ such that $f_{|[0,\varepsilon]} = 0\}$ is dense in E_o. Now let $f \in E_{oo}$. Then there exists $\varepsilon > 0$ such that $f_{|[0,\varepsilon]} = 0$. It is easy to see that there exists a sequence $(f_n) \subset AC(\mathbb{R})$ such that $f_n' \in L^1(\mathbb{R})$, $f_{n|[0,\varepsilon/2]} = 0$ and

$||f - f_n||_1 \to 0$ for $n \to \infty$. Then $f_n \in D(A)$. Since $|| \ ||_1$ and $|| \ ||_o$ are equivalent norms on $E_{\varepsilon/2} := \{f \in L^1(\mathbb{R}) : f(x) = 0$ for $x \in [0,\varepsilon/2]$ a.e.$\} \subset E_o$, it follows that $\lim_{n\to\infty} f_n = f$ in E_o. Finally, we show that A is not a generator. In fact, assume that there exists a semigroup $(T_o(t))_{t>0}$ on E_o, which is strongly continuous for $t > 0$, such that $R(\lambda,A)f = \int_0^\infty e^{-\lambda t}T_o(t)f\,dt$ for large $\lambda > 0$. For $f \in E_\varepsilon$, $T(t)f$ is continuous in t for the norm $|| \ ||_o$ and $R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\ T(t)f\,dt$. So it follows from the uniqueness theorem for Laplace transforms, that $T_o(t)f = T(t)f$ $(t>0)$ for all $f \in E_{oo}$. Let $t > 1$ and $f = 2^n\ 1_{[t+2^{-n},\ t+2^{-n+1}]}$. Then $||f||_o = ||f||_1 = 1$. But $T_o(t)f = 2^n\ 1_{[2^{-n},2^{-n+1}]}$. Hence $||T_o(t)f||_o \geq p_n(T_o(t)f) = (3/2)^n$. Thus $T_o(t)$ is not continuous for $t > 1$. □

<u>Remark</u>. $-A$ is not a generator either (see Example 7.3).

3. Perturbation and Examples.

In this section we present two kinds of perturbations which demonstrate that there exist many natural resolvent positive operators which are not generators.

<u>Theorem</u> 3.1. Let A be a resolvent positive operator and $B : D(A) \to E$ a positive operator. If $r(BR(\lambda,A)) < 1$ for some $\lambda > s(A)$, then $A + B$ with domain $D(A)$ is a resolvent positive

operator and $s(A+B) < \lambda$.
Moreover, if $\sup\{\|\mu R(\mu,A)\| : \mu \geq \lambda\} < \infty$ (e.g., if A is the generator of a strongly continuous semigroup), then
$\sup\{\|\mu R(\mu,A+B)\| : \mu \geq \lambda\} < \infty$.
Note: By assumption, $BR(\lambda,A)$ is a positive, hence bounded operator on E; we denote by $r(BR(\lambda,A))$ its spectral radius.

Proof. Let $f \in D(A)$. Then $(\lambda - (A+B))f = (I - BR(\lambda,A))(\lambda - A)f$. Let $S_\lambda := (I - BR(\lambda,A))^{-1} = \sum_{n=0}^{\infty} (BR(\lambda,A))^n \geq 0$. Then $R(\lambda,A)S_\lambda(\lambda - (A+B))f = f$ for all $f \in D(A)$ and $(\lambda - (A+B))R(\lambda,A)S_\lambda g = g$ for all $g \in E$. Hence $\lambda \in \rho(A+B)$ and $R(\lambda,A+B) = R(\lambda,A)S_\lambda \geq 0$. If $\mu > \lambda$, then $BR(\mu,A) \leq BR(\lambda,A)$ by (1.2), and so $r(BR(\mu,A)) \leq r(BR(\lambda,A)) < 1$. Hence also $\mu \in \rho(A+B)$ and $R(\mu,A+B) \geq 0$. Moreover, $S_\mu \leq S_\lambda$ and $R(\mu,A) \leq R(\lambda,A)$ so that $\mu R(\mu,A+B) = \mu R(\mu,A)S_\mu \leq \mu R(\mu,A)S_\lambda$. Hence $\sup\{\|\mu R(\mu,A+B)\| : \mu \geq \lambda\} \leq \sup\{\|\mu R(\mu,A)\|\,\|S_\lambda\| : \mu \geq \lambda\} < \infty$ if the additional assumption is satisfied. □

The following examples show that even in rather simple and natural cases perturbations as in Theorem 3.1 may yield resolvent positive operators which are not generators.

Example 3.2. Let $\alpha \in (0,1)$. Define the operator A by

$$Af(x) = f'(x) + \frac{\alpha}{x} f(x) \qquad x \in (0,1]$$

on the space $E = C_0(0,1] := \{f \in C[0,1] : f(0) = 0\}$ with domain $D(A) = \{f \in C^1[0,1] : f'(0) = f(0) = 0\}$. Then A is resolvent positive but not a generator.

Moreover, $s(A) = -\infty$ and $\sup\{\|\mu R(\mu,A)\| : \mu \geq 0\} \leq 1/(1-\alpha)$.

Proof. Let $A_o f = -f'$ with domain $D(A_o) = D(A)$. Then A_o is the generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)f(x) = \begin{cases} f(x-t) & x \geq t \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $\sigma(A_o) = \emptyset$ and $R(\lambda,A_o)f(x) = e^{-\lambda x}\int_0^x e^{\lambda y} f(y)\,dy$ $(\lambda \in \mathbb{C},\ f \in E)$. Let $B : D(A_o) \to E$ be given by $Bf(x) = \alpha\ f(x)/x$ $(x > 0)$, $Bf(0) = 0$. Let $f \in E$ and $g = R(0,A)f$. Then $|Bg(x)| = |\alpha/x \int_0^x f(y)\,dy| \leq \alpha\ \|f\|_\infty$. Thus $\|BR(0,A_o)\| \leq \alpha < 1$. So Theorem 3.1 implies that $A = A_o + B$ is resolvent positive and $s(A) < 0$. Moreover, for $\mu \geq 0$ one has $\mu R(\mu,A) = \mu R(\mu,A_o)S_\mu \leq \mu R(\mu,A_o)S_o$, where $S_\mu = (I - BR(\mu,A_o))^{-1}$. Since $\|\mu R(\mu,A_o)\| \leq 1$ and $\|S_o\| \leq 1/(1-\alpha)$ it follows that $\sup\{\|\mu R(\mu,A)\| : \mu \geq 0\} \leq 1/(1-\alpha)$.

It remains to show that A is not a generator. One can easily check that for all $\lambda \in \mathbb{C}$ one has $\lambda \in \rho(A)$ and $R(\lambda,A)f(x) = e^{-\lambda x}\ x^\alpha \int_0^x y^{-\alpha}\ e^{\lambda y}\ f(y)\,dy = \int_0^x x^\alpha (x-t)^{-\alpha} f(x-t)e^{-\lambda t}\,dt$ $(f \in E)$. Suppose that there exists a semigroup $(T(t))_{t>0}$ which is strongly continuous for $t > 0$ such that $R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\ T(t)f\,dt$ for all $f \in E$ and all sufficiently large real λ. Then by the uniqueness theorem for Laplace transforms (Theorem C5), for $0 < t < 1$, one would have $T(t)f(x) = (x^\alpha/(x-t)^\alpha)\ f(x-t)$ for $x \geq t$ and $T(t)f(x) = 0$ otherwise. This does not define a bounded operator on $C_o(0,1]$. □

Remark. It follows from a result of Benyamini [9] that $C_o(0,1]$ is isomorphic as a Banach space to a space $C(K)$ (K compact). Thus Example 3.2 yields an operator B on $C(K)$ such that $\sigma(B) = \emptyset$ and the resolvent satisfies $\sup\{\|\lambda^{n-1}R(\lambda,B)^n\| : \lambda \geq 0,\ n \in \mathbb{N}\} < \infty$; $\sup\{\|\lambda R(\lambda,B)\| : \lambda \geq 0\} < \infty$. But B is not a generator. Of course, B is not resolvent positive by Corollary 2.4.

Example 3.3. Let $E = L^p[0,1]$, where $1 < p < \infty$. Choose $\alpha \in (0, (p-1)/p)$. Define the operator A by

$$Af(x) = -f'(x) + (\alpha/x)f(x)$$

with domain $D(A) = \{f \in AC[0,1] : f' \in L^p[0,1],\ f(0) = 0\}$. Then A is resolvent positive. Moreover, $s(A) < 0$ and $\sup\{\|\lambda R(\lambda,A)\| : \lambda \geq 0\} < \infty$. But A is not a generator.

Proof. Let $A_o f = -f'$ with domain $D(A_o) = D(A)$. Then A_o generates the semigroup $(T_o(t))_{t\geq 0}$ on E given by $T_o(t)f(x) = f(x-t)$ for $x \geq t$ and $T_o(t)f(x) = 0$ otherwise. Moreover, $s(A_o) = -\infty$ and $R(0,A_o)f(x) = \int_0^x f(y)\,dy$. Let $B : D(A) \to E$ be defined by $Bf(x) = (\alpha/x)f(x)$. Then by [11, Lemma 1] $BR(0,A_o) \in \mathcal{L}(E)$ and $\|BR(o,A_o)\| = \alpha p/(p-1)$. Hence Theorem 3.1 implies the first assertions stated above. It remains to show that A is not a generator. It is not difficult to check that the resolvent of A is given by $R(\lambda,A)f(x) = x^{\alpha} e^{-\lambda x} \int_0^x e^{\lambda y} y^{-\alpha} f(y)\,dy$ $= \int_0^x x^{\alpha}(x-t)^{-\alpha} f(x-t) e^{-\lambda t}\,dt$ $(\lambda > o)$. Let $E_o = \{f \in E :$ there exists $\delta > 0$ such that $f(y) = 0$ for almost all $x \in [0,\delta)\}$. Define $T(t) : E_o \to E$ by $T(t) = 0$ if $t \geq 1$ and

$$T(t)f(x) = \begin{cases} x^{\alpha}(x-t)^{-\alpha} f(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise} \end{cases}$$

if $0 \leq t < 1$. Then $T(.)f$ is continuous from $[0,\infty)$ into E and $R(\lambda,A)f = \int_0^\infty e^{-\lambda t} T(t)f\,dt$ for all $\lambda \geq 0$ if $f \in E_o$. Thus if there exists a semigroup $(T_1(t))_{t>0}$ which is strongly continuous for $t > 0$ and such that $R(\lambda,A)f = \int_0^\infty e^{-\lambda t} T_1(t)f\,dt$ $(f \in E)$ for large λ, it follows from the uniqueness theorem for Laplace transforms that $T_1(t)f = T(t)f$ for all $f \in E_o$, $t \geq 0$. But for $t \in (0,1)$ the mapping $T(t)$ is not continuous (from E_o with the induced norm into E). In fact, let $\beta > 0$ such that $1-\alpha p < \beta p < 1$, and for $n \in \mathbb{N}$, let $f_n(x) = 1_{[1/n,1]}(x)\, x^{-\beta}$. Then $f_n \in E_o$ and $\sup\{\|f_n\|_p : n \in \mathbb{N}\} < \infty$. But $\|T(t)f_n\|_p^p = \int_t^1 x^{\alpha p}/(x-t)^{\alpha p} f_n(x-t)^p\,dx \geq t^{\alpha p} \int_0^{1-t} f_n(y)^p/y^{\alpha p}\,dy = t^{\alpha p} \int_{1/n}^{1-t} y^{-(\alpha+\beta)p}\,dy \to \infty$ for $n \to \infty$ since $(\alpha+\beta)p > 1$. □

<u>Proposition</u> 3.4. Let (X,μ) be a σ-finite measure space and $E = L^p(X,\mu)$ $(1 \leq p < \infty)$ [resp., X locally compact and $E = C_o(X)$]. Let A be a resolvent positive operator. Suppose that $m : X \to [0,\infty)$ is measurable (resp., continuous) such that $m(x) > 0$ a.e. (resp., $m(x) > 0$ for all $x \in X$) and $(1/m)f \in E$ for all $f \in D(A)$.

Let $D(A^\#) = \{g \in E : m\cdot g \in D(A),\ (1/m)A(m\cdot g) \in E\}$ and $A^\# g = (1/m)A(m\cdot g)$. Then $A^\#$ is a resolvent positive operator and $s(A^\#) \leq s(A)$.

<u>Proof</u>. For $\lambda > s(A)$ let $R^\#(\lambda)f = (1/m)R(\lambda,A)(m\cdot f)$ $(f \in E)$. Then $R^\#(\lambda)$ is a positive, hence bounded operator. It is easy to show that $R^\#(\lambda) = (\lambda - A^\#)^{-1}$. □

Example 3.5. Let $E = L^p[0,1]$, $1 \leq p < \infty$, and A be given by $Af = f'$ with $D(A) = \{f \in AC[0,1] : f' \in L^p[0,1],\ f(1) = 0\}$. Then A is the generator of the semigroup $(T(t))_{t\geq 0}$ given by $T(t)f(x) = f(x+t)$ if $x+t \leq 1$ and $T(t)f(x) = 0$ if $x+t > 1$. Moreover, $s(A) = -\infty$ and $R(\lambda,A)f(x) = e^{\lambda x} \int_x^1 e^{-\lambda y} f(y)\,dy \quad (f \in E)$.

Let $\alpha \in (0,1/p)$ and $m(x) = x^\alpha$. Then $1/m \in L^p[0,1]$ and since $D(A) \subset C[0,1]$ it follows that $(1/m)f \in L^p[0,1]$ for all $f \in D(A)$. By Proposition 3.4 the operator $A^\#$ is resolvent positive, where $D(A^\#) = \{f \in E : m\cdot f \in D(A),\ (1/m)A(m\cdot f) \in E\}$ and $A^\# f = (1/m)A(m\cdot f) = x^{-\alpha}(x^\alpha f)' = f' + (\alpha/x)f$.

$D(A^\#)$ is dense in $L^p[0,1]$. In fact, $D(A) \cap \{ f \in L^p[0,1] : f|_{[0,\varepsilon]} = 0$ for some $\varepsilon > 0\} \subset D(A^\#)$. But $D(A)$ is dense in $L^p[0,1]$, and it is easy to see that every $f \in D(A) = \{g \in AC[0,1] : g' \in L^p[0,1],\ g(1) = 0\}$ can be approximated by functions in $D(A)$ which vanish in a neighborhood of 0.

$A^\#$ is not the generator of a semigroup. In fact, assume that there exists a semigroup $(T(t))_{t>0}$ which is strongly continuous for $t > 0$ such that $R(\lambda,A^\#)f = \int_0^\infty e^{-\lambda t}\, T(t)f\,dt$ for sufficiently large λ. It is not difficult to see that

$$R(\lambda,A^\#)f(x) = x^{-\alpha} e^{\lambda x} \int_x^1 e^{-\lambda y} f(y)\, y^\alpha\, dy$$
$$= \int_0^{1-x} e^{-\lambda t}\, x^{-\alpha} f(x+t)(x+t)^\alpha\, dt.$$

Similarly as in Example 3.3 one shows that for $0 < t < 1$, $T(t)$ is given by $T(t)f(x) = x^{-\alpha}(x+t)^\alpha f(x+t)$ for $x+t \leq 1$. This does not define a bounded operator on $L^p[0,1]$.

Note: For $p > 1$, $R(\lambda, A^{\#})$ is the adjoint of $R(\lambda, A)$ in Example 3.3 on $L^q(0,1)$ (with $1/p + 1/q = 1$). Thus $\lambda R(\lambda, A^{\#})$ is norm-bounded for $\lambda \to \infty$. This is also the case for $p = 1$. One can argue as follows. $\|\lambda R(\lambda, A^{\#})\| = \|\lambda R(\lambda, A^{\#})'\| = \|\lambda R(\lambda, A^{\#})'1\|_\infty = \sup \{\lambda y^\alpha e^{-\lambda y} \int_0^y x^{-\alpha} e^{\lambda x} dx : y \in [0,1]\} = \|\lambda R(\lambda)\|$, where $R(\lambda)$ denotes the resolvent of the operator A on $C_o(0,1]$ in Example 3.2. Hence $\|\lambda R(\lambda, A^{\#})\| \leq 1/(1-\alpha)$ $(\lambda \geqq 0)$. □

Remark. In the literature, the first example of a resolvent positive operator which is not a generator was given by Batty and Davies [7] on $C_o(\mathbb{R})$. A similar example on $L^1(\mathbb{R})$ appears in [8, Example 2.2.11]. Independently, H.P. Lotz constructed an example by a renorming procedure similar to Example 2.8 (unpublished).

4. Positive Resolvent as Laplace-Stieltjes Transform

For definition and properties of the vector-valued Riemann-Stieltjes integral and the Laplace-Stieltjes transform we refer to Appendix B and C.

Let A be a densely defined resolvent positive operator.

Theorem 4.1. There exists a unique strongly continuous family $(S(t))_{t\geq 0}$ of positive operators satisfying $S(0) = 0$ and $S(s) \leq S(t)$ for $0 \leq s \leq t$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} \, dS(t) \qquad (\lambda > s(A)) \tag{4.1}$$

(where the integral converges in the operator norm).

For the proof of the theorem we use the following construction which is due to P.R. Chernoff (unpublished).

Construction 4.2. Let A be a resolvent positive operator satisfying $s(A) < 0$. Then for $\lambda > 0$ one obtains from the resolvent equation

$$R(0,A)\lambda R(\lambda,A) = R(0,A) - R(\lambda,A) \leq R(0,A). \tag{4.2}$$

Let $\|f\|_1 := \inf\{\|R(0,A)g\| : \pm f \leq g\}$ $(= \|R(0,A)|f|\|$ if E is a Banach lattice). Denote by E_1 the completion of E with respect to this norm. For $f \in E$, $\lambda > 0$ one has

$$\begin{aligned}
\|\lambda R(\lambda,A)f\|_1 &= \inf\{\|R(0,A)g\| : \pm \lambda R(\lambda,A)f \leq g\} \\
&\leq \inf\{\|R(0,A)\lambda R(\lambda,A)h\| : \pm f \leq h\} \\
&\leq \inf\{\|R(0,A)h\| : \pm f \leq h\} \qquad \text{(by (4.2))} \\
&= \|f\|_1 .
\end{aligned}$$

Thus $R(\lambda,A)$ has a unique continuous extension $R_1(\lambda)$ on E_1 which satisfies

(4.3) $\|\lambda R_1(\lambda)\| \leq 1 \quad (\lambda > 0)$.

It is obvious that $(R_1(\lambda))_{\lambda>0}$ is a pseudoresolvent. Since $D(A) \subset R_1(\lambda)E_1 \quad (\lambda > 0)$, it has a dense image, and so it is the resolvent of a densely defined operator A_1 on E_1 [15, Theorem 2.6]. It follows from the Hille-Yosida theorem that A_1 is the generator of a strongly continuous contraction semigroup $(T_1(t))_{t\geq 0}$ on E_1.

<u>Observation</u> 4.3. The operator $R(0,A)$ satisfies

(4.4) $\|R(0,A)f\| \leq \|f\|_1 \qquad (f \in E)$.

(In fact, let $f \in E$ and $\pm f \leq g$. Then $\pm R(0,A)f \leq R(0,A)g$. Hence $\|R(0,A)f\| \leq \|R(0,A)g\|$. Thus $\|R(0,A)f\| \leq \inf\{\|R(0,A)g\| : \pm f \leq g\} = \|f\|_1$.) Consequently, the extension $R_1(0)$ of $R(0,A)$ onto E_1 maps E_1 into E. Moreover, $(R_1(\lambda))_{\lambda\geq 0}$ ($\lambda = 0$ included) is a pseudoresolvent too. Thus $R_1(0) = R(0,A_1)$. This implies that

(4.5) $D(A_1) = R(0,A_1)E_1 \subset E$.

The closure E_{1+} of E_+ is a cone in E_1 which is invariant under $R(\lambda,A_1)$ for $\lambda \geq 0$. This cone is proper [in fact, let $f \in E_{1+} \cap (-E_{1+})$; then $R(0,A_1)f \in E_+ \cap (-E_+)$, hence $R(0,A_1)f = 0$, and so $f = 0$]. Thus (E_1,E_{1+}) is an ordered Banach space and the semigroup $(T_1(t))_{t\geq 0}$ is positive. (If E is a Banach lattice, then E_1 is a Banach lattice as well.)

Illustration 4.4. In order to illustrate the construction 4.2, consider the operator $A^\#$ on $L^1[0,1]$ given in Example 3.5. Then $R(0,A^\#)f(x) = x^{-\alpha}\int_x^1 f(y)\, y^\alpha\, dy$. Thus $\|f\|_1 = \|R(0,A^\#)|f|\| = \int_0^1 x^{-\alpha}\int_x^1 |f(y)|\, y^\alpha\, dy\, dx = \int_0^1 |f(y)|\, y^\alpha \int_0^y x^{-\alpha}\, dx\, dy = 1/(1-\alpha)\int_0^1 |f(y)|\, y\, dy$. Hence $E_1 = L^1([0,1], \frac{1}{1-\alpha}\, ydy)$ and

$$T_1(t)f(x) = \begin{cases} x^{-\alpha}(x+t)^\alpha f(x+t) & \text{if } x \leq 1-t \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in E_1$, $t \geq 0$.

Proof of Theorem 4.1. Uniqueness follows from Theorem C5, so we have to show the existence of the representation (4.1).

a) We assume that $s(A) < 0$. Using the construction 4.2 we define $S(t)f = \int_0^t T_1(s)f\, ds \in D(A_1) \subset E$ for $f \in E$. Then $S(t)$ is a positive operator on E and hence bounded $(t \geq 0)$. It is clear from the definition that $0 = S(0) \leq S(s) \leq S(t)$ for $0 < s < t$. Moreover, let $t > 0$. Then for $f \in E_{1+}$, $A_1\int_0^t T_1(s)f\, ds = T_1(t)f - f$. Hence $\int_0^t T_1(s)f\, ds = R(0,A_1)f - R(0,A_1)T_1(t)f \leq R(0,A_1)f$. Thus,

$$(4.6) \qquad S(t) \leq R(0,A) \qquad (t \geq 0).$$

In particular, $\sup\{\|S(t)\| : t \geq 0\} < \infty$. We now show that $S(.) : [0,\infty) \to \mathcal{L}(E)$ is strongly continuous. Let $f \in D(A)$, $g = Af$. Then $\|S(t+h)f - S(t)f\| = \|R(0,A)(S(t+h)g - S(t)g)\| \leq \|S(t+h)g - S(t)g\|_1 \to 0$ for $h \to 0$. Here we made use of (4.4).

Thus $S(.)$ is strongly continuous on a dense subspace. Since $S(.)$ is bounded, this implies the strong continuity on the whole space.

Since $S(.)$ is bounded, it follows from Proposition C3 that the integral in (4.1) converges in the operator norm for $\lambda > 0$. Let $f \in E$. Then $\int_0^\infty e^{-\lambda t}\, dS(t)f = \int_0^\infty e^{-\lambda t}\, T_1(t)f\, dt = R(\lambda,A_1)f = R(\lambda,A)f$. Thus (4.1) holds and the proof is finished in the case when $s(A) < 0$.

b) Let now $s(A)$ be arbitrary. For $w > s(A)$ consider the operator $B = A - w$. Then $s(B) < 0$, so by a), there exists a strongly continuous increasing function $S_w^\#(.) : [0,\infty) \to \mathcal{L}(E)_+$ satisfying $S_w^\#(0) = 0$, such that $R(\mu,B) = \int_0^\infty e^{-\mu t}\, dS_w^\#(t)$ for $\mu > 0$. Hence $R(\lambda,A) = R(\lambda-w, B) = \int_0^\infty e^{-\lambda t}\, e^{wt}\, dS_w^\#(t) = \int_0^\infty e^{-\lambda t}\, dS_w(t)$ for all $\lambda > w$, where $S_w(t) = \int_0^t e^{ws}\, dS_w^\#(s)$ (by Proposition B4). Clearly, $S_w(.)$ is strongly continuous, increasing and satisfies $S_w(0) = 0$. Because of the uniqueness theorem (Theorem C5), it follows that $S_w(t) = S_{w'}(t)$ $(t \geq 0)$ for all $w, w' > s(A)$. This proves the theorem in the general case. □

Example 4.5. a) Let A be the generator of a strongly continuous positive semigroup $(T(t))_{t\geq 0}$. Then $S(t)f = \int_0^t T(s)f\, ds$ for all $f \in E$, $t \geq 0$.

b) If A is the operator in Example 3.2, then

$$S(t)f(x) = \begin{cases} x^\alpha \int_0^x y^{-\alpha}\, f(y)\, dy & \text{if } x \leq t \\ x^\alpha \int_{x-t}^x y^{-\alpha}\, f(y)\, dy & \text{if } x > t \end{cases}$$

$(f \in C_o(0,1],\ x \in (0,1],\ t \geq 0)$.

5. Approach via Bernstein's theorem.

The representation of a positive resolvent by a Laplace-Stieltjes integral can also be obtained using Bernstein's theorem instead of the Hille-Yosida theorem if additional assumptions on the space are made. On the other hand, it is not necessary to assume that A has dense domain.

Definition 5.1. We say that E is an ideal in E'' if for $f \in E$, $g \in E''$, $0 \leq g \leq f$ implies $g \in E$.

Note: Here we identify E with a subspace of E'' (via evaluation). Then by (A1), $E_+'' \cap E = E_+$ (i.e. E is an ordered subspace of E'').

Lemma 5.2. Suppose that E is an ideal in E''. Then the norm is order continuous in the following sense. If $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence in E_+, then $(f_n)_{n \in \mathbb{N}}$ converges strongly (and $\lim_{n \to \infty} f_n = \inf_{n \in \mathbb{N}} f_n$).

Proof. (cf.[42,II 5.9]) Let $F_0(\phi) = \inf_{n \in \mathbb{N}} \langle f_n, \phi \rangle$ $(\phi \in E_+')$. Then $F_o : E_+' \to \mathbb{R}$ is additive and positive homogeneous. For $\psi \in E'$ let $F(\psi) = F_o(\psi_1) - F_o(\psi_2)$ where $\psi_1, \psi_2 \in E_+'$ such that $\psi_1 - \psi_2 = \psi$. Then F is well-defined (since F_o is additive), linear and positive. Thus $F \in E''$ and $0 \leq F \leq f_n$ for all $n \in \mathbb{N}$. Since E is an ideal in E'', it follows that $F \in E$. Moreover, by Dini's theorem $\lim_{n \to \infty} \langle f_n, \phi \rangle = \langle F, \phi \rangle$ uniformly on $U_+^o := \{\phi \in E_+' : \|\phi\| \leq 1\}$. Hence $\lim_{n \to \infty} N(f_n - F) = 0$, where N denotes the canonical half-norm

(A2). Since $f_n - F \geq 0$, $N(-(f_n-F)) = 0$ $(n \in \mathbb{N})$. Thus $\lim_{n\to\infty} \|f_n - F\|_N = 0$, which implies that $\lim_{n\to\infty} f_n = F$ (by (A4), since the cone E_+ is normal). □

<u>Examples</u> 5.3. a) If E is reflexive, then E is trivially an ideal in E''.

b) A Banach lattice E is an ideal in E'' if and only if the norm is order continuous (in the sense of Lemma 5.2.) (see[42,II§5]). For example, $L^p(X,\mu)$ ((X,μ) a σ-finite measure space and $1 \leq p < \infty$) and c_o have an order continuous norm, but $C[0,1]$ has not.

<u>Definition</u> 5.4. A function $f : (a,\infty) \to E$ is called <u>completely monotonic</u> if f is infinitely differentiable and

$$(5.1) \quad (-1)^n f^{(n)}(\lambda) \geq 0 \quad \text{for all } \lambda > a, \quad n = 0,1,2,\ldots$$

<u>Theorem</u> 5.5. Assume that E is an ideal in E''.
Let $f : (a,\infty) \to E$ be a completely monotonic function. Then there exists a uniquely determined normalized increasing function $\alpha : (0,\infty) \to E$ such that

$$(5.2) \quad f(\lambda) = \int_o^\infty e^{-\lambda t} d\alpha(t) \quad (\lambda > a) .$$

<u>Proof</u>. Let $\phi \in E_+'$. Then $\lambda \to \langle f(\lambda), \phi\rangle$ is completely monotonic. So by Bernstein's theorem there exists a unique normalized increasing function $\alpha_\phi : (0,\infty) \to \mathbb{R}$ such that

$$(5.3)\quad \langle f(\lambda),\phi\rangle = \int_0^\infty e^{-\lambda t} d\alpha_\phi(t) \qquad (\lambda > a).$$

From the uniqueness theorem (Theorem C.5) it follows that $\alpha_\phi(t)$ is additive and positive homogeneous in ϕ for every $t \geq 0$. Thus for every $t \geq 0$ there exists a unique $\alpha_\phi(t) \in (E')'_+$ such that $\langle\phi,\alpha(t)\rangle = \alpha_\phi(t)$ for all $\phi \in E'_+$. Let $\lambda > \max\{a,0\}$. Then for every $\phi \in E'_+$, $\langle f(\lambda),\phi\rangle \geq \int_0^t e^{-\lambda s} d\alpha_\phi(s) = e^{-\lambda t}\alpha_\phi(t) + \lambda\int_0^t e^{-\lambda s}\alpha_\phi(s)ds \geq e^{-\lambda t}\alpha_\phi(t)$. Consequently, $\langle\alpha(t),\phi\rangle \leq \langle e^{\lambda t} f(\lambda),\phi\rangle$. Hence $\alpha(t) \leq e^{\lambda t} f(\lambda)$, and our assumption on E implies that $\alpha(t) \in E_+$. It follows from (A1) that $\alpha : (0,\infty) \to E_+$ is increasing. Since the integral in (5.3) converges for every $\lambda > a$ and $\phi \in E'_+$, we conclude from Proposition C.1 that the integral $\int_0^\infty e^{-\lambda t} d\alpha(t)$ converges in the norm for every $\lambda > a$. Finally, (5.3) implies that $\langle f(\lambda),\phi\rangle = \int_0^\infty e^{-\lambda t} d\langle\alpha(t),\phi\rangle = \langle\int_0^\infty e^{-\lambda t} d\alpha(t),\phi\rangle$ for all $\phi \in E'_+$. Hence (5.2) holds. This proves the existence. Uniqueness follows from Theorem C5. □

<u>Remark</u> 5.6. a) It is not difficult to see that the converse of Theorem 5.5 holds as well, i.e., if f is representable as in (5.2) then f is completely monotonic.

b) There are other results related to Theorem 5.5. Schaefer [40] obtained a characterisation of completely monotonic sequences with values in an ordered locally convex space as moments of an increasing function on $[0,1]$ (Hausdorff moment problem).

Another vector-valued version of Bernstein's theorem has been obtained by Bochner [10]. He defines the Stieltjes integral purely in terms of the ordering (and the precise definition can only be seen in the proof). In our context we need that the Riemann-Stieltjes sums converge strongly to the integral.

Theorem 5.7. Suppose that E is an ideal in E''. Let A be a resolvent positive operator. Then there exists a unique strongly continuous family $(S(t))_{t\geq 0}$ of operators on E such that

$$0 = S(0) \leq S(s) \leq S(t) \qquad (0\leq s\leq t) \quad \text{and}$$

$$(5.4) \qquad R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\, dS(t) \qquad (\lambda > s(A))$$

(where the integral converges in the operator norm).

Proof. Uniqueness follows from Theorem C5. We show the existence of the representation (5.4). Let $f \in E_+$. Then $R(.,A)f$ is a completely monotonic function from $(s(A),\infty)$ into E . By Theorem 5.5 there exists a unique normalized increasing function $S(.,f) : (0,\infty) \to E$ such that

$$(5.5) \qquad R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\, dS(t,f) \qquad (\lambda > s(A)).$$

From the uniqueness theorem (Theorem C.5) it follows that for every $t\geq 0$ the mapping $f \to S(t,f)$ from E_+ into E_+ is additive and positive homogeneous. Since $E = E_+ - E_+$, there exists a unique linear operator $S(t)$ on E such that $S(t)f = S(t,f)$ for all $f \in E_+$.
Since $S(.)f$ is increasing for all $f\in E_+$, $S(.)$ is increasing

Moreover, $S(.)$ is normalized (since $S(.,f)$ is for all $f \in E_+$). Let $\mu > s(A)$. Then it follows immediately from the definition that $R(\mu,A)S(.)$ and $S(.)R(\mu,A)$ are also normalized. Moreover, for all $\lambda > s(A)$, $\int_0^\infty e^{-\lambda t}\, d(R(\mu,A)S(t))$ $= R(\mu,A)R(\lambda,A) = R(\lambda,A)R(\mu,A) = \int_0^\infty e^{-\lambda t}\, d(S(t)(R(\lambda,A)))$. Hence it follows from Theorem C.5 that

$$(5.6) \qquad S(t)R(\mu,A) = R(\mu,A)S(t) \qquad (t \geq 0).$$

Now let $f \in D(A)$. Then for all $\lambda > \max\{s(A),0\}$,
$\int_0^\infty \lambda e^{-\lambda t}\, d(tf) = f = R(\lambda,A)(\lambda f - Af) = \int_0^\infty \lambda e^{-\lambda t} dS(t)f -$
$\int_0^\infty e^{-\lambda t} d(S(t)Af) = \int_0^\infty \lambda e^{-\lambda t}\, dS(t)f - \int_0^\infty \lambda e^{-\lambda t} S(t)Af\, dt =$
$\int_0^\infty \lambda e^{-\lambda t}\, dS(t)f - \int_0^\infty \lambda e^{-\lambda t}\, d(\int_0^t S(s)Af ds)$.
[In order to justify the last step, we first observe that $Af = g_1 - g_2$ for some $g_1, g_2 \in E_+$. Hence $t \to \int_0^t S(s)Afds$ is of bounded variation as the difference of increasing functions. Moreover, the Riemann integral $\int_0^t S(s)Afds$ exists in the norm by the remark following (B8). Hence $\langle S(.)Af,\phi\rangle$ is Riemann- and so Lebesgue integrable on every interval $[0,t]$ $(t>0)$ for every $\phi \in E'_+$. It follows from [50,I Theorem 6a] that
$\int_0^b \lambda e^{-\lambda t} \langle S(t)Af,\phi\rangle dt = \int_0^b \lambda e^{-\lambda t}\, d(\int_0^t \langle S(s)Af,\phi\rangle ds)$ for all $b>0$.]
Thus $\int_0^\infty e^{-\lambda t} d(tf) = \int_0^\infty e^{-\lambda t} dS(t)f - \int_0^\infty e^{-\lambda t} d(\int_0^t S(s)Afds)$ for all $\lambda > \max\{0,s(A)\}$. Consequently, by the uniqueness theorem,

$$(5.7) \qquad tf = S(t)f - \int_0^t S(s)Afds \qquad (t \geq 0).$$

This implies that $S(.)f$ is continuous for all $f \in D(A)$.
Now let $g \in E_+$, $t>0$. Then $\lim_{s \downarrow t} S(s)g =: h_+$ and

$\lim_{s\uparrow t}$ =: h_- exist by Lemma 5.2. We have to show that $h_+ = h_-$. Let $\lambda > s(A)$. Then by (5.6), $R(\lambda,A)h_+ = \lim_{s\downarrow t} R(\lambda,A)S(s)g = \lim_{s\downarrow t} S(s)R(\lambda,A)g = S(t)R(\lambda,A)g$ (since $R(\lambda,A)g \in D(A)$) $= \lim_{s\uparrow t} S(s)R(\lambda,A)g = R(\lambda,A)(\lim_{s\uparrow t} S(s)g) = R(\lambda,A)h_-$. Since $R(\lambda,A)$ is injective, it follows that $h_+ = h_-$. □

Remark 5.8 Suppose that A is a resolvent positive operator such that a normalized increasing function $S : [0,\infty) \to \mathcal{L}(E)_+$ exists such that (5.4) holds. Then the proof of Theorem 5.7 shows that S is strongly continuous.

Remark 5.9 It is not difficult to deduce the Hille-Yosida theorem (in the form stated in the introduction to this chapter) from Theorem 5.7 .

6. The integrated semigroup.

Let A be a resolvent positive operator. We assume that there exists a strongly continuous increasing function $S : [0,\infty) \to \mathcal{L}(E)$ satisfying $S(0) = 0$ such that

$$(6.1) \qquad R(\lambda,A) = \int_0^\infty e^{-\lambda t}\, dS(t) \qquad (\lambda > \lambda_0)$$

(in the weak operator topology) for some $\lambda_0 \geq s(A)$. By the results of the last section such a representation of $R(\lambda,A)$ exists when either A is densely defined or F is an ideal in E'' .

Note that $(S(t))_{t \geq 0}$ is uniquely defined, and we call $(S(t))_{t \geq 0}$ the <u>integrated semigroup generated by</u> A. (Of course, this terminology is motivated by the case when A is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$, because then $S(t) = \int_0^t T(s)\, ds$.)

The following proposition shows that the spectral bound $s(A)$ is determined by the asymptotic behavior of $S(t)$ for $t \to \infty$.

<u>Proposition</u> 6.1. a) For all $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\lambda > s(A)$,

$$(6.2) \qquad R(\lambda, A) = \int_0^\infty e^{-\lambda t}\, dS(t)$$

where the integral converges in the operator norm.

b) If $s(A) \geq 0$, then $s(A) = \inf\{w > 0$: there exists $M \geq 0$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0\}$.

c) If $s(A) < 0$, then $\lim_{t\to\infty} S(t) = R(0,A)$ and $s(A) = \inf\{w<0$: there exists $M \geq 0$ such that $\|R(0,A) - S(t)\| \leq Me^{wt}$ for all $t \geq 0\}$.

<u>Proof</u>: Let $f \in E_+$, $\phi \in E'_+$. Denote by s the abscissa of convergence of the integral $\int_0^\infty e^{-\lambda t}\, d\langle S(t)f, \phi\rangle$. Then by [51, ch. 5, Theorem 10.1] s is a singular point of the analytic function $\lambda \to \int_0^\infty e^{-\lambda t}\, d\langle S(t)f,\phi\rangle$ $(\mathrm{Re}\lambda > s)$. By the uniqueness of analytic extensions this implies that $s \leq s(A)$ and $\langle R(\lambda,A)f,\phi\rangle = \int_0^\infty e^{-\lambda t}\, d\langle S(t)f,\phi\rangle$ $(\mathrm{Re}\lambda > s(A))$.
From this a) follows by Proposition C3. Assertion b) follows from a), Proposition C3 and the uniqueness of analytic extensions.

Now we show c). Let $s(A) < 0$ and $w \in (s(A),0)$. Let $S_1(t) = \int_0^t e^{-wr}\, dS(r)$. Then $\lim_{t\to\infty} S_1(t) = R(w,A)$ (by (6.2)). In particular, $S_1(t) \leq R(w,A)$ for $t \geq 0$. So $0 \leq R(0,A) - S(t) = \int_t^\infty dS(s) = \int_t^\infty e^{ws}e^{-ws}\, dS(s) = \int_t^\infty e^{ws}\, dS_1(s) = \lim_{s\to\infty} e^{ws}S_1(s) - e^{wt}S_1(t) - w\int_t^\infty e^{ws}S_1(s)\, ds$
$= -e^{wt}S_1(t) - w\int_t^\infty e^{ws}S_1(s)ds \leq (-w)\int_t^\infty e^{ws}S_1(s)\, ds$
$\leq (-w)\int_t^\infty e^{ws}ds\; R(w,A) = e^{wt}R(w,A)$. Thus
$\|R(0,A) - S(t)\| \leq Me^{wt}$ for some $M \geq 0$ and all $t \geq 0$.
Conversely assume that $w < 0$ such that $\|R(0,A) - S(t)\| \leq Me^{wt}$ $(t \geq 0)$. Let $\lambda > w$. Let $S_1(t) = \int_0^t e^{-\lambda s}\, dS(s)$ $(t \geq 0)$. It follows from (6.2) that

$$S(t) = \int_0^t dS(s) \leq R(0,A) \quad \text{for all } t \geq 0. \tag{6.3}$$

Let $t \geq 0$. Then for all $r \geq t$, $0 \leq S_1(r) - S_1(t) = \int_t^r e^{-\lambda s}\, dS(s)$ $\leq e^{-\lambda t}\int_t^r dS(s) = e^{-\lambda t}(S(r) - S(t)) \leq e^{-\lambda t}(R(0,A) - S(t))$.
Consequently $\|S_1(r) - S_1(t)\| \leq Me^{-(\lambda-w)t}$ for all $r \geq t$. Thus $S_1(t) = \int_0^t e^{-\lambda s}\, dS(s)$ converges in the operator norm. We have proved that the integral (6.2) converges for all $\lambda > w$ (in the operator norm). By the uniqueness of analytic extensions this implies that $s(A) \leq w$. This finishes the proof of c). □

<u>Remark</u> 6.2. If A is the generator of a positive strongly continuous semigroup, then Proposition 6.1 implies that $R(\lambda,A) = \int_0^\infty e^{-\lambda t}T(t)dt$, where the integral converges in the <u>operator norm</u>, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > s(A)$. (Here $\int_0^b e^{-\lambda t}T(t)\, dt$ is defined strongly.) However, it may happen that $s(A) < \omega(A)$ (see [23]).

Now we establish the relations between A and the integrated semigroup. The operators A and $S(t)$ commute. In fact,

(6.4) $\quad S(t)R(\lambda,A) = R(\lambda,A)S(t) \qquad (\lambda > s(A), t>0)$.

[This is proved as (5.6).] As a consequence,

(6.5) $\quad f \in D(A)$ implies $S(t)f \in D(A)$ and
$AS(t)f = S(t)Af \qquad (t \geq 0)$.

<u>Proposition</u> 6.3. Let $t > 0$. If $f \in D(A)$, then

(6.6) $\quad \int_0^t S(s)Af\, ds = S(t)f - tf$ for all $t > 0$.

Moreover, $\int_0^t S(s)f\, ds \in D(A)$ for all $f \in \overline{D(A)}$ and

(6.7) $\quad A\int_0^t S(s)f\, ds = S(t)f - tf$.

<u>Proof</u> : (6.6) is shown as (5.7). Then (6.7) follows since A is closed. □

It follows from (6.6) that

(6.8) $\quad \lim_{t\to 0} \frac{1}{t}S(t)f = f \qquad$ for all $f \in D(A)$.

The integrated semigroup can be characterized as the solution of an inhomogeneous Cauchy problem.

Proposition 6.4. Let $f \in D(A)$.

a) Let $v(t) = S(t)f$ $(t \geq 0)$. Then v is continuously differentiable and

$$(6.9) \quad \begin{aligned} v'(t) &= Av(t) + f \qquad (t \geq 0) \\ v(0) &= 0 . \end{aligned}$$

b) Conversely, if $v : [0,\infty) \to E$ is continuously differentiable such that $v(t) \in D(A)$ for all $t \geq 0$ and such that (6.9) holds, then $v(t) = S(t)f$ $(t \geq 0)$.

Proof. a) follows immediately from (6.6) and (6.5). So let v satisfy the assumption of b). Let $w(t) = v(t) - S(t)f$. Then w is continuously differentiable, satisfies $w(0) = 0$, $w(t) \in D(A)$ and $w'(t) = Aw(t)$ for all $t \geq 0$.
Let $F = \overline{D(A)} \subset E$. Then $R(\lambda,A)F \subset F$ for all $\lambda \in \rho(A)$. Let the operator A_o on F be defined by $A_of = Af$, $D(A_o) = \{f \in D(A) : Af \in F\}$. Then it is easy to see that $\rho(A) \subset \rho(A_o)$ and $R(\lambda,A_o) = R(\lambda,A)|_F$ for $\lambda \in \rho(A)$. Since $Aw(t) = w'(t) = \lim_{h\to 0} \frac{1}{h}(w(t+h)-w(t)) \in \overline{D(A)} = F$, it follows that $w(t) \in D(A_o)$ for all $t \geq 0$. Thus w is a solution in F of the Cauchy problem $w'(t) = A_ow(t)$ $(t \geq 0)$, $w(0)=0$. But $D(A_o)$ is dense in F by definition and $\lim_{\lambda\to\infty}\|R(\lambda,A_o)\| < \infty$ (by (1.3)). So it follows from [36, Chapter 4, 1.2] that $w(t) = 0$ $(t \geq 0)$. □

Proposition 6.5. Let $s,t > 0$. Then

(6.10) $S(s)S(t) = \int_0^{s+t} S(r)dr - \int_0^s S(r)dr - \int_0^t S(r)dr$.

In particular, $S(s)S(t) = S(t)S(s)$.
If $f \in \overline{D(A)}$, then $S(s)S(t)f \in D(A)$ and

(6.11) $AS(s)S(t)f = S(s+t)f - S(s)f - S(t)f$.

Proof Let $s > 0$, $f \in D(A)$ and $v(t) = \int_0^t (S(s+r)f-S(r)f)dr = \int_0^{s+t} S(r)f\,dr - \int_0^s S(r)fdr - \int_0^t S(r)fdr$ $(t \geq 0)$. We show that $v'(t) = Av(t) + S(s)f$. Then it follows from Proposition 6.4.b) that

(6.12) $\int_0^{s+t} S(r)fdr - \int_0^s S(r)fdr - \int_0^t S(r)fdr = S(t)S(s)f$

We have by Proposition 6.4., $\frac{d}{dr}S(s+r)f = AS(s+r)f + f$ and $\frac{d}{dr}S(r)f = AS(r)f + f$. This implies that $Av(t) = \int_0^t (AS(s+r)f-AS(r)f)\,dr = \int_0^t \frac{d}{dr}(S(s+r)f-S(r)f)dr = S(s+t)f - S(t)f - S(s)f$. Hence, $v'(t) = S(s+t)f - S(t)f = Av(t)+S(s)f$. Thus (6.12) is proved and so (6.10) holds on $D(A)$. Let $f \in E$ be arbitrary. Applying (6.12) to $g := R(\lambda,A)f$ (where $\lambda > s(A)$) one obtains (using (6.5)) that
$R(\lambda,A)[\int_0^{t+s} S(r)fdr - \int_0^s S(r)f\,dr - \int_0^t S(r)fdr] = R(\lambda,A)S(t)S(s)f$.
Since $R(\lambda,A)$ is injective, (6.12) follows. The remaining assertion (6.11) is a consequence of (6.10) and (6.7). □

Remark 6.6. a) Formula (6.10) corresponds to the semigroup property. In fact, suppose that $(T(t))_{t>0}$ is a strongly continuous family of positive operators such that $\int_0^t T(s)\,ds$ exists strongly. Let $S(t) := \int_0^t T(s)ds$ $(t \geq 0)$. Then $(T(t))_{t>0}$ is a semigroup if and only if $(S(t))_{t>0}$ satisfies (6.10) (this is easy to show by differentiating (6.10); resp., integrating the semigroup formula).

b) The preceding procedure can be reversed. One can start from an "integrated semigroup" and obtain a resolvent positive operator. More precisely, let $S : [0,\infty) \to \mathcal{L}(E)_+$ be a strongly continuous function such that

(i) $S(0) = 0$;

(ii) $\|S(t)\| \leq Me^{wt}$ $(t \geq 0)$ for some $w \in \mathbb{R}$, $M \geq 0$;

(iii) (6.10) holds;

(iv) for all $f \in E_+$ there exists $t \geq 0$ such that $S(t)f \neq 0$.

Then S is increasing and there exists a (unique) resolvent positive operator A such that $s(A) \leq w$ and

$R(\lambda,A) = \int_0^\infty e^{-\lambda t}\, dS(t)$ for all $\lambda > w$.

The proof can be given by showing that the operators $R(\lambda) := \int_0^\infty \lambda e^{-\lambda t} S(t)dt$ $(\lambda > w)$ form a pseudoresolvent. We omit the details.

7 The Abstract Cauchy Problem.

Let A be a resolvent positive operator. We assume that either $D(A)$ is dense or that E is an ideal in E'' (see section 5).

Theorem 7.1. For every $f \in D(A^2)$ there exists a unique continuously differentiable function $u : [0,\infty) \to E$ such that $u(t) \in D(A)$ for all $t \geq 0$ and

$$(7.1) \qquad \begin{aligned} u'(t) &= Au(t) \\ u(0) &= f \ . \end{aligned}$$

If $f \geq 0$, then $u(t) \geq 0$ for all $t \geq 0$. Moreover, the solution of (7.1) depends continuously on the initial value in the following sense: Let $f_n \in D(A^2)$ such that $\lim_{n\to\infty} f_n = f$ in the graph norm. Denote by u_n the solution of (7.1) for the initial value f_n. Then $u_n(t)$ converges to $u(t)$ in the norm uniformly on bounded intervals.

Proof. Uniqueness is proved as in Proposition 6.4. In order to prove existence we assume that $s(A) < 0$ (otherwise one considers $A-w$ instead of A for some $w > s(A)$). Denote by $(S(t))_{t\geq 0}$ the integrated semigroup generated by A. Let $f \in D(A^2)$ and define $u(t) = S(t)Af + f$ $(t\geq 0)$. Then by Proposition 6.4 $u'(t) = AS(t)Af + Af = Au(t)$ $(t\geq 0)$. Thus u is a solution of (7.1).

Now let $f_n \in D(A^2)$ such that $\lim_{n\to\infty} f_n = f$ in the graph norm. Let $u_n(t) = S(t)Af_n + f_n$. Since $(S(t))_{t\geq 0}$ is strongly continuous, it follows that $u_n(t)$ converges in the norm to $u(t)$ uniformly on bounded intervals. Finally, assume that $0 \leq f \in D(A^2)$. Then using (6.8) and (6.11) one obtains $u(t) = S(t)Af + f = \lim_{s\to 0} \frac{1}{s}(S(s)S(t)Af + S(s)f) = \lim_{s\to 0} \frac{1}{s}(S(s+t)f - S(t)f)$. Hence $u(t) \geq 0$, since $S(.)$ is increasing. □

Remarks 7.2. a) If $D(A)$ is dense, then also $D(A^2)$ is dense. In fact, let $\lambda \in \rho(A)$, then $E = \overline{D(A)} = \overline{(R(\lambda,A)E)} = \overline{(R(\lambda,A)\overline{D(A)})}$ $\subset \overline{(R(\lambda,A)D(A))} = \overline{((R(\lambda,A))^2E)} = \overline{D(A^2)}$.

b) In general, there does not exist a continuously differentiable solution of (7.1) for every initial value in $D(A)$. In fact, if $D(A)$ is dense, this would imply that A is the generator of a strongly continuous semigroup (see [30, ch.I Thm. 2.12] or [33]).

c) The continuous dependence of the solutions on the initial values is no longer guaranteed if in Theorem 7.1 one replaces the graph norm by the norm. In fact, if $D(A)$ is dense, this implies that for every $t \geqq 0$, the operator $T_o(t)$ given by $T_o(t)f = S(t)Af + f$ (from $D(A^2)$ into E) has a continuous extension $T(t)$ on E. It is not difficult to see that then $(T(t))_{t \geqq 0}$ is a strongly continuous semigroup whose generator is A.

d) If A is densely defined, an alternative proof of Theorem 7.1 can be given using the construction 4.2.

Under more restrictive assumptions Theorem 7.1 can be sharpened. We need the following correspondence between the asymptotic behavior of $S(t)$ for $t \to 0$ and $\lambda R(\lambda,A)$ for $\lambda \to \infty$.

Proposition 7.2 Let $(S(t))_{t \geqq 0}$ be the integrated semigroup generated by A . The following are equivalent:

(i) $\sup_{0 \leqq t \leqq 1} \frac{1}{t} \|S(t)\| < \infty$.

(ii) $\sup_{\lambda \geqq \lambda_o} \|\lambda R(\lambda,A)\| < \infty$ where $\lambda_o > s(A)$.

Moreover, if $D(A)$ is dense and (ii) holds, then

$$(7.2) \qquad \lim_{t\to 0} \frac{1}{t} S(t)f = f \qquad \text{for all } f\in E .$$

<u>Proof.</u> Let $w\in\mathbb{R}$. Then condition (i) as well as condition (ii) holds for A if and only if it holds for $A-w$ (observe that the integrated semigroup $(S_w(t))_{t\geq 0}$ generated by $A-w$ is given by $S_w(t) = \int_0^t e^{-ws}\, dS(s) = e^{-wt}S(t) + w\int_0^t e^{-sw}S(s)\, ds$). Thus we can assume that $s(A)<0$.
Assume that (i) holds. Then $M := \sup_{0<t<\infty} (1/t)\|S(t)\| < \infty$. Hence $\|\lambda R(\lambda,A)\| = \|\int_0^\infty \lambda^2 e^{-\lambda t} S(t)\,dt\| \leq \int_0^\infty \lambda^2 t e^{-\lambda t} (\frac{1}{t}) \|S(t)\|dt \leq M\int_0^\infty \lambda^2 t e^{-\lambda t} dt = M$ for all $\lambda>0$.
Conversely, assume that $\sup_{\lambda\geq 0}\|\lambda R(\lambda,A)\|<\infty$. Let $t>0$. Choose $\lambda = \frac{1}{t}$. Then $0\leq\frac{1}{t}S(t) = \frac{1}{t}\int_0^t dS(s) = e\lambda\int_0^t e^{-1}dS(s) \leq e\lambda\int_0^t e^{-\lambda s}dS(s) \leq e\lambda R(\lambda,A)$. This implies that $\sup_{t\geq 0}(\frac{1}{t})\|S(t)\| < \infty$. The last assertion follows from (6.8). □

<u>Remark.</u> The argument in the proof is due to G. Greiner (unpublished) who used it to prove a corresponding statement for the behavior of $\frac{1}{t}S(t)$ for $t\to\infty$ and $\lambda R(\lambda,A)$ for $\lambda\to 0$ when A is the generator of a positive semigroup (cf. [19,2.10.c) i]).

<u>Example</u> 7.3. a) The operators given in Example 3.2, 3.3 and 3.5 satisfy the equivalent conditions of Proposition 7.2. Moreover, they all are densely defined.
b) Consider the operator $-A$, where A is defined as in Example 2.8. on $E_0 \subset L^1(\mathbb{R})$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by $-A$. Then $S(t)f(x) = \int_0^t f(x-s)\,ds = \int_{x-t}^x f(y)\,dy$. We

show that $\lim_{t\to 0} \frac{1}{t}\|S(t)\| = \infty$. Let $t_n = 2^{-n}$ and $f_n = 2^n 1_{[-2^{-n},0]}$. Then $\|f\|_o = \|f\|_1 = 1$. Moreover,

$$S(t_n)f(x) = \begin{cases} 0 & \text{for } x \leq -2^{-n} \\ 2^n x + 1 & \text{for } -2^{-n} < x \leq 0 \\ 1 - 2^n x & \text{for } 0 < x \leq 2^{-n} \\ 0 & \text{for } 2^{-n} < x . \end{cases}$$

Hence $p_{n+1}(S(t_n)f) = (\frac{3}{2})^{n+1} \int_{2^{-n-1}}^{2^{-n}} S(t_n)f(x)dx =$

$(\frac{3}{2})^{n+1}(1/8)2^{-n}$. Hence, $\|(1/t_n)S(t_n)\| \geq t_n^{-1} p_{n+1}(S(t_n)f) =$ $= \frac{1}{8}(\frac{3}{2})^{n+1} \to \infty$ for $n\to\infty$. □

Lemma 7.4. Suppose that A is densely defined and $\sup_{\lambda\geq\lambda_o} \|\lambda R(\lambda,A)\| < \infty$ where $\lambda_o > s(A)$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A . Let $f\in E$ and $t>0$. Then $S(t)f \in D(A)$ if and only if $S(.)f$ is differentiable in t . In that case,

$$\frac{d}{ds}\Big|_{s=t} S(s)f = AS(t)f + f . \tag{7.3}$$

Proof. By Proposition 6.5 we have $S(s)S(t)f \in D(A)$ and $\frac{1}{s}AS(s)S(t)f + \frac{1}{s}S(s)f = \frac{1}{s}(S(t+s)f - S(t)f)$ $(s>0)$. Since by (7.2) $\lim_{s\to 0} \frac{1}{s}S(s)g = g$ for all $g\in E$, the assertion follows because A is closed.

Proposition 7.5. Suppose that E is reflexive. If $\sup_{\lambda\geq\lambda_o} \|\lambda R(\lambda,A)\| < \infty$ for some $\lambda_o > s(A)$ then $D(A)$ is dense.

Proof. Let $f \in E$. Since norm bounded sets in E are relatively weakly compact, there exists a limit point g of $\lambda R(\lambda,A)f$ for $\lambda \to \infty$. Since by (1.5) $\lim_{\lambda\to\infty} \lambda R(\lambda,A)R(\mu,A)f = R(\mu,A)f$ (where $\mu > s(A)$ is fixed), it follows that $R(\mu,A)f = R(\mu,A)g$. Hence $g = f$. Consequently, $\lim_{\lambda\to\infty} \lambda R(\lambda,A)f = f$ weakly. Hence $f \in \overline{D(A)}^{\sigma(E,E')} = \overline{D(A)}$. □

Let $I(A) := \{g \in E: \pm g \leq f \text{ for some } f \in D(A)_+\}$. Since $D(A) = D(A)_+ - D(A)_+$, one has $D(A) \subset I(A)$ (in fact, $I(A)$ is the ideal generated by $D(A)$).

Theorem 7.6. Assume that E is reflexive and $\sup_{\lambda \geq \lambda_0} \|\lambda R(\lambda,A)\| < \infty$ (where $\lambda_0 > s(A)$).
Then for every $f \in D(A)$ such that $Af \in I(A)$ there exists a unique differentiable function $u : [0,\infty) \to E$ satisfying $u(t) \in D(A)$ for all $t \geq 0$ such that

$$(7.4) \qquad \begin{aligned} u'(t) &= Au(t) \\ u(0) &= f\,. \end{aligned}$$

Proof. Uniqueness is shown as in the proof of Proposition 6.4. We show the existence. It follows from Proposition 7.5 that $D(A)$ is dense.

a) Let $g \in I(A)$, $t>0$. We claim that $S(t)g \in D(A)$. There exists $h \in D(A)_+$ such that $\pm g \leq h$. Since $S(.)h$ is differentiable, one has $\sup\{\|\frac{1}{s}(S(t+s)-S(t))h\| : 0<s\leq 1\} < \infty$. But $\pm\frac{1}{s}(S(t+s)-S(t))g \leq \frac{1}{s}(S(t+s)-S(t))h$ for $0<s\leq 1$. Hence also $\sup\{\|\frac{1}{s}(S(t+s)-S(t))g\| : 0<s\leq 1\} < \infty$. Using (6.11) and (7.2) we conclude that $\sup\{\|\frac{1}{s}AS(s)S(t)g\| : 0<s\leq 1\} < \infty$. It follows from (6.10) that $\lim_{s\to 0} \frac{1}{s}S(s)S(t) = S(t)$ strongly. Let $k_1,k_2 \in E$ be weak limit points of $\frac{1}{s}AS(s)S(t)g$ for $s\to 0$. Let $\mu > s(A)$ be fixed. Then $R(\mu,A)k_1 = \lim_{s\to 0} \frac{1}{s}S(s)S(t)AR(\mu,A)g = R(\mu,A)k_2$. Consequently, $k_1=k_2$. Since norm-bounded sets in E are weakly relatively compact, $\frac{1}{s}AS(s)S(t)g$ has exactly one limit point for the weak topology; hence $\frac{1}{s}AS(s)S(t)g$ converges weakly for $s\to 0$. Since $\lim_{s\to 0} \frac{1}{s}S(s)S(t)g = S(t)g$ and A is closed, it follows that $S(t)g \in D(A)$.

b) Let $f \in D(A)$ such that $Af \in I(A)$. Let $u(t) = S(t)Af + f$. Then by a) $S(t)Af \in D(A)$ for all $t\geq 0$. It follows from Lemma 7.4 that u is differentiable and $u'(t) = AS(t)Af + Af = Au(t)$ $(t\geq 0)$. □

Remark 7.7 The solutions depend continuously on the initial values in the same sense as stated in Theorem 7.1.

8 Kato's Inequality and the existence of a positive resolvent.

Up to this point we assumed that a resolvent positive operator was given. Now we find conditions on A which imply that A is resolvent positive.

Throughout this section we assume that E is a Banach lattice with order continuous norm and that there exists a strictly positive linear form ϕ on E . Then $\|f\|_\phi := \langle |f|, \phi\rangle$ defines a norm on E . We denote by (E,ϕ) the completion of E with respect to this norm. (E,ϕ) is an AL-space (and so isomorphic to a space of type L^1 [42,II Theorem 8.5]). Moreover, E is an <u>ideal</u> in (E,ϕ) ; that is, if $f,g \in (E,\phi)$, $|g| \leq f$ and $f \in E$, then also $g \in E$ (see [42, IV 9.3]). For example, let $E = L^p(X,\mu)$ $(1 \leq p < \infty)$, where (X,μ) is a σ-finite measure space. Let $\phi \in L^q(X,\mu)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and $\phi(x) > 0$ μ-a.e. Then $(E,\phi) = L^1(X,\phi\mu)$.

<u>Theorem</u> 8.1. Let A be a densely defined operator on E such that the following two assertions hold.

(i) There exist a strictly positive $\phi \in D(A')$ and
$\lambda_o \in \mathbb{R}$ such that $A'\phi \leq \lambda_o\phi$ and
$\langle(\text{sign } f)Af,\phi\rangle \leq \langle|f|,A'\phi\rangle$ $(f \in D(A))$ (Kato's inequality).

(ii) $(\mu_o - A)D(A) = E$ for some $\mu_o > \lambda_o$ (range condition).

Then A is resolvent positive and $s(A) \leq \mu_o$.
Moreover A is closable in (E,ϕ) and its closure is the generator of a positive strongly continuous semigroup on (E,ϕ) .

<u>Remark</u>. The theorem is in some aspects similar to the Lumer-Phillips theorem [15,Theorem 2.24]. The condition that A be dissipative is replaced by Kato's inequality and the existence of a strictly positive subeigenvector of A' . In contrast to dissipativity, this condition is non-metric; in particular, it is satisfied by A if and only if it holds for $A+w$ $(w \in \mathbb{R})$. The conclusion is weaker than that of the Lumer-Phillips theorem, but sufficient to yield solutions of the abstract Cauchy problem for

all initial values in $D(A^2)$ (cf. section 7).

Proof. Considering $(A-\lambda_o)$ instead of A we can assume that $\lambda_o=0$. Denote by N the canonical half-norm on (E,ϕ) . Then $N(f) = \langle f^+,\phi\rangle$ for all $f\in E$. By Proposition I.2.4 it follows from Kato's inequality that A is N-dissipative. Since $D(A)$ is dense in E , it is also dense in (E,ϕ) . Thus it follows from [5,Theorem 2.4] that A is closable in (E,ϕ) and the closure A_1 of A is N-dissipative. Since $E = (\mu_o-A)D(A) \subset (\mu_o-A_1)D(A_1)$, μ_o-A_1 also has dense range. So it follows from [5 ,Remark 4.2] (see also [39]) that A_1 generates an N-contraction semigroup, i.e. a positive contraction semigroup on (E,ϕ) . In particular, A_1 has a positive resolvent and $s(A_1) \leqq 0$. It follows from (ii) that $\mu_o\in\rho(A)$ and $R(\mu_o,A) = R(\mu_o,A_1)_{|E}$. Moreover,

(8.1) $Af = A_1f$ and $D(A) = \{f\in D(A_1)\cap E : A_1f\in E\}$.

Let $\mu \geqq \mu_o$. Then for $f\in E_+$ by (1.2), $R(\mu,A_1)f \leqq R(\mu_o,A_1)f \in E$. Since E is an ideal in (E,ϕ), it follows that $R(\mu,A_1)E \subset E$ for all $\mu\geqq\mu_o$. This together with (8.1) implies that $\mu\in\rho(A)$ and $R(\mu,A)=R(\mu,A_1)_{|E}$ for all $\mu\geqq\mu_o$. Thus A is resolvent positive and $s(A) \leqq \mu_o$. □

Remark 8.2. Also a converse version of Theorem 8.1 holds. In fact, assume that A is a densely defined resolvent positive operator. It is obvious from the proof of Propositon I.1.5 that for every $\lambda > s(A)$ there exists a strictly positive $\phi \in D(A')$ such that $A'\phi \leqq \lambda\phi$. Moreover, assume that

$$(8.2) \qquad \sup_{\lambda \geqq \lambda_0} \|\lambda R(\lambda, A)\| < \infty$$

where $\lambda_o > s(A)$. Then

$$(8.3) \qquad <(\text{sign } f)Af, \psi> \leqq <|f|, A'\psi>$$

holds for all $f \in D(A)$, $\psi \in D(A')_+$.
The proof can be done in the same way as that of Proposition I.1.1 if $T(t)$ is replaced by $(1-tA)^{-1}$ ($t>0$ small), because (8.2) implies that $Af = \frac{d}{dt}\Big|_{t=0} (1-tA)^{-1}f$ for all $f \in D(A)$.

<u>Remark</u> 8.3. Let A be a densely defined operator on E . Suppose that there exists $0 \leqq \lambda \in \rho(A)$ such that $R(\lambda, A) \geqq 0$. Then by the proof of Proposition I.1.5 there exists a strictly positive subeigenvector of A'. Thus it follows from Theorem 8.1 that A is resolvent positive (in the sense of our definition) if in addition Kato's inequality holds. Example [23,3.10] shows that this last condition cannot be omitted.

Appendix A Ordered Banach Spaces

A general reference is chapter V in Schaefer's monograph [41]; we also refer to the article by Batty and Robinson [8]. We confine ourselves to list some notions and results which are used in the text.

Let E be a Banach space. A subset C of E is called a <u>cone</u> if $\mathbb{R}_+ \cdot C \subset C$ and $C + C \subset C$. The cone C is called <u>proper</u> if $C \cap (-C) = \{0\}$. An ordered Banach space is a Banach space E together with a closed proper cone E_+ . The ordering in E is then defined by setting $f \leq g$ if and only if $g-f \geq 0$.

Let E be an ordered Banach space with positive cone E_+ . A linear form ϕ on E is called <u>positive</u> if $\langle f, \phi \rangle \geq 0$ for all $f \in E_+$. We denote the dual space of E by E' and by E'_+ the <u>dual cone</u>, i.e. the set of all positive continuous linear forms on E . Then (E', E'_+) is also an ordered Banach space. Note that

(A1) $$E_+ = \{f \in E : \langle f, \phi \rangle \geq 0 \text{ for all } \phi \in E'_+\} .$$

The cone E_+ is called <u>generating</u> if $E_+ - E_+ = E$. If E_+ is generating, then there exists a constant $c > 0$ such that every $f \in E$ can be written as $f = f_1 - f_2$ where $f_1, f_2 \in E_+$ such that $\|f_1\| + \|f_2\| \leq c\|f\|$ [41, Chapter V, 3.5 Corollary] and [8,1.1.2]. As a consequence one obtains that every positive linear form on E is continuous [41,Chapter V, 5.5 Theorem] (see also

[12, Theorem A2]). Using this we obtain the following result on automatic continuity of positive linear mappings (which we could not find in the literature in this generality).

<u>Theorem</u> A1. Let E,F be ordered Banach spaces with positive cones E_+, resp., F_+. Assume that E_+ is generating. Let $T : E \to F$ be a positive linear mapping (i.e. T is linear and satisfies $TE_+ \subset E_+$). Then T is continuous.

<u>Proof</u>. a) If $g \in F$ such that $\langle g,\phi\rangle = 0$ for all $\phi \in F'_+$, then $g = 0$. In fact, since $F_+ \cap (-F_+) = \{0\}$ it follows from [41, Chapter IV, 1.5 Corollary] that $\overline{(F'_+ - F'_+)}^{\sigma(F',F)}$ $= (F_+ \cap (-F_+))^{\circ} = F'$. So the assumption implies that $\langle g,\phi\rangle = 0$ for all $\phi \in F'$. Hence $g = 0$.

b) We show that T has a closed graph (which implies continuity). Let $f_n \to f$ in E and $Tf_n \to g$ in F. We have to show that $Tf = g$. Let $\phi \in F'_+$. Then $\psi := \phi\, T$ is a positive linear form on E. Thus ψ is continuous. Consequently, $\langle g,\phi\rangle = \lim_{n\to\infty} \langle Tf_n,\phi\rangle = \lim_{n\to\infty} \langle f_n,\psi\rangle = \langle f,\psi\rangle = \langle Tf,\phi\rangle$. It follows from a) that $g = Tf$. □

Let E be an ordered Banach space with positive cone E_+. The notion of a <u>half-norm</u> (see also Chapter I) was introduced in [5]; for further information see also [8]. We denote by N the <u>canonical half-norm</u> on E; i.e. $N : E \to \mathbb{R}$ is given by

(A2) $$N(f) = \mathrm{dist}(-f,E_+) = \inf\{\|f+g\| : g \in E_+\}$$
$$= \sup\{\langle f,\phi\rangle : \phi \in E'_+,\ \|\phi\| \le 1\}$$

If E is a Banach lattice then $N(f) = \|f^+\|$.

The canonical half-norm defines a <u>norm</u> $\| \|_N$ on E by

(A3) $\|f\|_N = N(f) + N(-f)$.

The cone E_+ is called <u>normal</u> if there exists an equivalent <u>monotone</u> <u>norm</u> on E (that is, a norm $\| \|_o$ which satisfies $0 \leqq f \leqq g$ implies $\|f\|_o \leqq \|g\|_o$).

(A4) The following assertions are equivalent [41,V.3.5], [8,1.2.1].

(i) E_+ is normal.

(ii) $\| \|_N$ is an equivalent norm.

(iii) $E' = E'_+ - E'_+$.

<u>In Chapter II we assume throughout that the cone E_+ is generating and normal.</u> We now state some consequences of this general assumption.

Choosing a suitable equivalent norm on E we can assume that

(A5) $\pm f \leqq g$ implies $\|f\| \leqq \|g\|$.

[In fact, $\|f\|_o := \max\{N(f),N(-f)\}$ $(f \in E)$ defines a norm doing this. To see this let $\pm f \leqq k$ then $N(f) \leqq \|f + (k-f)\| = \|k\|$ and $N(-f) \leqq \|-f + (f+k)\| = \|k\|$. Hence $\|f\|_o = \max\{N(f),N(-f)\} \leqq \|k\|$. Let now $\pm f \leqq g$. Then $g \geqq 0$. Hence $\|g\|_o = N(g) = \inf\{\|k\| : k \geqq g\}$. Since $\pm f \leqq k$ for all $k \geqq g$, we conclude that $\|f\|_o \leqq \|g\|_o$.]

A set M is called <u>order bounded</u> if there exists $f \in E$ such that $M \subset [-f,f]$ where the <u>order interval</u> is defined by $[-f,f] = \{g \in E : -f \leq g \leq f\}$. Thus by (A5) order bounded sets are norm bounded. Another consequence of the general assumption that E_+ be generating and normal is the following.

(A6) Let $L \subset \mathcal{L}(E)$ be a set of bounded operators. If $\sup \{|\langle Tf,\phi\rangle| : T \in L\} < \infty$ for all $f \in E_+$, $\phi \in E'_+$, then $\sup \{\|T\| : T \in L\} < \infty$.

This follows from the uniform boundedness principle since E_+ and E'_+ are generating.

(A7) Let S,T be linear operators on E and T be positive. If $|\langle Sf,\phi\rangle| \leq \langle Tf,\phi\rangle$ for all $f \in E_+$, $\phi \in E'_+$ .Then $\|S\| \leq \|T\|$.
[In fact, the assumption implies that $\pm Sf \leq Tf$ for all $f \in E_+$. Then by (A5), $\|Sf\| \leq \|Tf\|$.]
Occasionally, the complexification $E_{\mathbb{C}}$ of E is considered without further comments. For example, if A is an operator on E, then for $\lambda \in \rho(A) \setminus \mathbb{R}$,by definition, $R(\lambda,A) := (\lambda-A)^{-1}$ is an operator on $E_{\mathbb{C}}$.

(A8) Similar to (A5), we have for $f \in E_+$, $g \in E_{\mathbb{C}}$, $|\langle g,\phi\rangle| \leq \langle f,\phi\rangle$ for all $\phi \in E'_+$ implies $\|g\| \leq c\|f\|$, where $c > 0$ is a fixed constant.

Note that every Banach lattice and the hermitian part of a C*-algebra have a generating and normal positive cone.

Appendix B The Vector-valued Stieltjes Integral

Here we collect definition and properties of the vector-valued Stieltjes integral. We follow [25,Chapter III] closely, but emphasize the integral of increasing functions.

Let G,H be Banach spaces and E be an ordered Banach space with generating and normal cone.

Definition B1 [25, Definition 3.2.4]. A function $f:[a,b]\to E$ is of bounded variation if $\sup \|\sum_i [f(t_i)-f(s_i)]\| < \infty$ over every choice of a finite number of non-overlapping intervals (s_i,t_i) in $[a,b]$.

Proposition B2.a) Let $f : [a,b] \to E$ be increasing. Then f is of bounded variation.

b) Let $S : [a,b] \to \mathcal{L}(E)$ be increasing. Then S is of bounded variation.

Proof. Let $(s_i,t_i) \subset [a,b]$ $(i=1,\ldots,n)$ be a finite number of non-overlapping intervals. Then $0 \leq \sum_i (f(t_i)-f(s_i)) \leq f(b)-f(a)$. This implies $\|\sum_i (f(t_i)-f(s_i))\| \leq \|f(b)-f(a)\|$. Thus a) holds. Similarly, $0 \leq \sum_i (S(t_i)-S(s_i)) \leq S(a)-S(b)$ and b) follows from (A7).

Let $f : [a,b] \to G$ and $u : [a,b] \to \mathbb{R}$. We denote the subdivision $(t_o=a \leq t_1 \leq t_2 \leq \ldots \leq t_n=b)$ together with points

$s_i \in (t_{i-1}, t_i)$ by π and let $|\pi| = \max_i |t_i - t_{i-1}|$. Let

(B1) $\qquad \sum_\pi (f,u) = \sum_{i=1}^n f(s_i)(u(t_{i+1}) - u(t_i))$.

If $\lim_{|\pi|\to 0} \sum_\pi (f,u)$ exists in a given topology τ , this limit is denoted by the integral

(B2) $\qquad \int_a^b f(t)\,du(t)$

and we say that the integral exists in the topology τ . Let

(B3) $\qquad \sigma_\pi (f,u) = \sum_{i=1}^n u(s_i)(f(t_{i+1}) - f(t_i))$.

If $\lim_{|\pi|\to 0} \sigma_\pi (f,u)$ exists in a given topology τ , then this limit is denoted by the integral

(B4) $\qquad \int_a^b u(t)\,df(t)$,

and we say the integral exists in the topology τ .

Proposition B3 [25 ,Theorem 3.3.1 and Theorem 3.3.2]. Suppose that either (1) $f:[a,b]\to G$ is strongly continuous and $u:[a,b]\to \mathbb{R}$ is of bounded variation or (2) f is of bounded variation and u is continuous. Then the integrals (B2) and (B4) exist in the norm topology and

(B5) $\qquad \int_a^b u(t)\,df(t) = u(t)f(t)\Big|_a^b - \int_a^b f(t)\,du(t)$.

Further, if A is a closed operator from $D(A) \subset G$ into H and if $f(t) \in D(A)$, and if $(Af)(.)$ is strongly continuous in the case (1) or of bounded variation in the case (2) then $\int_a^b u(t)\,df(t) \in D(A)$ and $\int_a^b f(t)\,du(t) \in D(A)$ and

$$\text{(B6)} \qquad A\int_a^b f(t)\,du(t) = \int_a^b Af(t)\,du(t) \qquad \text{and}$$

$$\text{(B7)} \qquad A\int_a^b u(t)\,df(t) = \int_a^b u(t)\,d(Af(t)) \ .$$

Applying Proposition B3 for $G=\mathscr{L}(E)$ we obtain in particular the following. Let $S : [a,b] \to \mathscr{L}(E)$ be increasing and $u \in C[a,b]$. Then the integrals $\int_a^b u(t)\,dS(t)$ and $\int_a^b S(t)\,du(t)$ exist in the <u>operator</u> <u>norm</u> and

$$\text{(B8)} \qquad \int_a^b S(t)\,du(t) = S(t)u(t)\Big|_a^b - \int_a^b u(t)\,dS(t) \ .$$

Choosing $u(t) = t$ one obtains that the Riemann-integral $\int_a^b S(t)\,dt$ exists in the operator norm.

Let $f \in E, \phi \in E'$. Then (B7) applied twice (for $A: \mathscr{L}(E) \to E$ given by $T \to Tf$ and then $A=\phi$) gives $\langle (\int_a^b u(t)\ dS(t))f, \phi\rangle = \int_a^b u(t)\ d\langle S(t)f,\phi\rangle$. This and the Hahn-Banach theorem allow us to carry over the rules for the classical Riemann-Stieltjes integral to the vector-valued case. For example, the following corresponds to [50,Chapter I, Theorem 6b].

<u>Proposition</u> B4. Let $S : [a,b] \to \mathcal{L}(E)$ be increasing and $v \in C[a,b]$ such that $v(t) \geqq 0$ for all $t \in [a,b]$. Then $S_1(t) = \int_0^t v(s)\, dS(t)$ is increasing and

$$\text{(B9)} \qquad \int_a^b u(t)v(t)\, dS(t) = \int_a^b u(t)\, dS_1(t)$$

for all $u \in C[a,b]$.

<u>Appendix</u> C <u>The vector-valued Laplace-Stieltjes transform</u>

Let E be an ordered Banach space with generating and normal cone E_+. In the following we discuss properties of the Laplace-Stieltjes transform with values in E or $\mathcal{L}(E)$.

We need the notion of the <u>improper</u> Riemann-Stieltjes integral. Let G be a Banach space, $a \in \mathbb{R}$ and $f : [a,\infty) \to G$, $u : [a,\infty) \to \mathbb{R}$. Assume that either (1) f is strongly continuous and u is of bounded variation or (2) f is of bounded variation and u is continuous. Then by Propositon B3 the integral $\int_a^b u(t)\, df(t)$ exists in the norm for all $b \geqq a$.
Let τ be a topology on G. We say that the integral $\int_a^\infty u(t)\, df(t)$ converges in the topology τ if $\lim_{b\to\infty} \int_a^b u(t)\, df(t)$ exists for the topology τ. In that case we define $\int_a^\infty u(t)\, df(t) := \lim_{b\to\infty} \int_a^b u(t)\, df(t)$. The definition is analoguous for $\int_a^\infty f(t)\, du(t)$.

The following proposition shows that for the Laplace-Stieltjes strong and weak convergence are essentially equivalent.

Proposition C1. Let $\alpha : [0,\infty) \to E_+$ be an increasing function such that $\alpha(0) = 0$ and let $w \in \mathbb{R}$. Consider the following assertions.

(i) $\int_0^\infty e^{-wt} d\langle\alpha(t),\phi\rangle$ converges for all $\phi \in E'_+$.

(ii) For all $\lambda > w$ there exists $M \geqq 0$ such that $\|\alpha(t)\| \leqq Me^{\lambda t}$ for all $t \geqq 0$.

(iii) $\int_0^\infty e^{-\lambda t} d\alpha(t)$ converges in the norm whenever $\mathrm{Re}\lambda > w$.

Then (i) implies (iii). Moreover, if $w \geqq 0$, then (i) implies (ii), and (ii) implies (iii). Finally, if (i) holds, then

(C1) $$\int_0^\infty e^{-\lambda t} d\alpha(t) = \int_0^\infty \lambda e^{-\lambda t} \alpha(t)\,dt \qquad (\mathrm{Re}\lambda > \max\{0,w\}) .$$

Proof. a) We consider the case when $w \geqq 0$. Assume (i). We show (ii). Let $\lambda > w$. Then for $\phi \in E'_+$, $t \geqq 0$ one has:
$0 \leqq e^{-\lambda t}\langle\alpha(t),\phi\rangle \leqq e^{-\lambda t}\langle\alpha(t),\phi\rangle + \lambda\int_0^t e^{-\lambda s}\langle\alpha(s),\phi\rangle ds =$
$\int_0^t e^{-\lambda s}\, d\langle\alpha(s),\phi\rangle \leqq \int_0^\infty e^{-s\lambda}\, d\langle\alpha(s),\phi\rangle$. Hence
$\sup_{t\geqq 0} e^{-\lambda t}\langle\alpha(t),\phi\rangle < \infty$. Since $E'_+ - E'_+ = E'$, it follows that $(e^{-\lambda t}\alpha(t))_{t\geqq 0}$ is weakly bounded, hence norm bounded. This proves (ii). Now assume (ii). Let $\mathrm{Re}\lambda > w$. We first consider the case when λ is real. Choose $u \in (w,\lambda)$. Then there exists $M \geqq 0$ such that $\|\alpha(t)\| \leqq Me^{ut}$ $(t \geqq 0)$. Let $r \geqq t \geqq 0$. Then $\|\int_t^r e^{-\lambda s} d\alpha(s)\| =$
$\|e^{-\lambda r}\alpha(r) - e^{-\lambda t}\alpha(t) + \lambda\int_t^r e^{-\lambda s}\alpha(s)ds\| \leqq$
$M\, e^{-(\lambda-u)r} + M\, e^{-(\lambda-u)t} + \lambda M \frac{1}{\lambda-u} (e^{-(\lambda-u)t} - e^{-(\lambda-u)r}) \leqq$
$2Me^{-(\lambda-u)t} + M \frac{\lambda}{\lambda-u} e^{-(\lambda-u)t} \to 0$ for $t \to \infty$. Hence
$\lim_{t\to\infty} \int_0^t e^{-\lambda s} d\alpha(s)$ exists in the norm. This proves (iii) in the

case when λ is real. In the case when λ is arbitrary let $r \geq t \geq o$. Then $|\int_t^r e^{-\lambda s} d\alpha(s)| \leq \int_t^r e^{-Re\lambda s} d\alpha(s)$. So (iii) follows from the real case by (A8).

b) We consider the case when w is arbitrary. Assume that (i) holds. We show (iii). Let $\beta(t) = \int_o^t e^{-ws} d\alpha(s)$. Then by hypothesis, the integral $\int_o^\infty d<\beta(t),\phi> = \int_o^\infty e^{-ws}\, d<\alpha(s),\phi>$ converges for every $\phi \in E_+$. Hence by a) (for $w=0$), $\int_o^\infty e^{-\lambda t} d\beta(t)$ converges in the norm for all $\lambda > 0$. Using Proposition B4 we obtain that for $\mu > w$, the integral $\int_o^\infty e^{-\mu t} d\alpha(t) = \int_o^\infty e^{-(\mu-w)t} e^{-wt} d\alpha(t) = \int_o^\infty e^{-(\mu-w)t} d\beta(t)$ converges in the norm.

c) Finally, we prove the last assertion. Assume that (i) holds. Then by b) also (ii) and (iii) are satisfied for $Re\lambda > \max\{0,w\}$. Hence, $\lim_{t\to\infty} e^{-\lambda t}\alpha(t) = 0$. Thus by Proposition B4, $\int_o^\infty e^{-\lambda t} d\alpha(t) = \lim_{b\to\infty} \int_o^b e^{-\lambda t} d\alpha(t) = \lim_{b\to\infty} e^{-\lambda b}\alpha(b) + \lim_{b\to\infty} \int_o^b \lambda e^{-\lambda t}\alpha(t)dt = \int_o^\infty \lambda e^{-\lambda t}\alpha(t)dt$. □

Remark C2. Assume that in Proposition C1 assertion (i) holds. Then $g(\lambda) = \int_o^\infty e^{-\lambda t} d\alpha(t)$ defines an analytic function on $H := \{\lambda \in \mathbb{C} : Re\lambda > w\}$. In fact, it follows from the corresponding classical result that g is weakly analytic, hence g is strongly analytic.

To Proposition C1 corresponds a result for operator-valued functions. The proof is the same apart from minor modifications.

Proposition C3. Let $S : [0,\infty) \to \mathcal{L}(E)$ be increasing satisfying $S(0) = 0$ and let $w \in \mathbb{R}$. Consider the following assertions.

(i) $\int_0^\infty e^{-wt}\, d\langle S(t)f,\phi\rangle$ converges for every $f \in E_+$, $\phi \in E'_+$ and $\lambda > w$.

(ii) For every $\lambda > w$ there exists $M \geq 0$ such that $\|S(t)\| \leq Me^{\lambda t}$ for all $t \geq 0$.

(iii) $\int_0^\infty e^{-\lambda t}\, dS(t)$ converges in the operator norm for $\mathrm{Re}\lambda > w$.

Then (i) implies (iii). Moreover, if $w \geq 0$, then (i) implies (ii), and (ii) implies (iii). Finally, if (i) holds, then

$$\int_0^\infty e^{-\lambda t} dS(t) = \int_0^\infty \lambda e^{-\lambda t} dS(t) \qquad (\mathrm{Re}\lambda > \max\{0,w\}) \ . \tag{C2}$$

Next we reformulate the uniqueness theorem in the vector-valued case.

Definition C4. a) Let $\alpha : [0,\infty) \to \mathbb{R}$ be increasing. Then α is said to be normalized if $\alpha(0)=0$ and for every $t>0$

$$\alpha(t) = \frac{1}{2}(\alpha(t+)+\alpha(t-)) \ .$$

b) An increasing function $\alpha : [0,\infty) \to E$ is normalized if for every $\phi \in E'_+$ the numerical function $t \to \langle\alpha(t),\phi\rangle$ is normalized.

c) An increasing function $S : [0,\infty) \to \mathcal{L}(E)$ is normalized if for every $f \in E_+$ the function $t \to S(t)f$ is normalized in the sense of b).

Now the classical uniqueness theorem [51, 7.2] gives the following.

Theorem C5 (uniqueness theorem). Let $\alpha : [0,\infty) \to F$ (where $F = \mathbb{R}$, E or $\mathcal{L}(E)$) be an increasing normalized function. Let $\lambda_o \in \mathbb{R}$. If

$$\int_o^\infty e^{-\lambda t} d\alpha(t) = 0 \qquad \text{for all} \quad \lambda > \lambda_o$$

then $\alpha(t) = 0$ for all $t \geqq 0$.

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