Entropy methods for diffusive PDEs

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1 Introduction

The term *entropy*, which will be (as the Reader may expect) the leading concept, the key idea of the course, was born in 1865 by the mind of Rudolf Clausius, who used this word to denote the amount of energy which is no longer usable for physical work (e.g. heat produced by friction). In 1877, Ludwig Boltzmann suggested a statistical interpretation of the concept of entropy, stating the the entropy S of an ideal gas is proportional to the logarithm of the number of microstates W corresponding to the gas macrostate, i.e. $S = k_B \log W$ (carved on his gravestone in Vienna's Zentralfriedhof). Here $k_B \approx 1.38064852 \times 10^{-23}\, m^2\, kg\, s^{-2}\, K^{-1}$ is the well-known Boltzmann constant, a recurring object in statistical mechanics. Willard Gibbs gave a similar definition of entropy: $S = -k_B \sum_i p_i \log p_i$, where p_i is the probability of the microstate i, and the summation is on all the microstates associated to the macrostate.

One of the most celebrated results by Boltzmann is the H-theorem, which states that the H-function:

$$H[f] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, v, t) \log f(x, v, t) dx dv. \tag{1}$$

is nonincreasing in time along the solutions f os the Boltzmann equation. However, since the entropy S of the system is proportional to $-H[f]$, this implies that S is nondecreasing in time, which can be seen as a formulation of the second law of thermodynamics (the physical entropy of a closed system cannot decrease in time). We will see that the H-theorem is a kind of prototype for the entropy methods for PDEs.

Before we go on, let us introduce the mathematical convention of "putting a minus in front of the entropy": the mathematical entropy equals *minus* the physical entropy. For us, an entropy will be, tipically, a convex Lyapounov functional for some PDE, that is, a functional which will be nonincreasing along the solutions of some partial differential equation (or system of them).

The results hereby presented can be found in [32].

1.1 A few applications of the concept of entropy

There are many examples which display the usefulness of the concept of entropy in mathematics. We discuss here some of them.

Hyperbolic conservation laws. When dealing with systems of hyperbolic conservation laws, i.e. PDEs like

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ is the flux, the fact that the equation has no strong solution, while weak solutions are not unique, can be really annoying. Luckily, the entropy lends us a hand in such a difficult predicament. An *entropy solution* of (2) is a weak solution $u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^n$ of (2) such that for all convex functions $h : \mathbb{R}^n \to \mathbb{R}$ a function $q : \mathbb{R}^n \to \mathbb{R}^d$ exists such that

$$\partial_t h(u) + \operatorname{div} q(u) \le 0$$

in some distributional sense. The function h is called *entropy density*, while q is the *entropy flux*. Moreover, the entropy $H(u) = \int_{\mathbb{R}^d} h(u)dx$ is nonincreasing in time. It turns out that the entropy solutions are unique.

Kinetic theory. Take the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \qquad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \tag{3}$$

where $f = f(x, v, t)$ is the system distribution function and $Q(f, f)$ is the (binary) collision operator. Recall the definition (1) of H, which we will call simply "entropy". The entropy production $P[f]$ is defined as $P[f] := -\frac{d}{dt}H[f]$. The properties of Q ensure that $P[f] \geq 0$, and that $P[f] = 0$ if and only if $f = M$, where

$$M(v) = \rho \left(\frac{m}{2\pi k_B T} \right)^{d/2} e^{-m|v-u|^2/2k_B T}$$

is the so-called Maxwellian, ρ is the particle density, u is the mean velocity, and T is the temperature. As a consequence $\frac{d}{dt}H[f] \leq 0$, i.e. $H[f]$ is nonincreasing in time.

You, curious Reader, may ask: *is it true what they say, that $f(t) \to M$ as $t \to \infty$? If so, how fast?* The usual idea is to consider the relative entropy $H[f|M] := H[f] - H[M] \geq 0$ and see if it can be dominated by the entropy production, that is

$$\Phi(H[f|M]) \leq P[f] \quad \text{for some increasing } \Phi : \mathbb{R} \to \mathbb{R} \text{ such that } \Phi(0) = 0. \tag{4}$$

Replacing $P[f]$ inside (4) with its definition leads to

$$\frac{d}{dt}H[f] + \Phi(H[f] - H[M]) \leq 0 \qquad t > 0.$$

Let us assume that $H[f(t)] > H[M]$ (otherwise $f = M$, since M can be shown to be a strict minimum point). Therefore

$$\int_{H[f(t)]-H[M]}^{H[f_0]-H[M]} \frac{ds}{\Phi(s)} \geq t \qquad t > 0.$$

The integral on the left-hand side of the above inequality must explode as $t \to \infty$. Since the only singularity of $1/\Phi(s)$ is at $s = 0$, we conclude that $H[f] \to H[M]$, and therefore $f \to M$ since $H[M] = \min_f H[f]$. If $\Phi(s) = \lambda s$, then we deduce that $H[f(t)] - H[M] \leq e^{-\lambda t}(H[f_0] - H[M])$, $t > 0$.

Bakry-Emery technique. An idea developed first by Barky and Emery in 1985 consists in estimating the second time derivative of the entropy by means of the first time derivative. Assume we can prove

$$\frac{d^2}{dt^2}H[f] \geq -\kappa \frac{d}{dt}H[f] \qquad t > 0,$$

for some constant $\kappa > 0$. Assume moreover that both $\frac{d}{dt}H[f]$, $H[f]$ tend to zero as $t \to \infty$. Integrating the above inequality in the time interval (t, ∞) leads to

$$-\frac{d}{dt}H[f] \geq \kappa H[f], \qquad t > 0.$$

Again, Gronwall's Lemma implies that $H[f(t)] \leq e^{-\kappa t}H[f_0]$.

Cross-diffusion systems. A system of PDEs of the form

$$\partial_t u = \operatorname{div}(A(u)\nabla u) \qquad x \in \Omega \subset \mathbb{R}^d, \quad t > 0, \tag{5}$$

where $u : \Omega \times (0, \infty) \to \mathbb{R}^n$ is the vector of the unknowns and $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the diffusion matrix, is called a *cross-diffusion system.* In applications u models chemical concentrations or population densities, and the matrix A is often not symmetric nor positive (semi)definite, making any analytical study of (5) tricky. For example, tools like maximum principles or parabolic regularity theory cannot be applied. However, not everything is lost, as sometimes cross-diffusion systems have an entropy, i.e. a Lyapounov functional. This fact provides us with useful a-priori estimates, while it also allows us to make a change of variables which yields a (often symmetric) positive definite diffusion matrix and makes it possible to prove nonnegativity and even uniform boundedness of the original variables u without exploiting any maximum principle. These new variables, which we will call *entropy variables*, are simply what is thermodynamics is termed *chemical potentials.*

1.2 Some ideas involving entropy

We sketch here a few mathematical uses of the entropy.

Long-time behaviour. Let us consider, as a toy problem, the heat equation on the $d-$dimensional torus $\mathbb{T}^d = (0, 1)^d$:

$$\partial_t u = \Delta u \quad x \in \mathbb{T}^d, \quad t > 0, \qquad u(x, 0) = u_0(x) \quad x \in \mathbb{T}^d, \tag{6}$$

where $u_0 \in L^1(\mathbb{T}^d)$, $u_0 \geq 0$ in $\mathbb{T}^d$ is the initial datum. It is well known that (6) has a unique smooth solution $u = u(x, t)$ having the same mass as u_0, i.e. $\int_{\mathbb{T}^d} u(x, t)dx = \int_{\mathbb{T}^d} u_0(x)dx$, $t > 0$. Let $u_\infty = \int_{\mathbb{T}^d} u_0(x)dx$ be this mass, which coincides with the average of u_0 since $\mathbb{T}^d$ has measure 1. We call u_∞ the *steady state* of the system. We want to show that $u(t) \to u_\infty$ as $t \to \infty$ in some sense. How do we proceed?

We can, for example, define the convex, nonnegative functional

$$H_2[u] = \int_{\mathbb{T}^d} (u - u_\infty)^2 dx. \tag{7}$$

Clearly $H_2[u]$ is just the square L^2 norm of the difference between u and the steady state u_∞. If we can find an upper bound for $H_2[u(t)]$, where $u(t)$ is the solution of (7), and such upper bound tends to 0 as $t \to \infty$, then we have achieved our goal.

In order to find this upper bound for $H_2[u(t)]$, we ask ourselves: *How does H_2 vary along the solutions of* (6)*?* The answer is easily found by computing the time derivative of $H_2[u(t)]$:

$$\frac{d}{dt}H_2[u(t)] = 2\int_{\mathbb{T}^d}(u(t)-u_\infty)\Delta u(t)dx = -2\int_{\mathbb{T}^d}|\nabla u(t)|^2dx \leq 0, \tag{8}$$

where, in order to obtain the second equality, we integrated by parts and exploited the periodic boundary conditions. Now we should find an upper bound for the right-hand side of (8). Poincaré-Wirtingen inequality provides us with the following result:

$$\int_{\mathbb{T}^d}(u(t)-u_\infty)^2dx \leq C_P\int_{\mathbb{T}^d}|\nabla u(t)|^2dx. \tag{9}$$

Now we just have to put (8), (9) together to obtain

$$\frac{d}{dt}H_2[u(t)] \leq -\frac{2}{C_P}H_2[u(t)] \qquad t>0. \tag{10}$$

The Gronwall Lemma implies

$$\|u(t)-u_\infty\|^2_{L^2(\mathbb{T}^d)} = H_2[u(t)] \leq H_2[u_0]e^{-2t/C_P} \qquad t>0. \tag{11}$$

Therefore, we proved that $u(t)\to u_\infty$ in $L^2(\mathbb{T}^d)$ as $t\to\infty$ exponentially with a rate equal to $1/C_P$.

What about other metrics? Can we prove, for example, a convergence result in $L^1(\mathbb{T}^d)$ with a different (possibly bigger) rate? Well, of course we can.

Let us consider the Boltzmann entropy:

$$H_1[u] = \int_{\mathbb{T}^d}u\log\frac{u}{u_\infty}dx = \int_{\mathbb{T}^d}\left(u\log\frac{u}{u_\infty}-u+u_\infty\right)dx. \tag{12}$$

Since $u\log\frac{u}{u_\infty}-u+u_\infty = h(u)-h(u_\infty)-h'(u_\infty)(u-u_\infty)$ with $h(u)=u\log u-u$ and h is convex in $(0,\infty)$, we deduce that $u\log\frac{u}{u_\infty}-u+u_\infty\geq 0$ and therefore $H_1[u]\geq 0$. We take the time derivative of $H_1[u(t)]$ and get

$$\frac{d}{dt}H_1[u(t)] = \int_{\mathbb{T}^d}\log\frac{u(t)}{u_\infty}\Delta u(t)\,dx = -4\int_{\mathbb{T}^d}|\nabla\sqrt{u(t)}|^2dx \leq 0.$$

Now we need to bound the left-hand side of the above inequality by something proportional to $-H_1[u(t)]$. This time we exploit the logarithmic Sobolev inequality:

$$\int_{\mathbb{T}^d}u(t)\log\frac{u(t)}{u_\infty}dx \leq C_L\int_{\mathbb{T}^d}|\nabla\sqrt{u(t)}|^2dx. \tag{13}$$

So we deduce

$$\frac{d}{dt}H_1[u(t)] \leq -\frac{4}{C_L}H_1[u(t)] \qquad t>0,$$

which, thannk to Gronwall's Lemma, implies

$$H_1[u(t)] \leq e^{-4t/C_L} H_1[u_0] \qquad t > 0.$$

To get a convergence rate in $L^1(\mathbb{T}^d)$ we can apply Csiszár-Kullback inequality:

$$\|u(t) - u_\infty\|_{L^1(\Omega)} \leq u_\infty \sqrt{2H_1[u(t)]},$$

and conclude

$$\|u(t) - u_\infty\|_{L^1(\Omega)} \leq Ce^{-2t/C_L} \qquad t > 0,$$

where $C > 0$ is a suitable constant, depending on $H_1[u_0]$ and u_∞. Therefore, we proved that $u(t) \to u_\infty$ in $L^1(\mathbb{T}^d)$ as $t \to \infty$ exponentially with a rate equal to $2/C_L$.

What about nonlinear equations? The technique works in this context, too. Consider for example the DLSS (Derrida-Lebowitz-Speer-Spohn) equation, modeling quantum electron transport in a semiconductor under suitable assumptions:

$$\partial_t u = -\text{div}\left(u\nabla \frac{\Delta\sqrt{u}}{\sqrt{u}}\right) \quad t > 0, \qquad u(0) = u_0 \geq 0 \quad \text{in } \mathbb{T}^d. \tag{14}$$

Differentiate $H_1[u]$ in time along a (nonnegative) solution $u(t)$ of (14):

$$\begin{aligned}\frac{d}{dt}H_1[u(t)] &= -\int_{\mathbb{T}^d} \log u \,\text{div}\left(u\nabla \frac{\Delta\sqrt{u}}{\sqrt{u}}\right) dx \\ &= \int_{\mathbb{T}^d} \nabla u \cdot \nabla \frac{\Delta\sqrt{u}}{\sqrt{u}}\, dx = -\int_{\mathbb{T}^d} \Delta u \frac{\Delta\sqrt{u}}{\sqrt{u}}\, dx.\end{aligned}$$

It is possible to prove that

$$\int_{\mathbb{T}^d} \Delta u \frac{\Delta\sqrt{u}}{\sqrt{u}}\, dx \geq \kappa \int_{\mathbb{T}^d} (\Delta\sqrt{u})^2 dx,$$

where $\kappa = \frac{4d-1}{d(d+2)}$. At this point we employ the higher-order log-Sobolev inequality

$$\int_{\mathbb{T}^d} u \log \frac{u}{u_\infty} dx \leq \frac{1}{8\pi^4} \int_{\mathbb{T}^d} (\Delta\sqrt{u})^2 dx.$$

Let us put everything together to get

$$\frac{d}{dt}H_1[u(t)] \leq -4\pi^4\kappa H_1[u(t)] \qquad t > 0,$$

which, by Gronwall's lemma, implies that $H_1[u(t)] \to 0$ exponentially as $t \to \infty$ with rate $4\pi^4\kappa$.

At this point we can summarize the above ideas into a general strategy to prove convergence of solutions to PDEs towards a steady state. Let us imagine we have an evolution equation with the form

$$\partial_t u + A(u(t)) = 0 \quad t > 0, \qquad u(0) = u_0,$$

where $u : (0, \infty) \to B$, B is some Banach space with dual B^*, and $A : B \to B^*$ is some (nonlinear) mapping. Furthermore imagine that we are given some (relative) entropy functional $H = H[u]$ and a steady state u_∞ (i.e. a solution of $A(u) = 0$). What we have to do is to compute the entropy production, i.e. (minus) the time derivative of $H[u]$ along the solutions on the evolution equation, and then find a (possibly linear) relation between the entropy production and the entropy itself. Finally, Gronwall's lemma will allow us to conclude that $H[u(t)] \to 0$ with some rate (exponential, algebraic...) as $t \to \infty$.

Global existence and boundedness of weak solutions. Entropy methods can also be employed to prove existence of (positive, uniformely bounded) weak solutions to systems of reaction-diffusion and cross-diffusion PDEs. For example, consider (5) with $n = 2$ $A(u)$ given by

$$A(u) = \frac{1}{2 + 4u_1 + u_2} \begin{pmatrix} 1 + 2u_1 & u_1 \\ 2u_2 & 2 + u_2 \end{pmatrix}.$$

The considered equations are a special case of a Maxwell-Stefan system and describe a fluid mixture of 3 components with equal molar masses under isobaric, isothermal conditions; u_i is the mass fraction of the component i, for $i = 1, 2, 3$. The mass fractions must be nonnegative, i.e. the constraints $u_1 \geq 0$, $u_2 \geq 0$, $u_1 + u_2 \leq 1$ must be satisfied. One can prove the nonnegativity of u_1, u_2 with a minimum principle, but no maximum principle is available which allows for the proof of the uniform boundedness of $u_1 + u_2$.

But we need no maximum principle. First we have to derive an entropy balance inequality for the entropy

$$H[u] = \int_{\mathbb{R}^d} h(u)dx, \qquad h(u) = \sum_{i=1}^{3}(u_i \log u_i - u_i).$$

One can show (the main obstacle is to find a uniform lower bound for some $u-$dependent quadratic form) that

$$\frac{d}{dt}H[u] \leq -2\int_{\mathbb{R}^d}(|\nabla\sqrt{u_1}|^2 + |\nabla\sqrt{u_2}|^2)dx \leq 0.$$

So we have a nice a-priori estimate which would prove itself to be quite useful in an analytical study of the system. However, having an entropy also means that we can define new variables, called *entropy variables*:

$$w_i = \frac{\partial h}{\partial u_i} = \log\frac{u_i}{u_3} \qquad i = 1, 2.$$

What's the use of these new variables? First, (5) can be rewritten as

$$\partial_t u = \operatorname{div}(B(w)\nabla w), \qquad B(w) \equiv A(u(w))(h''(u(w)))^{-1}.$$

It turns out that B *is symmetric and positive definite.* This, of course, would prove invaluable in the analysis.

Second, let us have a look at $u(w)$:

$$u_i(w) = \frac{e^{w_i}}{1 + e^{w_i} + e^{w_2}}, \qquad i = 1, 2.$$

Clearly $u_i \geq 0$ for $i = 1, 2, 3$, that is, we would have positivity and uniform boundedness for the physical variables *without using any maximum (or minimum) principle.*

Therefore, if we can turn these a-priori estimates and smart ideas into a rigorous proof, we can have *global existence of nonnegative, uniformely bounded weak solutions for a system of nonlinear PDEs with a diffusion matrix which is neither symmetric nor positive semidefinite.* That doesn't sound too bad, does it?

Uniqueness of *weak* solutions. Let us spend a few words about how entropy can help in showing uniqueness of *weak* solutions to PDEs. We can e.g. consider the drift-diffusion model in the $d-$dimensional torus

$$\partial_t u = \operatorname{div}(\nabla u + u \nabla V) \quad x \in \mathbb{T}^d, \quad t > 0, \qquad u(x, 0) = u_0(x) \quad x \in \mathbb{T}^d. \tag{15}$$

The model describes the (semiclassical) transport of electrons in a semiconductor under certain simplifying assumptions. Here $u \geq 0$ is the electron density and V is a given electric potential.

We want to prove that (15) has at most one solution. The most natural thing to do would be to test (15) against $u_1 - u_2$ (the difference of two solutions with the same initial datum) and try to control the drift term by means of the diffusion term (plus some Sobolev embedding). However, this strategy works only as long as the potential is smooth enough, e,g. $\nabla V \in L^p(\mathbb{T}^d)$ with $p > 2$ big enough. But what if this is not true?

Actually, we can achieve our goal by means of an entropy-based idea. Let us define

$$F[u_1, u_2] = H[u_1] + H[u_2] - 2H\left[\frac{u_1 + u_2}{2}\right], \qquad H[u] = \int_{\mathbb{T}^d} (u \log u - u)\, dx.$$

Taking the time derivative of $F[u_1, u_2]$ leads to

$$\frac{d}{dt} F[u_1, u_2] = -4 \int_{\mathbb{T}^d} \left(|\nabla \sqrt{u_1}|^2 + |\nabla \sqrt{u_2}|^2 - |\sqrt{u_1 + u_2}|^2 \right) dx. \tag{16}$$

However, it is possible to prove that the so-called *Fisher information* $\mathcal{F}[u] \equiv \int_{\mathbb{T}^d} |\nabla \sqrt{u}|^2 dx$ is a convex functional, ands therefore the right-hand side of (16) is nonpositive. This means that, for $t > 0$, $F[u_1(t), u_2(t)] \leq F[u_1(0), u_2(0)] = 0$ since $u_1 = u_2$ at initial time. However, the strict convexity of H also implies that $F[u_1, u_2] \geq 0$ and the equality holds if and only if $u_1 = u_2$. Therefore $u_1(t) = u_2(t)$ for $t > 0$.

2 The Bakry-Emery approach

The method that we are going to present in this section was first developed by D. Bakry and M. Emery in the 1980s [5] and consists in computing the second derivative of the entropy (with respect to time) and estimating it by means of the first derivative of the entropy, that is, the entropy production.

2.1 The linear Fokker-Planck equation

The Barky-Emery method is usually explained by applying it to the linear Fokker-Planck equation:

$$u_t = \mathrm{div}\,(\nabla u + u\nabla V), \quad t > 0, \quad x \in \mathbb{R}^d, \tag{17}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^d, \tag{18}$$

which arises from many applications, e.g. semiconductor transport, plasma physics, stellar dynamics. The function V depends only on x and represents a potential, while the unknown function $u = u(x,t) \geq 0$ is a density.

Existence results for (17), (18) are available in literature; see e.g. [45]. For the purpose of these lecture notes, we will assume that the solution exists and is sufficiently smooth to justify the computations that will be carried out.

We assume that the initial datum is nonnegative and has mass equal to 1: $\int_{\mathbb{R}^d} u_0 dx = 1$. Since (17) is in divergence form the mass is conserved:

$$\int_{\mathbb{R}^d} u(x,t)dx = \int_{\mathbb{R}^d} u_0(x)dx = 1 \qquad \text{for } t > 0.$$

Concerning the potential V, we assume that V is smooth enough and[1]

$$e^{-V} \in L^1(\mathbb{R}^d), \quad \exists \lambda > 0 : \sum_{i,j=1}^{d} \partial_{x_i x_j} V(x) w_i w_j \geq \lambda |w|^2 \quad x \in \mathbb{R}^d, \quad w \in \mathbb{R}^d. \tag{19}$$

The steady state u_∞ of (17) is defined as the only positive constant-in-time solution $u = u_\infty$ of $\nabla u + u\nabla V = 0$ (in $\mathbb{R}^d$) such that $\int_{\mathbb{R}^d} u_\infty dx = \int_{\mathbb{R}^d} u_0 dx$. Since $\nabla u + u\nabla V = u\nabla(\log u + V)$ and we are looking for positive solutions, this implies that $\log u_\infty + V$ must be constant in $\mathbb{R}^d$, and therefore

$$u_\infty(x) = \frac{e^{-V(x)}}{\int_{\mathbb{R}^d} e^{-V(y)} dy} \qquad x \in \mathbb{R}^d. \tag{20}$$

Let now $\phi : (0,\infty) \to [0,\infty)$ a smooth function such that

$$\phi(1) = \phi'(1) = 0, \qquad \phi''(1) = 1, \qquad \phi'' > 0, \quad (1/\phi'')'' \leq 0 \quad \text{in } [0,\infty). \tag{21}$$

[1] Constraint $(19)_2$ on the Hessian of V is called *Bakry-Emery condition*.

Examples of functions ϕ satisfying (21) are

$$\phi(s) = s\log s - s + 1, \qquad \phi(s) = \frac{s^\alpha - 1}{\alpha(\alpha-1)} \qquad \alpha \in (1,2].$$

The function ϕ generates the following relative entropy

$$H_\phi[u] = \int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty dx. \tag{22}$$

We are going to prove the following

Theorem 2.1. *Assume* (19)–(21) *hold, and let* $H[u_0] < \infty$. *Then any smooth solution* $u : \mathbb{R}^d \times (0,\infty) \to [0,\infty)$ *to* (17), (18) *converges exponentially to the steady state* u_∞, *in the sense that*

$$\|u(t) - u_\infty\|_{L^1(\mathbb{R}^d)} \le e^{-\lambda t}\sqrt{2H_\phi[u_0]} \qquad t > 0.$$

Proof. We begin by differentiating $H_\phi[u]$ with respect to t. Let $\rho = u/u_\infty$, so that (17) takes the form

$$u_t = \operatorname{div}(u_\infty \nabla\rho) = u_\infty\Delta\rho + \nabla u_\infty \cdot \nabla\rho. \tag{23}$$

The entropy production can be computed through an integration by parts as follows:

$$P_\phi[u] = -\frac{d}{dt}H_\phi = -\int_{\mathbb{R}^d}\phi'(\rho)u_t dx = \int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2 u_\infty dx \ge 0. \tag{24}$$

The nonnegativity of P results from the convexity of ϕ.

It is now time to compute the second time derivative of the entropy:

$$\frac{d^2}{dt^2}H_\phi = -\int_{\mathbb{R}^d}(\phi'''(\rho)|\nabla\rho|^2\partial_t u + 2\phi''(\rho)\nabla\rho\cdot\nabla(\partial_t\rho)\,u_\infty)dx = I_1 + I_2, \tag{25}$$

$$I_1 \equiv -\int_{\mathbb{R}^d}\phi'''(\rho)|\nabla\rho|^2\partial_t u\,dx, \qquad I_2 \equiv -2\int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho\cdot\nabla(\partial_t\rho)\,u_\infty\,dx. \tag{26}$$

In the following we will denote with $\nabla^2 f$ the Hessian with respect to x of a scalar function $f = f(x,\ldots)$. Moreover, for any matrix $A \in \mathbb{R}^{d\times d}$, we define $|A|^2 \equiv \operatorname{tr}(A^\top A) = \sum_{i,j=1}^d A_{ij}^2$.

Let us compute I_1 by using (23) and integrating by parts:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d}\nabla(\phi'''(\rho)|\nabla\rho|^2)\cdot\nabla\rho\,u_\infty dx \\ &= \int_{\mathbb{R}^d}(\phi''''(\rho)|\nabla\rho|^4 + 2\phi'''(\rho)\nabla\rho\cdot(\nabla^2\rho)\nabla\rho)u_\infty dx. \end{aligned}$$

Before computing I_2, let us first deal with the term $\nabla\rho\cdot\nabla(\partial_t\rho)$:

$$\partial_t\rho = \frac{1}{u_\infty}\operatorname{div}(u_\infty\nabla\rho) = \Delta\rho + \nabla\rho\cdot\nabla\log u_\infty = \Delta\rho - \nabla\rho\cdot\nabla V,$$

$$\nabla\rho\cdot\nabla(\partial_t\rho)=\nabla\rho\cdot\nabla\Delta\rho-\nabla\rho\cdot(\nabla^2\rho)\nabla V-\nabla\rho\cdot(\nabla^2 V)\nabla\rho,$$

and by writing $\nabla\rho\cdot\nabla\Delta\rho=\mathrm{div}\,((\nabla^2\rho)\nabla\rho)-|\nabla\rho|^2$ we get

$$\nabla\rho\cdot\nabla(\partial_t\rho)=\mathrm{div}\,((\nabla^2\rho)\nabla\rho)-|\nabla\rho|^2-\nabla\rho\cdot(\nabla^2\rho)\nabla V-\nabla\rho\cdot(\nabla^2 V)\nabla\rho.$$

Therefore I_2 becomes

$$I_2=-2\int_{\mathbb{R}^d}\phi''(\rho)\mathrm{div}\,((\nabla^2\rho)\nabla\rho)u_\infty dx+2\int_{\mathbb{R}^d}\phi''(\rho)(|\nabla\rho|^2+\nabla\rho\cdot(\nabla^2\rho)\nabla V)u_\infty dx \qquad (27)$$
$$+2\int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho\cdot(\nabla^2 V)\nabla\rho\,u_\infty dx.$$

We integrate by parts the first integral on the right-hand side of (27), while using (19) to estimate the third integral:

$$\begin{aligned}I_2\geq\,&2\int_{\mathbb{R}^d}\phi'''(\rho)\nabla\rho\cdot(\nabla^2\rho)\nabla\rho)u_\infty dx\\&+2\int_{\mathbb{R}^d}\phi''(\rho)(u_\infty|\nabla\rho|^2+\nabla\rho\cdot(\nabla^2\rho)(\nabla u_\infty+u_\infty\nabla V))dx\\&+2\lambda\int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2u_\infty dx.\end{aligned}$$

Since u_∞ is the steady state, then $\nabla u_\infty+u_\infty\nabla V=0$; moreover the last integral on the right-hand side of the above equation equals the entropy production. Therefore

$$I_2\geq 2\int_{\mathbb{R}^d}\phi'''(\rho)\nabla\rho\cdot(\nabla^2\rho)\nabla\rho)u_\infty dx+2\int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2u_\infty\,dx+2\lambda P_\phi[u].$$

Summing I_1, I_2 and applying (25) leads to

$$-\frac{d}{dt}P_\phi[u]\geq\int_{\mathbb{R}^d}(\phi''''(\rho)|\nabla\rho|^4+4\phi'''(\rho)\nabla\rho\cdot(\nabla^2\rho)\nabla\rho+2\phi''(\rho)|\nabla\rho|^2)u_\infty dx.+2\lambda P_\phi[u]$$

By adding and subtracting $2\int_{\mathbb{R}^d}\frac{\phi'''(\rho)}{\phi''(\rho)}|\nabla\rho|^4u_\infty dx$ to the right-hand side of the above inequality we deduce

$$\begin{aligned}-\frac{d}{dt}P_\phi[u]\geq\,&2\int_{\mathbb{R}^d}\phi''(\rho)\left|\nabla^2\rho+\frac{\phi'''(\rho)}{\phi''(\rho)}\nabla\rho\otimes\nabla\rho\right|^2u_\infty dx\\&+\int_{\mathbb{R}^d}\left(\phi''''(\rho)-2\frac{\phi'''(\rho)}{\phi''(\rho)}\right)|\nabla\rho|^4u_\infty dx+2\lambda P_\phi[u].\end{aligned}$$

However, thanks to (21),

$$\phi''''(\rho)-2\frac{\phi'''(\rho)}{\phi''(\rho)}=-\phi''(\rho)\left(\frac{1}{\phi''(\rho)}\right)''\geq 0,$$

so

$$-\frac{d}{dt}P_\phi[u] \geq 2\lambda P_\phi[u]. \tag{28}$$

Gronwall's lemma allows us to deduce that $\lim_{t\to\infty} P[u(t)] = 0$. We would like to deduce that $\lim_{t\to\infty} H[u(t)] = 0$, too, but the proof of this claim is quite technical; see [4, Sect. 2] for details.[2] Therefore let us just assume that $\lim_{t\to\infty} H[u(t)] = 0$ and go on with the proof...

Let us integrate (28) in the time interval $[t, \infty)$ and use the definition of $P_\phi[u]$ and well as the relations $\lim_{t\to\infty} P[u(t)] = \lim_{t\to\infty} H[u(t)] = 0$:

$$P_\phi[u(t)] = -\frac{d}{dt}H_\phi[u(t)] \geq 2\lambda H_\phi[u(t)]. \tag{29}$$

Applying Gronwall's Lemma leads to

$$H_\phi[u(t)] \leq H_\phi[u_0]e^{-2\lambda t} \qquad t > 0.$$

The above estimate, together with Csiszár-Kullback-Pinsker inequality [47]

$$\|u - u_\infty\|^2_{L^1(\mathbb{R}^d)} \leq 2H_\phi[u],$$

allows us to conclude the proof. □

2.2 Convex Sobolev inequalities

The Reader, lost in the computational details of the proof of Theorem 2.1, may have not noticed that in the previous section we actually proved something more than the exponential decay of the solution to the linear Fokker-Planck equation towards the steady state. As a matter of fact, we showed also the following

Corollary 2.1 (Convex Sobolev inequalities)**.** *Let ϕ, V satisfy* (19)–(21)*, and let u_∞ be given by* (20)*. Then*

$$\int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty dx \leq \frac{1}{2\lambda}\int_{\mathbb{R}^d} \phi''\left(\frac{u}{u_\infty}\right)\left|\nabla\frac{u}{u_\infty}\right|^2 u_\infty dx, \tag{30}$$

for all functions $u : \mathbb{R}^d \to [0, \infty)$ such that the above integrals are convergent.

Proof. The left-hand side of (30) equals $H_\phi[u]$, while the right-hand side equals $P_\phi[u(t)]$ (see eq. (24)). Therefore eq. (30) can be obtained from (29) by replacing the solution $u(t)$ of (17), (18) with a generic function $u : \mathbb{R}^d \to [0, \infty)$. □

Ineq. (30) is actually a family of integral inequalities. By choosing ϕ is suitable ways we can obtain specific inequalities.

[2] There is, however, an intuitive argument which allows us to understand why that limit holds. Eq. (24) provides us with a handy expression for $P_\phi[u]$. If we assume that $\lim_{t\to\infty} \rho(t)$ exists in some suitable sense, then it should be equal to some function $\bar\rho$ such that $\int_{\mathbb{R}^d} \phi''(\bar\rho)|\nabla\bar\rho|^2 u_\infty dx = 0$. Since $\phi'' > 0$ in $[0, \infty)$, this implies that $\bar\rho$ must be constant, and given the fact that $\int_{\mathbb{R}^d} \bar\rho u_\infty dx = 1$ this means that $\bar\rho = 1$, i.e. $\lim_{t\to\infty} u(t) = u_\infty$. Therefore, we expect that $\lim_{t\to\infty} H[u(t)] = 0$.

2.2.1 Logarithmic Sobolev inequality

Let $\phi(s) = s\log s - s + 1$, $s > 0$. Let $u \in L^1(\mathbb{R}^d)$, $u \geq 0$. Assume that $\int_{\mathbb{R}^d} u dx = 1$. Then (30) becomes the *log-Sobolev inequality*:

$$\int_{\mathbb{R}^d} u \log \frac{u}{u_\infty} dx \leq \frac{2}{\lambda} \int_{\mathbb{R}^d} \left| \nabla \sqrt{\frac{u}{u_\infty}} \right|^2 u_\infty dx. \tag{31}$$

Letting $f = \sqrt{u/u_\infty}$, $d\mu = u_\infty dx$ into (31) yields the so-called "Gaussian form" of the log-Sobolev inequality:[3]

$$\int_{\mathbb{R}^d} f^2 \log \frac{f^2}{\int_{\mathbb{R}^d} f^2 d\mu} d\mu \leq \frac{2}{\lambda} \int_{\mathbb{R}^d} |\nabla f|^2 d\mu. \tag{32}$$

We point out that we added the factor $1/\int_{\mathbb{R}^d} f^2 d\mu$ inside the logarithm in order to get rid of the constraint $\int f^2 d\mu = 1$. Moreover, the Reader should notice an interesting fact: the constant $2/\lambda$ inside (32) does not depend on the dimension d of the space. However, the measure $d\mu$ depends on d through its normalization factor.

If $V(x) = |x|^2/2$ (and therefore $\lambda = 1$), then the log-Sobolev inequality can be rewritten in another, more explicit form. In fact, in this case $u_\infty = (2\pi)^{-d/2} e^{-\lambda |x|^2/2}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} u \log \frac{u}{u_\infty} dx = \int_{\mathbb{R}^d} u \log u dx + \frac{d}{2} \log(2\pi) + \int_{\mathbb{R}^d} \frac{|x|^2}{2} u dx,\\
&2 \int_{\mathbb{R}^d} \left| \nabla \sqrt{\frac{u}{u_\infty}} \right|^2 u_\infty dx = 2 \int_{\mathbb{R}^d} \left| \frac{\nabla \sqrt{u}}{\sqrt{u_\infty}} - \frac{\sqrt{u}}{\sqrt{u_\infty}} \nabla \log \sqrt{u_\infty} \right|^2 u_\infty dx\\
&= 2 \int_{\mathbb{R}^d} |\nabla \sqrt{u}|^2 dx - 2 \int_{\mathbb{R}^d} \nabla u \cdot \nabla \log \sqrt{u_\infty} dx + 2 \int_{\mathbb{R}^d} u |\nabla \log \sqrt{u_\infty}|^2 dx\\
&= 2 \int_{\mathbb{R}^d} |\nabla \sqrt{u}|^2 dx + 2 \int_{\mathbb{R}^d} (\Delta \log \sqrt{u_\infty} + |\nabla \log \sqrt{u_\infty}|^2) u \, dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^d} u \log u dx + \frac{d}{2} \log(2\pi) + \int_{\mathbb{R}^d} u \left(\frac{|x|^2}{2} - \Delta \log u_\infty - \frac{1}{2} |\nabla \log u_\infty|^2 \right) dx\\
\leq \int_{\mathbb{R}^d} |\nabla \sqrt{u}|^2 dx.
\end{aligned}$$

However, since

$$\frac{|x|^2}{2} - \Delta \log u_\infty - \frac{1}{2} |\nabla \log u_\infty|^2 = d,$$

we conclude

$$\int_{\mathbb{R}^d} u \log u dx + \frac{d}{2} \log(2\pi) + d \leq 2 \int_{\mathbb{R}^d} |\nabla \sqrt{u}|^2 dx. \tag{33}$$

Ineq. (33) constitutes the so-called "Euclidean form" of the log-Sobolev inequality.

[3] The reason why it is called "Gaussian form" probably lies in the fact that the simplest possible choice for potential, i.e. $V(x) = |x|^2/2$, leads to the Gaussian measure $d\mu = (2\pi)^{-d/2} e^{-|x|^2/2} dx$.

2.2.2 Weighted Poincaré inequality

What happens if we choose $\phi(s) = s^2 - 1$ in (30)? We obtain

$$\int_{\mathbb{R}^d} \left(\frac{u^2}{u_\infty^2} - 1 \right) u_\infty dx \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} \left| \nabla \frac{u}{u_\infty} \right|^2 u_\infty dx.$$

However, since $f = u/u_\infty$ and $\int_{\mathbb{R}^d} u_\infty dx = 1 = \int_{\mathbb{R}^d} f u_\infty dx = \left(\int_{\mathbb{R}^d} f u_\infty dx \right)^2$ it follows

$$\int_{\mathbb{R}^d} f^2 u_\infty dx - \left(\int_{\mathbb{R}^d} f u_\infty dx \right)^2 \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |\nabla f|^2 u_\infty dx, \tag{34}$$

which is known as the *weighted Poincaré inequality.*

2.2.3 Beckner inequality

More in general, what if we choose $\phi(s) = s^\alpha - 1$ in (30) with $\alpha \in (1, 2]$? With the same procedure as before we find the *Beckner inequalities*:

$$\frac{1}{\alpha - 1} \left(\int_{\mathbb{R}^d} f^\alpha u_\infty dx - \left(\int_{\mathbb{R}^d} f u_\infty dx \right)^\alpha \right) \leq \frac{\alpha}{2\lambda} \int_{\mathbb{R}^d} f^{\alpha-2} |\nabla f|^2 u_\infty dx. \tag{35}$$

Clearly by choosing $\alpha = 2$ we recover (34). On the other hand, by taking the limit $\alpha \to 1$ in (35)[4] we obtain the log-Sobolev inequality (31).

2.3 The heat equation: convergence to the self-similar solution

I bet the Reader has seen this thing before:

$$u_t = \Delta u \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \mathbb{R}^d. \tag{36}$$

It's the heat equation (of course). We take the initial datum u_0 to be nonnegative (of course) and with unit mass. From the well-known explicit expression for u:

$$u(x, t) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/2} u_0(y) dy, \qquad t > 0,$$

it follows immediately that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = 1, \qquad \|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq (4\pi t)^{-d/2},$$

and therefore

$$\int_{\mathbb{R}^d} u(t) \log u(t) dx \leq \int_{\mathbb{R}^d} u(t) \log \|u(t)\|_{L^\infty(\mathbb{R}^d)} dx \leq -\frac{d}{2} \log(4\pi t) \to -\infty \qquad \text{as } t \to \infty.$$

[4]To take the limit in the right-hand side of (35), one can use e.g. l'Hopital theorem.

So, the Boltzmann entropy of u tends to $-\infty$ as $t \to \infty$. In particular, the entropy method cannot be applied to determine a convergence rate for the solution towards the steady state (this is reasonable, since the steady state is 0, which does not have unit mass).

However, we can use the entropy to study the so-called intermediate asymptotic of the equation, that is, the rate of convergence of the solution towards the self-similar solution

$$U(x,t) = (2\pi(2t+1))^{-d/2} \exp\left(-\frac{|x|^2}{2(2t+1)}\right), \qquad x \in \mathbb{R}^d, \quad t > 0. \tag{37}$$

We are going to show the following

Theorem 2.2 (Relaxation to self-similarity). *Let $u_0 : \mathbb{R}^d \to [0,\infty)$ such that $\int_{\mathbb{R}^d} u_0 dx = 1$, $\int_{\mathbb{R}^d} |x|^2 u_0(x) dx < \infty$, $\int_{\mathbb{R}^d} u \log u dx < \infty$. Let u be the solution to* (36)*, U be given by* (37)*, $H[u]$ be the Boltzmann entropy. Then*

$$\|u(t) - U(t)\|_{L^1(\mathbb{R}^d)} \le \sqrt{\frac{2H[u_0]}{2t+1}}, \qquad t > 0. \tag{38}$$

Proof. Let us do the following rescaling:

$$y = \frac{x}{\sqrt{2t+1}}, \qquad s = \frac{1}{2}\log(2t+1), \qquad v(y,s) = e^{ds} u(e^s y, \frac{1}{2}(e^{2s}-1)).$$

As a consequence, v satisfies

$$v_s = \operatorname{div}_y(\nabla_y v + yv) \qquad s > 0, \qquad v(0) = u_0 \qquad \text{in } \mathbb{R}^d. \tag{39}$$

Eq. (39) is a linear Fokker-Planck with quadratic potential $V(y) = |y|^2/2$. The only steady state of (39) is the Gaussian

$$v_\infty(y) = (2\pi)^{-d/2} e^{-|y|^2/2} = (2t+1)^{2/d} U(x,t).$$

Theorem 2.1 implies that

$$\|v(s) - v_\infty\|_{L^1(\mathbb{R}^d)} \le e^{-s}\sqrt{2H[u_0]}, \qquad s > 0.$$

It is time to go back to the original variables (x,t). Since

$$\|v(s) - v_\infty\|_{L^1(\mathbb{R}^d)} = \|u(t) - U(t)\|_{L^1(\mathbb{R}^d)}, \qquad e^{-s} = (2t+1)^{-1/2},$$

ineq. (38) follows. This finishes the proof. □

2.4 Linear Fokker-Planck equation: generalizations.

The results presented in this Section about the long-time behaviour of the solution to the linear Fokker-Planck equation can be generalized in several ways. Here we present three possible generalizations: Fokker-Planck equation with variable diffusion, non-symmetric Fokker-Planck equation, degenerate Fokker-Planck equation.

2.4.1 Fokker-Planck equation with variable diffusion

Let us consider an equation of this kind:

$$u_t = \mathrm{div}\,(D(x)(\nabla u + u\nabla V)) \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \mathbb{R}^d, \tag{40}$$

where $D : \mathbb{R}^d \to (0, \infty)$ is a smooth function. We assume $u_0 : \mathbb{R}^d \to (0, \infty)$ is an $L^1(\mathbb{R}^d)$ function such that $\int_{\mathbb{R}^d} u_0 dx = 1$, while $e^{-V} \in L^1(\mathbb{R}^d)$. Moreover, we assume that

$$\begin{aligned}&\left(\frac{1}{2} - \frac{d}{4}\right)\frac{1}{D}\nabla D \otimes \nabla D + \frac{1}{2}(\Delta D - \nabla D \cdot \nabla V)\mathbb{I}\\ &\quad + D\nabla^2 V + \frac{1}{2}(\nabla D \otimes \nabla V + \nabla V \otimes \nabla D) - \nabla^2 D \geq \lambda \mathbb{I} \qquad \text{in } \mathbb{R}^d.\end{aligned} \tag{41}$$

The steady state of (40) is unique and given by (20). The entropy H_ϕ is again given by (22) where $\phi : (0, \infty) \to \mathbb{R}$ satisfies (21).

Theorem 2.3 (Long-time behaviour of (40))**.** *If the above-stated assumptions hold, and if $H_\phi[u_0] < \infty$, then any smooth solution $u(t)$ to* (40) *satisfies*

$$H_\phi[u(t)] \leq H_\phi[u_0]e^{-2\lambda t} \qquad t > 0. \tag{42}$$

Furthermore, the following convex Sobolev inequality holds:

$$H_\phi[u] \leq \frac{1}{2\lambda}\int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right)\left|\nabla\left(\frac{u}{u_\infty}\right)\right|^2 D(x)dx. \tag{43}$$

Hints of the proof. Let $\rho = u/u_\infty$, so that we can rewrite (40) as $u_t = \mathrm{div}\,(Du_\infty\nabla\rho)$. The first time derivative of the entropy reads as

$$\frac{d}{dt}H_\phi[u(t)] = -\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho|^2 u_\infty D(x)dx.$$

The second time derivative of $H_\phi[u]$ can be estimated as

$$\frac{d^2}{dt^2}H_\phi[u(t)] \geq \int_{\mathbb{R}^d} \mathrm{tr}(AB)u_\infty dx + 2\lambda\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho|^2 u_\infty D(x)dx,$$

where

$$A = \begin{pmatrix} 2\phi''(\rho) & 2\phi'''(\rho) \\ 2\phi'''(\rho) & \phi''''(\rho) \end{pmatrix}$$

and B is a suitable 2×2 matrix depending on D, ρ and their derivatives up to order 2. By using Sylvester's criterion it is straightforward to see that (21), (41) imply the positive semidefiniteness of A, B, respectively (actually, constraints $\phi'' > 0$, $(1/\phi'')'' \leq 0$ and (41) are equivalent to the nonnegativity of $\det(A)$, $\det(B)$). As a consequence $\mathrm{tr}(AB) \geq 0$ and therefore

$$\frac{d^2}{dt^2}H_\phi[u(t)] \geq 2\lambda\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho|^2 u_\infty D(x)dx = -2\lambda\frac{d}{dt}H_\phi[u(t)].$$

Integrating the above inequality in the time interval (t, ∞) and exploiting the relations $\lim_{t\to\infty} H_\phi[u(t)] = 0$, $\lim_{t\to\infty} \frac{d}{dt}H_\phi[u(t)] = 0$ (again, the second limit follows straightforwardly from Gronwall's lemma, while the first one is more difficult to prove) lead to $\frac{d}{dt}H_\phi[u(t)] + 2\lambda H_\phi[u(t)] \leq 0$, which imply both (42) and (43). This finishes the proof. □

2.4.2 Non-symmetric Fokker-Planck equation

What happens when we consider a Fokker-Planck equation with a nonconstant diffusion coefficient AND a non-conservative force? I'm talking about something like this:

$$u_t = \mathrm{div}\,(D(x)(\nabla u + u\mathcal{F})) \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \mathbb{R}^d, \tag{44}$$

with D a positive smooth function. The idea to deal with an equation like that is to decompose the force $\mathcal{F}$ as sum of a gradient term an a perturbation, i.e.

$$\mathcal{F} = \nabla V + F, \qquad \mathrm{div}\,(D(x)Fu_\infty) = 0 \qquad \text{in } \Omega, \quad t > 0, \tag{45}$$

where u_∞ is given again by (20). To this decomposition there corresponds a splitting of the Fokker-Planck operator L into a symmetric part $L_s[u]$ and a skew-symmetric part $L_{ss}[u]$, defined as

$$L_s[u] = \mathrm{div}\left(D(x)u_\infty\nabla\frac{u}{u_\infty}\right), \qquad L_{ss}[u] = \mathrm{div}\,(D(x)Fu).$$

It's clear that $L_s[u_\infty] = L_{ss}[u_\infty] = 0$, right?

Let us verify that L_s, L_{ss} are symmetric and skew-symmetric with respect to $L^2(u_\infty^{-1}dx)$, respectively. We start with L_s. For arbitrary u, v it follows

$$(L_s[u], v)_{L^2(u_\infty^{-1}dx)} = \int_{\mathbb{R}^d} L_s[u]vu_\infty^{-1}dx = -\int_{\mathbb{R}^d} u_\infty D(x)\nabla\frac{u}{u_\infty}\cdot\nabla\frac{v}{u_\infty}dx$$

which is symmetric in u, v. Therefore L_s is symmetric in $L^2(u_\infty^{-1}dx)$. Now let us deal with L_{ss}. Eq. (45) implies that $L_{ss}[u] = D(x)Fu_\infty\cdot\nabla(u/u_\infty)$, thus

$$(L_{ss}[u], v)_{L^2(u_\infty^{-1}dx)} + (L_{ss}[v], u)_{L^2(u_\infty^{-1}dx)} = \int_{\mathbb{R}^d} D(x)Fu_\infty\cdot\nabla\left(\frac{uv}{u_\infty^2}\right)dx = 0,$$

where the last equality follows from an integration by parts and (45). Therefore L_{ss} is skew-symmetric in $L^2(u_\infty^{-1}dx)$.

The decomposition $L = L_s + L_{ss}$ helps a lot in the proof of

Theorem 2.4 (Long-time behaviour of (44))**.** *Under the above-stated assumptions and the hypothesis that* (41) *holds with* ∇V *replaced by* $\nabla V - F$*, any smooth solution* $u(t)$ *to* (44) *having finite initial entropy* ($H_\phi[u_0] < \infty$) *satisfies*

$$H_\phi[u(t)] \le H_\phi[u_0]e^{-2\lambda t} \qquad t > 0.$$

Hints of the proof. Let $\rho = u/u_\infty$. The first time derivative of $H_\phi[u(t)]$ reads as

$$\frac{d}{dt}H_\phi[u(t)] = -\int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2u_\infty D(x)dx + \int_{\mathbb{R}^d}\phi'(\rho)\mathrm{div}\,(DFu)dx.$$

The second integral on the right-hand side of the above equation is actually zero. In fact,

$$\int_{\mathbb{R}^d}\phi'(\rho)\mathrm{div}\,(DFu)dx = \int_{\mathbb{R}^d}\phi'(\rho)D(x)Fu_\infty\cdot\nabla\rho\,dx = \int_{\mathbb{R}^d}D(x)Fu_\infty\cdot\nabla\phi(\rho)\,dx$$

which vanishes after an integration by parts since (45) holds.

So, it seems that the nonsymmetric perturbation does not change the form of the entropy dissipation. However, F plays a role in the computation of the second time derivative of $H_\phi[u(t)]$. Such a computation is similar to the one carried out in the proof of Thr. 2.1, but more involved, and will not be presented here. □

2.4.3 Degenerate Fokker-Planck equation

We present here a class of Fokker-Planck equations whose diffusion coefficient D is a (possibly singular) matrix:

$$u_t = \operatorname{div}(D\nabla u + uCx) \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \mathbb{R}^d. \tag{46}$$

We assume that $D \in \mathbb{R}^{d\times d}$ is constant and positive semidefinite, while $C \in \mathbb{R}^{d\times d}$ is a constant matrix. Since D has not full rank, the entropy production can vanish for functions other than the steady state, and the second time derivative of the entropy might not have a constant sign. The idea to deal with (46) is to employ a modified entropy; see [3] for details.

We assume that:

$$\{v \in \mathbb{R}^d \mid v \text{ is an eigenvector of } C^\top\} \cap \ker(D) = \emptyset, \tag{47}$$

$$\text{all eigenvalues of } C^\top \text{ have positive real part.} \tag{48}$$

Assumption (47) is a technical hypothesis which ensures existence of smooth, positive solutions to (46), provided that the initial datum u_0 is positive and L^1. Assumption (48) implies the existence of a confinement potential.

The steady state u_∞ is given by

$$u_\infty(x) = \frac{e^{-x\cdot Kx/2}}{\int_{\mathbb{R}^d} e^{-y\cdot Ky/2} dy} \qquad x \in \mathbb{R}^d,$$

where $K \in \mathbb{R}^{d\times d}$ is the unique symmetric and positive definite solution to the Lyapounov equation $CK + KC^\top = 2D$.

We decompose the Fokker-Planck operator L as $L = L_s + L_{ss}$ with $L_s[u] = \operatorname{div}(u_\infty D\nabla\rho)$ and $L_{ss}[u] = \operatorname{div}(u_\infty R\nabla\rho)$, where $R = \frac{1}{2}(CK - KC^\top)$. It is possible to show that L_s, L_{ss} are symmetric and skew-symmetric in $L^2(u_\infty^{-1}dx)$, respectively.

Furthermore, let us define

$$\mu = \min\{\Re(\lambda) \mid \lambda \text{ is an eigenvalue of } C\}.$$

Theorem 2.5 (Long-time behaviour for (46))**.** *Assume the same hypothesis of Thr. 2.3. Moreover assume that* (47)*,* (48) *hold. Let $u(t)$ be the smooth solution to* (46)*.*

(i) If for all eigenvalues λ of C such that $\Re(\lambda) = \mu$ it holds

$$\textit{geometric multiplicity of } \lambda = \textit{algebraic multiplicity of } \lambda$$

then there exists a constant $\kappa > 0$ *such that*

$$H_\phi[u(t)] \le \kappa H_\phi[u_0] e^{-2\mu t} \qquad t > 0.$$

(ii) If there exists an eigenvalue λ_0 *of* C *such that* $\Re(\lambda_0) = \mu$ *and*

$$\textit{geometric multiplicity of } \lambda_0 \neq \textit{algebraic multiplicity of } \lambda_0$$

then for every $\varepsilon > 0$ *there exists a constant* $\kappa_\varepsilon > 0$ *such that*

$$H_\phi[u(t)] \le \kappa_\varepsilon H_\phi[u_0] e^{-2(\mu-\varepsilon)t} \qquad t > 0.$$

Sketch of the proof. We only show (i). It is possible to prove that a symmetric, positive definite matrix $D_0 \in \mathbb{R}^{d\times d}$ exists such that

$$(KC^\top K^{-1})D_0 + D_0(KC^\top K^{-1})^\top \ge 2\mu D_0.$$

Let us define the functional

$$P^*[u] = \int_{\mathbb{R}^d} \phi''(\rho)\nabla\rho \cdot D_0 \nabla\rho \, u_\infty dx.$$

Since D_0 is positive definite, we can find a constant $\eta > 0$ such that $D_0 \ge \eta D$; therefore $P^*[u(t)] \ge \eta P[u(t)] = -\frac{d}{dt} H_\phi[u(t)]$. So, if we can find a suitable upper bound for $P^*[u(t)]$, then we are done.

It is possible to see that

$$\begin{aligned}\frac{d}{dt}P^*[u] = &- \int_{\mathbb{R}^d} \phi''(\rho)\nabla\rho \cdot ((D-R)K^{-1}D_0 + D_0K^{-1}(D+R))\nabla\rho \, u_\infty dx \\ &- 2\int_{\mathbb{R}^d} \mathrm{tr}(AB) u_\infty dx\end{aligned}$$

where

$$A = \begin{pmatrix} 2\phi''(\rho) & 2\phi'''(\rho) \\ 2\phi'''(\rho) & \phi''''(\rho) \end{pmatrix}$$

and B is a suitable 2×2 matrix depending on D, ρ and their derivatives up to order 2. Again, both the matrices A, B are positive semidefinite, so $\mathrm{tr}(AB) \ge 0$. Furthermore, it holds

$$(D-R)K^{-1}D_0 + D_0K^{-1}(D+R) = (KC^\top K^{-1})D_0 + D_0(KC^\top K^{-1})^\top \ge 2\mu D_0.$$

This leads to

$$\frac{d}{dt}P^*[u(t)] \le -2\mu P^*[u(t)] \qquad \Rightarrow \qquad P^*[u(t)] \le P^*[u(\delta)]e^{-2\mu(t-\delta)} \qquad t > \delta \ge 0.$$

A suitable convex Sobolec inequality implies

$$H_\phi[u(t)] \le \frac{1}{2\lambda_P} P^*[u(t)] \le \frac{1}{2\lambda_P} P^*[u(\delta)] e^{-2\mu(t-\delta)}.$$

We wish to set $\delta = 0$ in the above inequality, but if we do it, we get an estimate for $H_\phi[u(t)]$ depending on $P^*[u_0]$, which is not optimal. However, from [3, Thr. 4.8] it follows that

$$P^*[u(t)] \le c_1 t^{-(1+c_2)} H_\phi[u_0] \qquad t > 0,$$

for some $c_1, c_2 > 0$. It follows

$$H_\phi[u(t)] \le \frac{e^{-2\mu(t-\delta)}}{2\lambda_P} c_1 \delta^{-(1+c_2)} H_\phi[u_0] = c(\delta) e^{-2\mu t} H_\phi[u_0] \qquad t > \delta \tag{49}$$

with $c(\delta) = \frac{e^{2\mu\delta}}{2\lambda_P} c_1 \delta^{-(1+c_2)}$. Estimate (49) only holds for $t > \delta$; however, since $H_\phi[u(t)] \le H_\phi[u_0]$ and $e^{-2\mu t} \ge e^{-2\mu\delta}$ for $0 \le t \le \delta$, we conclude that

$$H_\phi[u(t)] \le \kappa e^{-2\mu t} H_\phi[u_0] \qquad t > 0$$

for some positive constant κ. This finishes the proof. □

2.5 Nonlinear Fokker-Planck equations

The Bakry-Emery method, applied successfully to the linear Fokker-Planck equation, can be extended to the case of *nonlinear* Fokker-Planck equations. We consider here equations of the form

$$u_t = \operatorname{div}\left(\nabla f(u) + u\nabla V\right) \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \Omega, \tag{50}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded and convex domain. We consider no-flux boundary conditions:

$$(\nabla f(u) + u\nabla V) \cdot \nu = 0 \qquad \text{on } \partial\Omega, \quad t > 0. \tag{51}$$

Equations similar to (50) are employed e.g. in the study of porous-media flow, charge transport in semiconductors and population dynamics.

While (50) has been studied for more general functions $f(u)$ and $V(x)$, we will consider here (for the sake of simplicity) the case of power functions, that is

$$V(x) = \lambda \frac{|x|^2}{2} \quad x \in \Omega, \qquad f(u) = u^m \quad u \ge 0, \quad m \ge 1 - \frac{1}{d}, \quad m \ne 1. \tag{52}$$

Such an ansatz leads e.g. to the porous medium equation [48]. The cases $m < 1$ and $m > 1$ are referred to as *slow diffusion* and *fast diffusion*, respectively. The steady state is unique and given by the *Barenblatt profile*:

$$u_\infty(x) = \left(N - \frac{\lambda(m-1)}{2m}|x|^2\right)_+^{1/(m-1)}, \tag{53}$$

where $(y)_+ \equiv \max\{y, 0\}$, $y \in \mathbb{R}$, while $N > 0$ is a constant which can be determined by imposing the constraint of mass conservation: $\int_\Omega u_\infty dx = \int_\Omega u_0 dx$. At this point, we would like to define the relative entropy as in (22), but we run into trouble: the steady state given by (53) has compact support. Therefore, we define the relative entropy $H[u|u_\infty]$ as

$$H[u|u_\infty] = H[u] - H[u_\infty], \qquad H[u] = \int_\Omega u \left(\frac{u^{m-1}}{m-1} + \frac{\lambda}{2}|x|^2 \right) dx. \tag{54}$$

The following result holds:

Theorem 2.6. *Let* (52), (53) *hold, and let* $u_0 \in L^1(\Omega)$ *be nonnegative such that* $H[u_0] < \infty$. *If* $u(t)$ *is a solution of* (50), (51), *then*

$$\|u(t) - u_\infty\|_{L^1(\Omega)} \leq Ce^{-\lambda t} \qquad t > 0, \tag{55}$$

where $C > 0$ *is a suitable positive constant.*

Hints of the proof. We are not going to present all the details of the proof, since the ideas are basically the same as in the proof of Thr. 2.1. For the complete picture see [32, pp. 33-36].

We define the so-called entropy variable $\mu = mu^{m-1}/(m-1) + \lambda|x|^2/2$. We point out that μ is simply the partial derivative of the integrand in (54) with respect to u. As a consequence (50) can be rewritten (in gradient-flow form) as

$$u_t = \operatorname{div}(u\nabla\mu) \qquad \text{in } \Omega, \quad t > 0. \tag{56}$$

Moreover, (51) implies $\nu \cdot \nabla\mu = 0$ on $\partial\Omega$, $t > 0$. Therefore by testing (56) against μ we obtain

$$\frac{d}{dt}H[u(t)|u_\infty] = -\int_\Omega u|\nabla\mu|^2 dx \leq 0, \qquad t > 0. \tag{57}$$

The next step in the proof is to compute the second time derivative of $H[u(t)|u_\infty]$. We omit the lengthy computations, which are quite similar (philosophically speaking) to the ones carried out to show (28), and simply state that the following inequality holds:

$$\frac{d^2}{dt^2}H[u(t)|u_\infty] + 2\lambda\frac{d}{dt}H[u(t)|u_\infty] \geq -\int_{\partial\Omega} u^m \nabla(|\nabla\mu|^2) \cdot \nu \, d\sigma. \tag{58}$$

So, it seems that working with a proper subdomain of $\mathbb{R}^d$ is not without consequences: now we have to deal with a surface integral. This is actually the main difference between this proof and the proof of Thr. 2.1. Is the right-hand side of (58) nonnegative? Of course it is. To see this, we just need to apply the following lemma, whose proof can be found e.g. in [26, Lemma 5.1]:

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^d$ *(*$d \geq 1$*) be a convex domain with* C^2 *boundary and let* $\mu \in H^3(\Omega)$ *satisfy* $\nabla\mu \cdot \nu = 0$ *on* $\partial\Omega$. *Then* $\nabla(|\nabla\mu|^2) \cdot \nu \leq 0$ *on* $\partial\Omega$.

Now we just have to integrate (58) in the time interval (t, ∞) and exploit the facts that $\lim_{t\to\infty} \frac{d}{dt} H[u(t)|u_\infty] = 0$ and $\lim_{t\to\infty} H[u(t)|u_\infty] = 0$. While the second limit follows directly from (58) through a Gronwall argument, the proof of the first limit is more technical and will be skipped in these lecture notes. The curious Reader can find more details in [32, p. 35].

Therefore we are left with

$$\frac{d}{dt} H[u(t)|u_\infty] + 2\lambda H[u(t)|u_\infty] \leq 0,$$

which implies, thanks to Gronwall's Lemma

$$H[u(t)|u_\infty] \leq e^{-2\lambda t} H[u_0|u_\infty] \qquad t > 0. \tag{59}$$

Now we wish to apply Csiszár-Kullback-Pinsker inequality, but there is a problem: the steady state u_∞ might vanish in a positive measure set. Therefore we cannot apply the aforementioned inequality in a straightforward way; however, it can be showed (see [38, pp. 30-31] for details) that a similar result holds:

$$\|u - u_\infty\|_{L^1(\Omega)} \leq C\sqrt{H[u|u_\infty]},$$

for some constant $C > 0$. This finishes the proof. □

It is now natural, given what we have seen in the linear case, to ask the question: *is the nonlinear Fokker-Planck equation with f, V given by* (52) *related to some functional inequality?* The answer is yes (of course):

Proposition 2.1 (Gagliardo-Nirenberg inequality (one of many))**.** *Let either $d = 2$, $p > 1$ or $d \geq 3$, $1 < p < d/(d-2)$. Moreover let $q = (p+1)/(p-1)$,*

$$C = \left(\frac{q(1-p)^2}{2\pi d}\right)^{\theta/2} \left(\frac{2q-d}{2q}\right)^{\theta/2} \left(\frac{\Gamma(q)}{\Gamma(q-d/2)}\right)^{\theta/2}, \quad \theta = \frac{d(1-1/p)}{d+2-(d-2)p},$$

where Γ is the Euler Gamma function. Then

$$\|v\|_{L^{2p}(\mathbb{R}^d)} \leq C\|\nabla v\|^{\theta}_{L^2(\mathbb{R}^d)} \|v\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}, \qquad v \in H^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d). \tag{60}$$

Furthermore, if $v(x) = (N + |x - x_0|^2)_+^{1/(1-p)}$ for any $N > 0$, $x_0 \in \mathbb{R}^d$, then equality holds in (60)*. In particular, the constant C is optimal.*

3 Cross-Diffusion PDEs

Many physical systems coming from the applied sciences (e.g. physics, biology, chemistry) can be modeled through a set of reaction-diffusion PDEs with cross-diffusion:

$$u_t = \operatorname{div}\left(A(u)\nabla u + D(u)\nabla\phi\right) + f(u), \qquad t > 0, \qquad u(0) = u_0 \qquad \text{in } \Omega. \tag{61}$$

In (61) $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with smooth boundary, the vector-valued function $u : \Omega \times (0, \infty) \to \mathcal{D} \subset \mathbb{R}^n$ is the unknown of the system, tipically representing densities or concentrations of a multicomponent physical system, $\mathcal{D}$ is the domain of the physical variables, $A : \mathcal{D} \to \mathbb{R}^{n\times n}$ is the diffusion matrix, $D : \mathcal{D} \to \mathbb{R}^{n\times n}$ is the drift matrix, $\phi : \Omega \to \mathbb{R}$ is a potential, and $f : \mathcal{D} \to \mathbb{R}^n$ is the reaction term. The notation $\operatorname{div}(A(u)\nabla u)$ is to be understood as

$$(\operatorname{div}(A(u)\nabla u))_i = \sum_{k=1}^{d}\sum_{j=1}^{n} \frac{\partial}{\partial x_k}\left(A_{ij}(u)\frac{\partial u_j}{\partial x_k}\right) \qquad i = 1, \ldots, n.$$

We impose homogeneous Neumann boundary conditions, which describe the conservation of the species:

$$(A(u)\nabla u + D(u)\nabla\phi)\cdot\nu = 0 \qquad t > 0, \quad \text{on } \partial\Omega. \tag{62}$$

3.1 Examples of cross-diffusion PDEs.

Let us see a few examples of cross-diffusion PDEs coming from the applied sciences.

3.1.1 Population dynamics: the SKT model.

Shigesada et alii proposed in [42] a famous model for a system of two populations which share the same environment and are subject to intra-specific and inter-specific population pressures. The evolution of the densities u_1, u_2 of the populations species is described by (61), (62) with

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + 2a_{21}u_1 + a_{22}u_2 \end{pmatrix}, \qquad D = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \tag{63}$$

while the reaction term is of Lotka-Volterra type:

$$f(u) = \begin{pmatrix} (b_{10} - b_{11}u_1 - b_{12}u_2)u_1 \\ (b_{20} - b_{21}u_1 - b_{22}u_2)u_2 \end{pmatrix}. \tag{64}$$

In (63), (64), the parameters a_{ij}, b_i are nonnegative. The potential ϕ describes inhomogeneities of the environment (e.g. if it is favorable to the species or not).

Under certain assumptions on the coefficients (roughly speaking, a_{10}, a_{20} should be small compared to a_{12}, a_{21}), eqs. (61) admit nonconstant steady state, which biologically represent pattern formation.

We point out that the diffusion matrix A given by (63) is in general not symmetric nor positive semidefinite; it has, however, positive eigenvalues. Due to this fact, the derivation of a-priori estimates is tricky, and the global existence of solutions has been an open problem for decades, until the 2000s [13, 14, 25].

3.1.2 Ion transport.

If you wish to describe the transport of ions in biological cells or in multicomponent fluid mixtures, you would probably use the Poisson-Nernst-Planck equations for the ion concentrations and the electric potential. The derivation of the equations works fine under the assumption that the concentrations levels are far from the saturation points (no volume-filling case); however, the concentrations are allowed to saturate (which may happen in reality), different equations are to be employed. An alternative set of equations is constituted by (61) with

$$A_{ij}(u) = D_i(u_i + u_n\delta_{ij}), \qquad D_{ij}(u) = u_i u_n \delta_{ij} \qquad i, j = 1, \ldots, n, \tag{65}$$

where $u_n \equiv 1 - \sum_{k=1}^{n-1} u_k$ and $D_1, \ldots, D_n$ are positive constants. The functions $u_1, \ldots, u_{n-1}$ are the ion concentrations, u_n is the solvent concentration, while ϕ is the electric potential, which is either a solution to the Poisson equation or a given function.

Again, A is not symmetric nor positive semidefinite. The upper bound $\sum_{i=1}^{n-1} u_i < 1$ should hold for consistency with the physics (volume-filling case), but in general no maximum principle is available for system of PDEs with cross-diffusion. Derivation of suitable a-priori estimates is also challenging. Global existence of bounded weak solutions to (61), (65) was proved in the two species case ($n = 2$) in [7] and for arbitrary n in absence of potential in [31, 52].

3.1.3 Tumor-growth models.

There are three stages in the process of tumor growth. The first stage is the *avascular growth*: the tumor cells proliferate by relying on the body's healthy blood vessels for oxygen and nutritional substances supply. However, as the tumor grows bigger, the amount of available oxygen at its center decreases, which means that the tumor cannot grow in size more than a millimeter or so without its own blood supply. The second phase of the tumor growth is the *vascular growth*: the tumor starts developing an independent blood supply by stimulating the formation of new blood vessels inside the tumor. The third and final stage of tumor growth is the *metastatic phase*, during which the tumor cells are able to escape from the tumor via the circulatory system and lead to the formations of other tumors in the body.

Avascular tumor growth can be described by fluid-dynamic models. For example, in [28] a continuous model is derived for avascular tumor growth in one space dimension, under the assumption that the tumor-host environment consists of tumor cells, the extracellular matrix (ECM), which provides support for the tumor cells, and water. The volume fractions of tumor cells, ECM and water are denoted with u_1, u_2, u_3 (respectively) and sum up to one: $u_1 + u_2 + u_3 = 1$ (volume-filling case). From the mass and momentum balance equations for the system a set of PDEs is derived, which has the structure (61) with A, f given by

$$A(u) = \begin{pmatrix} 2u_1(1-u_1) - \beta\theta u_1 u_2^2 & -2\beta u_1 u_2(1+\theta u_1) \\ -2u_1u_2 + \beta\theta(1-u_2)u_2^2 & 2\beta u_2(1-u_2)(1+\theta u_1) \end{pmatrix}, \tag{66}$$

$$f(u) = \begin{pmatrix} \alpha_1 u_1(1-u_1-u_2) - \alpha_2 u_1 \\ \alpha_3 u_1 u_2 (1-u_1-u_2) \end{pmatrix}, \tag{67}$$

where $\beta > 0$, $\theta > 0$, $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are parameters. There is no potential. Again, the diffusion matrix is in general neither symmetric not positive definite. Relation $u_1 + u_2 \leq 1$ should be fulfilled for consistency with the physics.

3.1.4 Multicomponent fluid mixtures.

The well-known Maxwell-Stefan equations describe the evolution of a multicomponent gaseous mixture under some suitable assumptions: ideal gas, zero baricentric velocity, isobaric and isothermal conditions, same molar mass for all components. They were suggested by J. C. Maxwell in 1866 for dilute gases and by J. Stefan in 1871 for fluids. The Maxwell-Stefan equations are constituted by the mass and reduced force balance equations for the mixture and read as

$$\partial_t u_i + \operatorname{div} J_i = f_i(u), \qquad \nabla u_i = \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{u_i J_k - u_k J_i}{D_{ij}}, \qquad i = 1, \ldots, n. \tag{68}$$

Maxwell-Stefan's model represents a generalization of Fick's law: while in the latter the flux J_i depends (linearly) only on ∇u_i, in the former ∇u_i depends on all the fluxes $J_1, \ldots, J_n$. The Maxwell-Stefan equations can predict phenomena that are beyond the reach of Fick's law, i.e. osmotic diffusion in multicomponent mixtures.

The mathematical difficulties that one has to deal with when solving (68) are three. First, the matrix associated with the linear relations $(J_1, \ldots, J_n) \mapsto (\nabla u_1, \ldots, \nabla u_n)$ is singular, and therefore expressing the fluxes $J_1, \ldots, J_n$ in terms of the concentrations gradients $\nabla u_1, \ldots, \nabla u_n$ is not straightforward. Second, the diffusion matrix $A(u)$ that one obtains after carrying out the aforementioned inversion process is in general not symmetric nor positive semidefinite, Third, nonnegativity and boundedness of $u_1, \ldots, u_n$ have to be proved for consistency with the physics, and this is far from simple for the lack of general minimum/maximum principles for systems of cross-diffusion PDEs.

3.2 Derivation of some cross-diffusion models.

At this point, the curious Reader might ask: *but how to derive the cross-diffusion presented above from other models?* What will now follow aims at (partially) answering this question. In fact, cross-diffusion PDEs can be obtained by performing all kind of nefarious activities on other, more basic models, like for example random walk lattices, fluid models, kinetic equations, stochastic PDEs, *et cetera.* We are going to focus on two methods, which involve taking suitable limits in space-discrete random walk equations and continuous fluid models, repsectively.

3.2.1 Derivation from random-walk lattice models.

Let us consider a one-dimensional lattice, whose cell $j \in \mathbb{Z}$ has a uniform size $h > 0$ and midpoint x_j, so that $x_j - x_{j-1} = h$, for $j \in \mathbb{Z}$. Moreover, let $u_1, \ldots, u_n$ population densities defined on the lattice $\{x_j \ : \ j \in \mathbb{Z}\}$, i.e. $u_i(x_j, t)$ represents the proportion of population i in the cell j at time t. The species can move from cell j to one of the neighbouring cell $j \pm 1$ with transition rates $T_i^{j\pm}$. The densities $u_1, \ldots, u_n$ evolve according to the following master equation:

$$\partial_t u_i(x_j) = T_i^{j-1,+} u_i(x_{j-1}) + T_i^{j+1,-} u_i(x_{j+1}) - (T_i^{j,+} + T_i^{j,-}) u_i(x_j), \tag{69}$$

for $i = 1, \ldots, n$, $j \in \mathbb{Z}$, $t > 0$. How to model the rates $T_i^{j,\pm}$? The basic idea is that, if the departure cell is more crowded than the arrival cell, then the tendency of the species to leave the cell is higher. Therefore, a possible expression for the rate is

$$T_i^{j,\pm} = \sigma_0(h) p_i(u_1(x_i), \ldots, u_n(x_i)) q_i(u_{n+1}(x_{j\pm 1})). \tag{70}$$

In the above equality, $\sigma_0(h) > 0$ is a suitable scaling constant, $u_{n+1} \equiv 1 - \sum_{i=1}^n u_i$ is the volume fraction unoccupied by the species, p_i, q_i are suitable functions. Expression $p_i(u_1(x_i), \ldots, u_n(x_i))$, $q_i(u_{n+1}(x_{j\pm 1}))$ measures the tendency of species i to leave cell j, while $q_i(u_{n+1}(x_{j\pm 1}))$ represents a damping of this tendency due to the crowding of the two neighbouring cells $j + 1$, $j - 1$. We are going to see that, if $\sigma_0(h)$ is chosen wisely, then (69), (70) converge in the limit $h \to 0$ to the cross diffusion system

$$\partial_t u = \partial_x (A(u) \partial_x u) \qquad x \in \mathbb{R}, \quad t > 0, \tag{71}$$

$$\begin{aligned} A_{ij}(u) &= \delta_{ij} p_i(u) q_i(u_{n+1}) + u_i p_i(u) q_i'(u_{n+1}) + u_i q_i(u_{n+1}) \frac{\partial p_i}{\partial u_j}(u) \\ &= q_i(u_{n+1})^2 \frac{\partial}{\partial u_j} \left(\frac{u_i p_i(u)}{q_i(u_{n+1})} \right) \qquad i, j = 1, \ldots, n. \end{aligned} \tag{72}$$

This argument is the same of [52, Appendix].

It is convenient to introduce the following abbreviations:

$$\begin{aligned} p_i^j &= p_i(u_1(x_j), \ldots, u_n(x_j)), \quad q_i^j = q_i(u_{n+1}(x_j)), \\ \partial_k p_i^j &= \frac{\partial p_i}{\partial u_k}(u_1(x_j), \ldots, u_n(x_j)), \quad \partial q_i^j = q_i'(u_{n+1}(x_j)). \end{aligned}$$

Thus, we can rewrite the master equation as

$$\sigma_0^{-1} \partial_t u_i^j = q_i^j (p_i^{j-1} u_i^{j-1} + p_i^{j+1} u_i^{j+1}) - p_i^j u_i^j (q_i^{j+1} + q_i^{j-1}). \tag{73}$$

Set $D = \partial_x$. We compute the Taylor expansions of p_i and q_i ($i = 1, \ldots, n$) and replace $u_k^{j\pm 1} - u_k^j$ by the Taylor expansion $\pm hDu_k^j + \frac{1}{2}h^2D^2u_i^j + O(h^3)$. Then, collecting all terms up to order $O(h^2)$, we arrive at

$$
\begin{aligned}
p_i^{j\pm 1} &= p_i^j + h\sum_{k=1}^n \partial_k p_i^j Du_k^j + \frac{h^2}{2}\left(\sum_{k=1}^n \partial_k p_i^j D^2u_k^j + \sum_{k,\ell=1}^n \partial_{k\ell}^2 p_i^j Du_k^j Du_\ell^j\right) + O(h^3), \\
q_i^{j\pm 1} &= q_i^j \pm h\partial p_i^j Du_{n+1}^j + \frac{h^2}{2}\big(\partial q_i^j D^2u_{n+1}^j + \partial^2 q_i^j (Du_{n+1}^j)^2\big) + O(h^3) \\
&= q_i^j \mp h\partial q_i^j \sum_{k=1}^n Du_k^j + \frac{h^2}{2}\left(-\partial q_i^j \sum_{k=1}^n D^2u_k^j + \partial^2 q_i^j \sum_{k,\ell=1}^n Du_k^j Du_\ell^j\right) + O(h^3).
\end{aligned}
$$

In the last step, we have used $u_{n+1} = 1 - \sum_{k=1}^n u_k$. We insert these expressions into (73) and rearrange the terms. It turns out that the terms of order $O(1)$ and $O(h)$ cancel, and we end up with

$$
\begin{aligned}
\sigma_0^{-1}h^{-2}\partial_t u_i^j &= \sum_{k=1}^n D^2u_k^j(q_i^j p_i^j \delta_{ik} + q_i^j u_i^j \partial_k p_i^j + p_i^j u_i^j \partial q_i^j) \\
&\quad + \sum_{k,\ell=1}^n Du_k^j Du_\ell^j (2q_i^j \partial_k p_i^j \delta_{i\ell} + q_i^j u_i^j \partial_{k\ell}^2 p_i^j - p_i^j u_i^j \partial^2 q_i^j).
\end{aligned}
$$

We choose $\sigma_0 = h^{-2}$ and pass to the limit $h \to 0$:

$$
\begin{aligned}
\partial_t u_i &= \sum_{k=1}^n D^2 u_k \left(q_i p_i \delta_{ik} + q_i u_i \frac{\partial p_i}{\partial u_k} + p_i u_i q_i'\right) \\
&\quad + \sum_{k,\ell=1}^n Du_k Du_\ell \left(2q_i \frac{\partial p_i}{\partial u_k}\delta_{i\ell} + q_i u_i \frac{\partial^2 p_i}{\partial u_k \partial u_\ell} - p_i u_i q_i''\right).
\end{aligned}
$$

A lenghty but straightforward computation shows that the last sum equals

$$
\sum_{k=1}^n Du_k D\left(q_i p_i \delta_{ik} + q_i u_i \frac{\partial p_i}{\partial u_k} + p_i u_i q_i'\right),
$$

and we end up with

$$
\partial_t u_i = D\sum_{k=1}^n Du_k \left(q_i p_i \delta_{ik} + q_i u_i \frac{\partial p_i}{\partial u_k} + p_i u_i q_i'\right),
$$

which is identical to (71), (72).

Volume-filling and non-volume-filling models. If we do not want to incorporate volume-filling effects in the model, then $q_i \equiv 1$ for $i = 1, \ldots, n$ is the right choice. Then $A_{ij}(u) = \frac{\partial}{\partial u_j}(u_i p_i(u))$ and therefore (71) becomes

$$\partial_t u_i = \Delta(u_i p_i(u)) \qquad i = 1, \ldots, n.$$

Choosing $n = 2$, $p_i(u) = a_{i0} + a_{i1}u_1 + a_{i2}u_2$ ($i = 1, 2$) yields the SKT model (61), (63) (with no potential). So we can state that the basic assumption of the SKT model is the linear dipendence of the transition rates on the species densities.

If we want to incorporate volume-filling effects, then q_i must be nonconstant and vanish at zero. A (relatively) simple model of this kind can be obtained by setting $p_i \equiv 1$, $q_i(s) = D_i s$ for $i = 1, \ldots, n$, $s > 0$, where $D_1, \ldots, D_n$ are positive constants. The equations we obtain constitute the ion transport model (61), (65) (without potential).

3.2.2 Derivation from fluid models.

Let us consider a fluid of n components. The mass and momentum balance equations for the fluid read as

$$\partial_t u_i + \operatorname{div}(u_i v_i) = r_i, \tag{74}$$

$$\partial_t(u_i v_i) + \operatorname{div}(u_i v_i \otimes v_i - S_i) = p\nabla u_i + u_i b_i + f_i \tag{75}$$

for $i = 1, \ldots, n$. Here u_i, v_i are the mass density and drift velocity of species i, respectively, r_i is the mass production rate (e.g. die to chemical reactions), S_i is the stress tensor, p is the phase pressure, f_i the momentum production rate. The sum $p\nabla u_i + u_i b_i$ represents the force acting on species i: $p\nabla u_i$ is the interphase force coming from the phase pressure, while $b_i u_i$ is the body force.

We are going to derive a cross-diffusion model from (74), (75). We impose the following hypothesis:

1. the total mass density is constant: $\sum_{i=1}^n u_i = 1$;
2. the baricentric velocity is zero: $\sum_{i=1}^n u_i v_i = 0$;
3. all species have the same molar masses;
4. the total body force is zero: $\sum_{i=1}^n b_i u_i = 0$;
5. the stress S_i is made up by the contributions of the phase pressure p and the partial pressures $P_i = P_i(u)$: $S_i = -u_i(p + P_i(u))\mathbb{I}$;
6. the partial pressure of the $n-$th components vanishes: $P_n(u) \equiv 0$;
7. the momentum production is proportional to the velocity differences:

   $$f_i = \sum_{j=1}^n k_{ij} u_i u_j (v_j - v_i) \qquad i = 1, \ldots, n,$$

   with symmetric positive coefficients $k_{ij} = k_{ji}$.

It follows that the total mass production rate and the total momentum production vanish, too: $\sum_{i=1}^n r_i = 0$, $\sum_{i=1}^n f_i = 0$. Furthermore

$$-\mathrm{div}\, S_i - p\nabla u_i = \nabla(u_i P_i) + u_i \nabla p \qquad i = 1, \ldots, n. \tag{76}$$

We consider now a large time scale and a small velocity scale, which means neglecting inertial effects: $t \mapsto \varepsilon^{-1} t$, $v \mapsto \varepsilon v$. Under this scaling (75) becomes

$$\varepsilon^2 \partial_t (u_i v_i) + \varepsilon^2 \mathrm{div}\,(u_i v_i \otimes v_i) - \mathrm{div}\, S_i = p\nabla u_i + u_i b_i + f_i, \qquad i = 1, \ldots n.$$

Taking the limit $\varepsilon \to 0$ in the above equation results in

$$-\mathrm{div}\, S_i - p\nabla u_i = u_i b_i + f_i \qquad i = 1, \ldots, n. \tag{77}$$

From (76), (77) it follows

$$\nabla(u_i P_i) + u_i \nabla p = u_i b_i + f_i \qquad i = 1, \ldots, n. \tag{78}$$

Summing the above equalities for $i = 1, \ldots, n$ and using assumptions 1, 6, as well as the fact that $\sum_{i=1}^n f_i = 0$, lead to

$$\nabla p = -\sum_{i=1}^{n-1} \nabla(u_i P_i).$$

Therefore,

$$\nabla(u_i P_i) - u_i \sum_{j=1}^{n-1} \nabla(u_j P_j) = u_i b_i + f_i. \tag{79}$$

The left-hand side of (79) can be explicitly computed to obtain the following relations:

$$\sum_{j=1}^{n} A_{ij}(u) \nabla u_j = u_i b_i + \sum_{j=1}^{n} k_{ij} u_i u_j (v_j - v_i), \tag{80}$$

$$A_{ii}(u) = (1 - u_i)\left(P_i + u_i \frac{\partial P_i}{\partial u_i}\right) - u_i \sum_{\substack{k=1 \\ k \neq i}}^{n-1} u_k \frac{\partial P_k}{\partial u_i}, \tag{81}$$

$$A_{ij}(u) = u_i \left((1 - u_i) \frac{\partial P_i}{\partial u_j} - P_i - \sum_{\substack{k=1 \\ k \neq i}}^{n-1} u_k \frac{\partial P_k}{\partial u_i} \right), \qquad j \neq i. \tag{82}$$

An even simpler set of equations can be obtained by setting $p = 0$:

$$\nabla(u_i P_i(u)) = u_i b_i + \sum_{j=1}^{n} k_{ij} u_i u_j (v_j - v_i). \tag{83}$$

Coupling (74) with (80)–(82) or (83) yields a cross-diffusion system in which the fluxes $J_i = u_i v_i$ are implicitly given in terms of the gradients of the mass densities. From this set of equations we will now derive, under simplifying assumptions, the tumor-growth model (61), (66), (67) and the Maxwell-Stefan model (68).

Tumor growth model. Let $n = 3$, $P_1(u) = u_1$, $P_2(u) = \beta u_2(1 + \theta u_1)$, $k_{ij} = k$, $b_i = 0$. Then the matrix A in (81), (82) coincides with (66). Furthermore,

$$u_i b_i + \sum_{j=1}^{n} k_{ij} u_i u_j (v_j - v_i) = k \sum_{j=1}^{n} u_i u_j (v_j - v_i) = -k u_i v_i,$$

and therefore we recover the tumor-growth model (61), (66), (67) .

Maxwell-Stefan equations. Let us now choose $p = 0$, $b_i = 0$, $P_i(u) = 0$, $k_{ii} = 0$ for $i = 1, \dots, n$, and $\min_{i \neq j} k_{ij} > 0$. Eqs. (80)–(82) become

$$\partial_t u_i + \operatorname{div} J_i = r_i, \qquad \nabla u_i = \sum_{j=1}^{n} k_{ij}(u_i J_j - u_j J_i), \qquad i = 1, \dots, n, \tag{84}$$

with $J_i = u_i v_i$. We can write the second relation in (84) as

$$\nabla u = MJ, \qquad M_{ij} = k_{ij} u_i - \delta_{ij} \sum_{s=1}^{n} k_{is} u_s \qquad i, j = 1, \dots, n. \tag{85}$$

In the following argument, we assume that $u_i > 0$ for $i = 1, \dots, n$.
We must now ask ourself: what's the rank of M? Clearly M is singular, since $Mu = 0$ (easy to verify). Thus (85) has a solution if and only if $\nabla u \in \operatorname{Ker}(M^\top)^\perp$ (Fredholm's alternative). So, let us find $\operatorname{Ker}(M^\top)$. Since $(M^\top)_{ij} = k_{ij} u_j - \delta_{ij} \sum_{s=1}^{n} k_{is} u_s$, there is no doubt that $\operatorname{Span}(\mathbf{1}) \subset \operatorname{Ker}(M^\top)$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Now, let $v \in \operatorname{Ker}(M^\top)$. It follows

$$v_i \sum_{s=1}^{n} k_{is} u_s = \sum_{j=1}^{n} k_{ij} u_j v_j, \qquad i = 1, \dots, n.$$

Assume $v \notin \operatorname{Span}(\mathbf{1})$, which means that it exists $\hat{\imath} \in \{1, \dots, n\}$ such that $v_{\hat{\imath}} < v_j$ for $j \neq \hat{\imath}$. We can assume w.l.o.g. that $\hat{\imath} = 1$. Since $k_{ii} = 0$ and $k_{ij} > 0$ for $i \neq j$, as well as $u_i > 0$, it holds

$$v_1 \sum_{s=1}^{n} k_{1s} u_s = \sum_{j=1}^{n} k_{1j} u_j v_j > \sum_{j=1}^{n} k_{1j} u_j v_1$$

which is absurd. Therefore $v \in \operatorname{Span}(\mathbf{1})$. This means that $\operatorname{Span}(\mathbf{1}) = \operatorname{Ker}(M^\top)$, that is, (85) has a solution if and only if $\nabla u \in \operatorname{Span}(\mathbf{1})^\perp$, i.e. if and only if $\sum_{i=1}^{n} \nabla u_i = 0$. However, $\sum_{i=1}^{n} u_i = 1$, so this constraint is satisfied.

As a consequence, the kernel of the matrix G defined as $G_{ij} = -u_i^{-1/2} M_{ij} u_j^{1/2}$ is generated by $\sqrt{u} \equiv (\sqrt{u_1}, \dots, \sqrt{u_n})$. Since $G_{ij} = k_{ij}\sqrt{u_i u_j} - \delta_{ij} \sum_{s=1}^{n} k_{is} u_s$, it is immediate to see that G is symmetric. Moreover, $u_i^{-1/2} G_{ij} u_j^{1/2} = (-M^\top)_{ij}$ is diagonally dominant, i.e. $(-M^\top)_{ii} - \sum_{j\neq i} |(-M^\top)_{ij}| = 0 \geq 0$ for $i = 1, \ldots, n$. Therefore all the complex eigenvalues of $-M^\top$ have nonnegative real part. However, since G is similar to $-M^\top$, the same property holds for G; being G symmetric, this implies that G is positive semidefinite; furthermore, again due to its symmetry, G is positive definite on $\mathrm{Span}(\sqrt{u})^\perp$. We can rewrite the relation between ∇u and J as

$$-\frac{\nabla u_i}{\sqrt{u_i}} = \sum_{s=1}^{n} G_{is} \frac{J_s}{\sqrt{u_s}} \qquad i = 1, \ldots, n. \tag{86}$$

To solve (86) we replace equation n (i.e. the one with $i = n$) with the constraint $\sum_{k=1}^{n} J_k = 0$. This means solving the linear problem:

$$\tilde{G} w = \begin{pmatrix} -\nabla u_1/\sqrt{u_1} \\ \vdots \\ -\nabla u_{n-1}/\sqrt{u_{n-1}} \\ 0 \end{pmatrix}, \qquad \tilde{G}_{ij} = \begin{cases} G_{ij} & i < n \\ \sqrt{u_j} & i = n \end{cases}, \qquad J_i = \sqrt{u_i} w_i. \tag{87}$$

We point out that $\tilde{G}$ is nonsingular since $\sqrt{u} \in \mathrm{Ker}(G) = \mathrm{Ker}(G^\top)$.

3.3 Entropy structure

We have seen some examples of cross-diffusion systems, i.e. nonlinear PDEs with the form

$$\partial_t u = \mathrm{div}\,(A(u)\nabla u) + f(u) \qquad \text{in } \Omega \subset \mathbb{R}^d, \quad t > 0, \tag{88}$$

with a diffusion matrix $A(u) \in \mathbb{R}^{n\times n}$ which might be not symmetric nor positive semidefinite. Such involved structure prevents the application of maximum/minimum principles to (88), so that proving lower and/or upper bounds (often required by physical arguments) for the solution $u : \Omega \times (0, \infty) \to \mathbb{R}^n$ to (88) is challenging. Moreover, unlike what happens for scalar PDEs, there is no regularity theory for (88): for instance, there are smooth solutions to certain cross-diffusion systems which exhibit blow-up in finite time [43]. This means that additional assumptions are needed to ensure that (88) has global-in-time weak solutions, that is, to show that the local-in-time solutions can be prolonged to the time interval $(0, \infty)$.

There are several approaches to this problem. Ladyženskaya et al. [35, Chap. VII] showed that a-priori estimates of local-in-time solutions to quasilinear parabolic systems follow from suitable L^∞ bounds for u and ∇u, and proved global existence of classical solutions under some growth conditions on the nonlinearity. Amann [2] defined the concept of $W^{1,p}$ weak solutions and showed their global existence under the hypothesis that their $W^{1,p}$ norm (with $p > d$) can be controlled. Pierre [39] proved that if the nonnegativity of the solution is preserved, the total mass of the components does not blowup in finite time, the reaction term $f(u)$ grows at most linearly in u *and the diffusion matrix* $A(u)$ *is diagonal*, then global existence of solutions follows.

We will present here another approach. The basic assumption is that (88) has a *gradient-flow (entropy) structure*:

$$\partial_t u = \operatorname{div}(B\nabla H'[u]) + f(u) \qquad \text{in } \Omega \subset \mathbb{R}^d, \quad t > 0, \tag{89}$$

where $B = B(u) \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix and $H'[u]$ is the Fréchet derivative of the entropy functional $H[u]$. If $H[u]$ has the usual structure

$$H[u] = \int_\Omega h(u)dx$$

then $H'[u] \equiv h'(u) = \left(\frac{\partial h}{\partial u_1}, \dots, \frac{\partial h}{\partial u_n}\right)$ by the Riesz representation theorem. Let $w \equiv h'(u)$. The vector-valued quantity w is called *entropy variable*. It is immediate to see that (88) has a gradient-flow structure if $B \equiv A(u)(h''(u))^{-1}$ is positive semidefinite (or, equivalently, if $h''(u)A(u)$ is positive semidefinite), since (88) can be rewritten as

$$\partial_t u = \operatorname{div}(B\nabla w) + f(u) \qquad \text{in } \Omega \subset \mathbb{R}^d, \quad t > 0. \tag{90}$$

We point out that in (90) u is a function of w; to be precise, it is the inverse of $u \mapsto h'(u)$. In light of this remark, we propose the following:

Definition 1 (Entropy)**.** We call the function $h : \mathcal{D} \to \mathbb{R}^n$ and *entropy density* for (88) and $H[u] = \int_\Omega h(u)dx$ the corresponding *entropy* if $h''(u)$ is positive definite and $h''(u)A(u)$ is positive semidefinite for a.e. $u \in \mathcal{D}$.

As a consequence of the above definition, the entropy $H[u]$ is nonincreasing in time along solutions (provided that suitable boundary conditions are imposed; possible choices are e.g. homogeneous Neumann and periodic boundary conditions):

$$\frac{d}{dt}H[u] = -\int_\Omega \nabla u \cdot h''(u)A(u)\nabla u\, dx = -\sum_{i,j,k=1}^{n}\sum_{\ell=1}^{d}\int_\Omega \frac{\partial u_i}{\partial x_\ell}\frac{\partial^2 h}{\partial u_i \partial u_j}A_{jk}(u)\frac{\partial u_k}{\partial x_\ell}dx \le 0.$$

Furthermore, if $h' : \mathcal{D} \to \mathbb{R}^n$ is invertible and $\mathcal{D}$ is bounded, then $u = (h')^{-1}(w) \in \mathcal{D}$ is uniformely bounded. That is, L^∞ bounds for u follow straightforwardly from the entropy structure of (88). A simple example is given by

$$h(u) = \sum_{i=1}^{n+1}(u_i \log(u_i) - u_i), \qquad u_{n+1} = 1 - \sum_{i=1}^{n} u_i,$$

$$\mathcal{D} = \left\{ u \in (0,\infty)^n \quad : \quad \sum_{i=1}^{n} u_i < 1 \right\}.$$

We will see that the above function constitutes an entropy for the ion transport, tumor growth, and Maxwell-Stefan models.
The entropy variables read as

$$w_i = \frac{\partial h}{\partial u_i} = \log\frac{u_i}{u_{n+1}}, \qquad i = 1, \dots, n.$$

Inverting the above relation leads to

$$u_i = \frac{e^{w_i}}{1 + \sum_{j=1}^n e^{w_j}}, \qquad i = 1, \ldots, n.$$

If the solution w to (90) belongs to some Sobolev or Lebesgue space, then it is a.e. finite, and so $u \in \mathcal{D}$. In particular u is bounded with bounds independent of time or any approximation/truncation parameters.

Clearly, the above argument is not true for all entropies. The function $h : (0, \infty) \to \mathbb{R}^n$, $h(u) = \sum_{i=1}^2 u_i \log u_i$, is an entropy for the SKT model. However, since $w_i = \log u_i$ for $i = 1, 2$, the variables u_1, u_2 are positive, but can be unbounded.

3.3.1 Relation to thermodynamics.

We are going to argue that entropy variables are strongly related to the chemical potentials from thermodynamics. To see this, consider a fluid consisting of n components with the same molar masses in isobaric and isothermal conditions. The system evolves according to the mass balance equations for the mass densities $u_1, \ldots, u_n$:

$$\partial_t u_i + \operatorname{div} J_i = 0 \qquad i = 1, \ldots, n.$$

In the above equation $J_1, \ldots, J_n$ are the diffusion fluxes. We assume that the baricentric velocity vanishes, there are no chemical reactions, and the total mass density is constant, say equal to one: $\sum_{i=1}^n u_i = 1$.

Let $s = s(u)$ be the thermodynamic entropy density of the system. The *chemical potentials* $\mu_1, \ldots, \mu_n$ are defined as

$$\frac{\mu_i}{T} = -\frac{\partial s}{\partial u_i} \qquad i = 1, \ldots, n,$$

where T is the (constant) system temperature. Writing $u_n = 1 - \sum_{i=1}^{n-1} u_i$, we can define the mathematical entropy density as

$$h(u_1, \ldots, u_{n-1}) = -Ts\left(u_1, \ldots, u_{n-1}, 1 - \sum_{i=1}^{n-1} u_i\right), \qquad u_1, \ldots, u_{n-1} > 0, \ \sum_{i=1}^{n-1} u_i < 1.$$

The entropy variables $w_1, \ldots, w_n$ relate to the chemical potentials through

$$w_i = \frac{\partial h}{\partial u_i} = -T\frac{\partial s}{\partial u_i} + T\frac{\partial s}{\partial u_n} = \mu_i - \mu_n, \qquad i = 1, \ldots, n-1.$$

For an ideal gas $\mu_i = \mu_i^0 + \log u_i$, $i = 1, \ldots, n$, where μ_i^0 is the Gibbs energy, which is a function of temperature and pressure. Since we are considering isobaric and isothermal conditions, μ_i^0 is constant. Therefore, $w_i = \log(u_i/u_n)$ (up to an additive constant) for $i = 1, \ldots, n-1$, which corresponds to the mathematical entropy $h(u) = \sum_{i=1}^n u_i \log u_i$, $u_n = 1 - \sum_{i=1}^{n-1} u_i$. As we have said at p. 33, this is a mathematical entropy for the Maxwell-Stefan, ion transport, tumor growth models.

3.3.2 Relation to hyperbolic conservation laws.

Now we will see that the entropy formulation of (88) is related also to the hyperbolic conservation laws

$$\partial_t u + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} f_j(u) = 0 \qquad \text{in } \mathbb{R}^d, \quad t > 0, \tag{91}$$

where $u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^n$ and $f_j : \mathbb{R}^n \to \mathbb{R}^n$, $j = 1, \ldots, d$. Let $h : \mathbb{R}^n \to \mathbb{R}$ smooth function such that $h''(u)$ is positive definite for $u \in \mathbb{R}^n$. Assume that the matrix/vector product $f_j'(u)^\top h'(u)$ (where f_j' is the Jacobian matrix of f_j) can be written as the gradient of some suitable scalar function $q_j : \mathbb{R}^n \to \mathbb{R}$, i.e.

$$q_j'(u) = f_j'(u)^\top h'(u) \qquad \text{for } u \in \mathbb{R}^n, \quad j = 1, \ldots, d. \tag{92}$$

Then $H[u] \equiv \int_{\mathbb{R}^d} h(u) dx$ is an entropy for (91) in the sense that

$$\begin{aligned} \frac{d}{dt} H[u] &= \int_{\mathbb{R}^d} h'(u) \cdot \partial_t u \, dx = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} h'(u) \cdot f_j'(u) \frac{\partial u}{\partial x_j} dx \\ &= -\sum_{j=1}^{d} \int_{\mathbb{R}^d} q_j'(u) \cdot \frac{\partial u}{\partial x_j} dx = -\int_{\mathbb{R}^d} \sum_{j=1}^{d} \frac{\partial q_j(u)}{\partial x_j} dx = 0. \end{aligned}$$

Therefore $H[u]$ is constant in time along the solutions of (91). Furthermore, assume that $h' : \mathbb{R}^n \to \mathbb{R}^n$ is (globally) invertible, and let us define the entropy variables $w = h'(u)$. It follows that (91) can be rewritten as

$$A_0(w) \partial_t w + \sum_{j=1}^{d} A_j(w) \frac{\partial w}{\partial x_j} = 0, \tag{93}$$

where $A_0(w) = h''(u)^{-1}|_{u=(h')^{-1}(w)}$, $A_j(w) = f_j'(u) h''(u)^{-1}|_{u=(h')^{-1}(w)}$, $j = 1, \ldots, d$. Clearly A_0 is symmetric and positive definite. Let us prove that A_j is symmetric, for $j = 1, \ldots, d$. It suffices to prove that $h''(u) A_j(w)|_{w=h'(u)} h''(u) = h''(u) f_j'(u)$ is symmetric. It holds

$$\begin{aligned} (h''(u) f_j'(u))_{\alpha\beta} &= \sum_{\gamma=1}^{n} \frac{\partial^2 h}{\partial u_\alpha \partial u_\gamma} \frac{\partial (f_j)_\gamma}{\partial u_\beta} = \frac{\partial}{\partial u_\alpha} \sum_{\gamma=1}^{n} \frac{\partial (f_j)_\gamma}{\partial u_\beta} \frac{\partial h}{\partial u_\gamma} - \sum_{\gamma=1}^{n} \frac{\partial^2 (f_j)_\gamma}{\partial u_\alpha \partial u_\beta} \frac{\partial h}{\partial u_\gamma} \\ &= \frac{\partial}{\partial u_\alpha} (f_j'(u)^\top h'(u))_\beta - \sum_{\gamma=1}^{n} \frac{\partial^2 (f_j)_\gamma}{\partial u_\alpha \partial u_\beta} \frac{\partial h}{\partial u_\gamma}. \end{aligned}$$

From (92) it follows

$$(h''(u) f_j'(u))_{\alpha\beta} = \frac{\partial^2 q_j}{\partial u_\alpha u_\beta} - \sum_{\gamma=1}^{n} \frac{\partial^2 (f_j)_\gamma}{\partial u_\alpha \partial u_\beta} \frac{\partial h}{\partial u_\gamma}$$

and therefore $h''(u)f'_j(u)$ is symmetric, for $u \in \mathbb{R}^n$, $j = 1, \ldots, d$.

The property that (91) can be rewritten as (93) with A_0 symmetric and positive definite, and $A_1, \ldots, A_d$ symmetric, is referred to, in the theory of hyperbolic conservation laws, as *symmetrizability.* Thus we have seen that, if (91) admits an entropy, then it is symmetrizable. This is interesting, since symmetrizable systems of conservation laws have good properties (i.e. uniqueness of smooth solutions, energy estimates).

3.3.3 About the symmetry of B and the eigenvalues of A.

Remember that the transformed matrix $B(w) = A(u)h''(u)|_{u=(h')^{-1}w}$ is positive semidefinite if h is an entropy density for (88). In many applications B is also symmetric. This is nice, but it's a case of good luck: it is not true in general. In fact, consider zero reaction $f \equiv 0$ and

$$A(u) = \begin{pmatrix} 1 & -u_1 \\ 1 & 1 \end{pmatrix}, \qquad h(u) = u_1(\log u_1 - 1) + \frac{1}{2}u_2^2.$$

The corresponding system of equations (88) is a modified Keller-Segel, with an additional diffusion term which prevents blowup of solutions. The entropy balance equation reads as

$$\frac{d}{dt}\int_\Omega h(u)dx = -\int_\Omega (4|\nabla\sqrt{u_1}|^2 + u_2^2)dx \le 0,$$

and so the entropy is decreasing along solutions as it should be; however,

$$A(u)h''(u)^{-1} = \begin{pmatrix} u_1 & -u_1 \\ u_1 & 1 \end{pmatrix}$$

is not symmetric (it is, of course, positive semidefinite).

The symmetry of B requires that A has additional properties. To see this, assume that B is symmetric and positive definite. Then $A(u) = h''(u)^{-1}(h''(u)A(u))$ is the product of the symmetric, positive definite matrix $h''(u)^{-1}$ and $h''(u)A(u) = h''(u)B(w(u))h''(u)$ which is symmetric (given the symmetry of B). However, the following result holds [41, Prop. 6.1]:

Lemma 3.1. *Let $H \in \mathbb{C}^{n\times n}$ Hermitian and positive definite, and let $K \in \mathbb{C}^{n\times n}$ Hermitian. Then the product HK (or KH as well) is diagonalizable with real eigenvalues. The number of positive (respectively, negative) eigenvalues of HK equals that for K.*

Applling the above lemma to our case implies that A is diagonalizable with real eigenvalues, and the number of positive eigenvalues of A equals that for $h''(u)A(u)$. However, being B symmetric and positive definite, so is $h''(u)A(u)$, meaning that $h''(u)A(u)$ has only real positive eigenvalues. Summarizing up, we have proved the following

Claim 3.1. *If $B(w)$ is symmetric and positive definite, than $A(u)$ is diagonalizable with real positive eigenvalues for $w = h'(u)$.*

3.4 The Boundedness-by-Entropy Method

In this section we will make more precise the idea that the entropy structure of a system of nonlinear PDEs can be exploited to get a-priori estimates and uniform L^∞ bounds for the solutions. The goal is to prove global-in-time existence of bounded weak solutions to cross-diffusion systems

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad t > 0, \qquad u(0) = u_0 \quad \text{in } \Omega, \tag{94}$$

with homogeneous Neumann boundary conditions

$$\nu \cdot \nabla u = 0 \qquad \text{on } \partial\Omega, \quad t > 0. \tag{95}$$

The domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded and the space dimension $d \geq 1$ is arbitrary. We make the following assumptions.

H1 There exist a bounded domain $\mathcal{D} \subset (0,1)^n$, $n \geq 1$, and a function $h \in C^2(\mathcal{D}, \mathbb{R})$ such that $h''(u)$ is positive definite for $u \in \mathcal{D}$ and $h' : \mathcal{D} \to \mathbb{R}^n$ is invertible.

H2' There exists $a \geq -1/2$ such that

$$z \cdot h''(u)A(u)z \geq \sum_{i=1}^n u_i^{2a} z_i^2 \qquad z \in \mathbb{R}^n, \quad u \in \mathcal{D}.$$

H2" For $i, j = 1, \ldots, n$, the function $u \mapsto A_{ij}(u)u_j^{-a}$ is $C^0(\overline{\mathcal{D}})$. We define

$$a^* \equiv \sup_{u \in \mathcal{D}\setminus\{0\}} \max_{i,j=1,\ldots,n} \frac{|A_{ij}(u)|}{|u_j|^a} < \infty.$$

H3 $A \in C^0(\overline{\mathcal{D}}, \mathbb{R}^{n\times n})$, $f \in C^0(\overline{\mathcal{D}}, \mathbb{R}^n)$, and

$$C_f \equiv \sup_{u \in \mathcal{D}} \frac{f(u) \cdot h'(u)}{1 + h(u)} < \infty.$$

Hypothesis H1 means that the transformation $u \mapsto w = h'(u)$ is invertible. In particular, if $w(t)$ solves the system in the entropy variable formulation, then $u(t) = (h')^{-1}(w(t)) \in \mathcal{D}$ for $t > 0$, implying a uniform (in t and in any truncation/regularization parameter) L^∞ bound for u. Constraint $\mathcal{D} \subset (0,1)^n$ is actually equivalent to the assumption of boundedness and positivity of $\mathcal{D}$, since a simple rescaling of the entropy density (i.e. $\tilde{h}(u) = h(\lambda u)$ for a suitable $\lambda > 0$) will transform an arbitrary bounded domain $\mathcal{D} \subset (0,\infty)^n$ into a domain $\tilde{\mathcal{D}} \subset (0,1)^n$.
Hypothesis H2' is required to obtain a gradient estimate for the solution. The power function u_i^{2a} can be replaced by a more general expression $\alpha_i(u_i)^2$, where $\alpha_i : \mathcal{D} \to (0,\infty)$ is a suitable monotone function, e.g. $\alpha_i(s) = s$ or $\alpha_i(s) = 1 - s$.
Hypothesis H2" is employed to find an estimate for the (discrete) time derivative of u, since

it allows to control the term $\operatorname{div}(A(u)\nabla u)$ in some Sobolev space with negative index. If the previous assumption H2' is generalized as explained above, then H2" is relaxed in a similar way by replacing the term $|u_j|^a$ with $\alpha_j(u_j)$.
Hypothesis H3 is a growth condition which is used to estimate the source term: for example, the contribution of $f(u)$ in the entropy balance equation is controlled, thanks to H3, by the entropy itself.

It is now time to state clearly what we mean for weak solution of (94).

Definition 2 (Weak solution to (94), (95))**.** We call $u : \Omega \times (0, \infty) \to \mathbb{R}^n$ a *weak solution* to (94), (95) if, for all $T > 0$,

1. $u^{a+1} \in L^2(0,T; H^1(\Omega, \mathbb{R}^n))$, $\partial_t u \in L^2(0,T; H^1(\Omega, \mathbb{R}^n)')$;
2. for all $\phi \in L^2(0,T; H^1(\Omega, \mathbb{R}^n))$,
$$\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_\Omega \nabla\phi : A(u)\nabla u \, dxdt = \int_0^T \int_\Omega f(u)\phi \, dxdt; \tag{96}$$
3. $\lim_{t\to 0} u(\cdot, t) = u_0$ in $H^1(\Omega)'$.

We point out that point 3 of the above definition makes sense thanks to the continuous Sobolev embedding $H^1(0,T; H^1(\Omega, \mathbb{R}^n)') \hookrightarrow C([0,T], H^1(\Omega, \mathbb{R}^n)')$. The bracket $\langle \cdot, \cdot \rangle$ denote the dual product between $H^1(\Omega, \mathbb{R}^n)'$ and $H^1(\Omega, \mathbb{R}^n)$.

The following result holds (and we are going to prove it).

Theorem 3.1 (Global existence of solutions to (94), (95))**.** *Let $u^0 : \Omega \to \mathcal{D}$ a Lebesgue-measurable function, and assume that hypothesis H1, H2', H2", H3 hold. Then there exists a weak solution to* (94), (95) *such that $u(x,t) \in \overline{\mathcal{D}}$ a.e. $x \in \Omega$, $t > 0$. In particular $u \in L^\infty(\Omega \times (0,\infty))$.*

This is a rather general result about the global existence and boundedness of weak solutions to a system of strongly coupled (cross-diffusion) nonlinear PDEs with entropy structure. We point out that the boundedness of the weak solutions derive from the entropy structure itself, rather than from a maximum principle (which is in general not available for systems like (94)).

3.4.1 Proof of the general existence theorem for cross-diffusion systems in the volume-filling case.

Let us first explain the key steps of the proof.

We first discretize (94) in time through an implicit Euler scheme. In this way we will deal with an elliptic, stationary problem, thus avoiding issues about time regularity of the solutions. Moreover, we rewrite the equation in terms of the entropy variable w; in particular, the physical variable u is to be considered a function of w.

We also add a higher order regularizing term proportional to $w + (-\Delta)^m w$, where w is the entropy variable and m is an integer. We choose $m > d/2$ so that the Sobolev embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ holds, which implies that the entropy variable w will be bounded and the physical variable u will lie strictly inside the domain $\mathcal{D}$.[5]

This discretization/regularization procedure will leave us with 2 parameters: a discrete timestep $\tau > 0$, and a regularization parameter $\varepsilon > 0$.

The next step consists in proving the existence of solutions to the discretized/regularized problem. Since this is a *nonlinear* elliptic problem, this result is achieved by first formulating the equations as a fixed-point problem for some operator $F : X \to X$, where X is a suitable Banach space, and then showing the existence of a fixed point through Leray-Schauder's theorem.

The problem of proving the well-posedness of the operator F is equivalent to the solution of a system of *linear* PDEs, which is achieved through a Lax-Milgram argument. Furthermore, suitable (uniform) estimates on the set of fixed points and other properties of F are showed (mainly) by means of a discrete entropy inequality. We point out that such discrete entropy inequality is available because the discretization we choose preserves the entropy structure.

At this point we have existence of sulutions to the discretized/regularized problem, as well as a discrete entropy inequality, which provides us with some estimates. The fact that some of these estimates are uniform in τ, ε allows us to apply a discrete version of the Aubin-Lions Lemma, therefore perfoming the limit $(\varepsilon, \tau) \to 0$ and obtaining a solution $u(x, t)$ the (94), (95). In particular, since u is a limit of functions having image contained into $\mathcal{D}$, it follows that $u(x, t) \in \overline{\mathcal{D}}$ for $x \in \Omega$, $t > 0$.

Step 1: definition of the approximated problem. Let $T > 0$, $n \in \mathbb{N}$, $\tau = T/N$, $m \in \mathbb{N} \cap (d/2, \infty)$. Assume $w^{k-1} \in L^\infty(\Omega, \mathbb{R}^n)$ is given (notice that for $k = 1$ the function w^0 is determinated by the initial datum, i.e. $w^0 = h'(u_0)$). We define the following nonlinear problem:

$$\text{Find } w^k \in H^m(\Omega, \mathbb{R}^n) \text{ such that, for all } \phi \in H^m(\Omega, \mathbb{R}^n), \tag{97}$$
$$\int_\Omega \left(\frac{u(w^k) - u(w^{k-1})}{\tau} \cdot \phi + \nabla\phi : B(w^k)\nabla w^k \right) dx + \varepsilon(w^k, \phi)_{H^m} = \int_\Omega f(u(w^k)) \cdot \phi \, dx,$$

where $(w^k, \phi)_{H^m} \equiv \int_\Omega \sum_{|\alpha| \le m} D^\alpha w^k \cdot D^\alpha \phi \, dx$ is the scalar product between w^k and ϕ in $H^m(\Omega, \mathbb{R}^n)$, and $u(w) = (h')^{-1}(w)$. Notice that the homogeneous Neumann boundary conditions are implicitly specified in (97).

Step 2: linearized approximated problem. Let $y \in L^\infty(\Omega, \mathbb{R}^n)$ and $\delta \in [0, 1]$ given. We define the following linear problem:

$$\text{Find } w \in H^m(\Omega, \mathbb{R}^n) \text{ such that, for all } \phi \in H^m(\Omega, \mathbb{R}^n), \tag{98}$$
$$\int_\Omega \left(\delta \frac{u(y) - u(w^{k-1})}{\tau} \cdot \phi + \nabla\phi : B(y)\nabla w \right) dx + \varepsilon(w, \phi)_{H^m} = \delta \int_\Omega f(u(y)) \cdot \phi \, dx.$$

[5]To be precise, the image of $\Omega \times [0, T]$ through u will be compactly contained into $\mathcal{D}$.

Clearly (98) can be simply rewritten as

$$a(w,\phi) = F(\phi) \qquad \phi \in H^m(\Omega,\mathbb{R}^n),$$

with

$$a(w,\phi) = \int_\Omega \nabla\phi : B(y)\nabla w\, dx + \varepsilon(w,\phi)_{H^m},$$
$$F(\phi) = \delta \int_\Omega \left(-\frac{u(y)-u(w^{k-1})}{\tau} + f(u(y)) \right) \cdot \phi\, dx$$

Since y is bounded and the functions f, B are smooth in $\mathcal{D}$ (Hypothesis H3) it follows that the forms a, F are bounded on $H^m(\Omega,\mathbb{R}^n)$. Moreover, the positive semidefiniteness of B implies that a is also coercive:

$$a(w,w) \geq \varepsilon(w,w)_{H^m} = \varepsilon\|w\|^2_{H^m} \qquad w \in H^m(\Omega,\mathbb{R}^n).$$

Hence Lax-Milgram lemma implies the existence of a unique solution $w \in H^m(\Omega,\mathbb{R}^n) \hookrightarrow L^\infty(\Omega,\mathbb{R}^n)$ to (98).

Step 3: solution of the nonlinear approximated problem. The previous step allows us to define an operator $S : L^\infty(\Omega,\mathbb{R}^n) \times [0,1] \to L^\infty(\Omega,\mathbb{R}^n)$ as follows: for $y \in L^\infty(\Omega,\mathbb{R}^n)$, $\delta \in [0,1]$, $w = S(y,\delta) \in H^m(\Omega,\mathbb{R}^n)$ is the solution to (98). The Reader has surely noticed that any fixed point of $S(\cdot,1)$, i.e. and solution w to $S(w,1) = w$, is a solution to the nonlinear problem (97). So, let us prove the existence of such a fixed point, shall we? We plan to apply Leray-Schauder's fixed point theorem. What we need is:

1. continuity of S;
2. compactness of S;
3. $S(\cdot,0)$ must be constant;
4. the set of the fixed points of $S(\cdot,\delta)$ must be bounded in $L^\infty(\Omega,\mathbb{R}^3)$ *uniformely w.r.t.* $\delta \in [0,1]$.

Clearly $S(\cdot,0) \equiv 0$ since the operator F of Step 2 vanishes. Let us now show that S is compact. We choose $\phi = w$ in (98).

$$\int_\Omega \left(\delta\frac{u(y)-u(w^{k-1})}{\tau} \cdot w + \nabla w : B(y)\nabla w \right) dx + \varepsilon(w,w)_{H^m} = \delta \int_\Omega f(u(y)) \cdot w\, dx. \quad (99)$$

The positive semidefiniteness of B and the boundedness of y, w^{k-1} implies (just use Cauchy-Schwartz):

$$\|w\|_{H^m(\Omega,\mathbb{R}^n)} \leq \frac{\delta}{\varepsilon}\left(\|f(u(y))\|_{L^2(\Omega,\mathbb{R}^n)} + \tau^{-1}\|u(y)\|_{L^2(\Omega,\mathbb{R}^n)} + \tau^{-1}\|u(w^{k-1})\|_{L^2(\Omega,\mathbb{R}^n)} \right) \leq C_{\varepsilon,\tau}.$$

The implies that $S : L^\infty(\Omega, \mathbb{R}^n) \times [0,1] \to H^m(\Omega, \mathbb{R}^n)$ is bounded. In particular $S : L^\infty(\Omega, \mathbb{R}^n) \times [0,1] \to L^\infty(\Omega, \mathbb{R}^n)$ is compact due to the Sobolev compact embedding $H^m(\Omega, \mathbb{R}^n) \hookrightarrow L^\infty(\Omega, \mathbb{R}^n)$.

The proof that S is (sequentially) continuous is quite standard. Take a sequence $(y_k, \delta_k)_{k\in\mathbb{N}} \subset L^\infty(\Omega, \mathbb{R}^n) \times [0,1]$. Assume that $y_k \to y$ in $L^\infty(\Omega, \mathbb{R}^n)$ and $\delta_k \to \delta$. Define $w_k = S(y_k, \delta_k)$. We know that S is compact, therefore w_k admits a convergent subsequence, which we will denote again (notation abuse!) as w_k. So, let $w_k \to w$ in $L^\infty(\Omega, \mathbb{R}^n)$. Moreover, since w_k is bounded in $H^m(\Omega, \mathbb{R}^n)$, then in particular $w_k \rightharpoonup w$ in $H^m(\Omega, \mathbb{R}^n)$ (up to subsequences). Finally, $u(y_k) \to u(y)$, $B(y_k) \to B(y)$, $f(u(y_k)) \to f(u(y))$ in $L^\infty(\Omega, \mathbb{R}^n)$ by (uniform) continuity. At this point, write (98) with y, δ, w replaced by y_k, δ_k, w_k (respectively) and take the limit $k \to \infty$. You will find that w solves (98), i.e. $w = S(y, \delta)$. So, S is continuous.

Now, the most interesting part (actually, the *key* part): let us show that the set of fixed points of $S(\cdot, \delta)$ is bounded in $L^\infty(\Omega, \mathbb{R}^n)$ uniformely w.r.t. $\delta \in [0,1]$. Let $0 \le \delta \le 1$, and $w \in H^m(\Omega, \mathbb{R}^n)$ be such that $w = S(w, \delta)$. Then (99) holds with $y = w$:

$$\int_\Omega \left(\delta \frac{u(w) - u(w^{k-1})}{\tau} \cdot w + \nabla w : B(w) \nabla w \right) dx + \varepsilon (w, w)_{H^m} = \delta \int_\Omega f(u(w)) \cdot w \, dx. \tag{100}$$

The convexity of h implies that $h(u) - h(v) \le h'(u) \cdot (u - v)$ for $u, v \in \mathbb{R}^n$. This implies

$$\int_\Omega \frac{u(w) - u(w^{k-1})}{\tau} \cdot w dx \ge \tau^{-1} \int_\Omega (h(u(w)) - h(u(w^{k-1}))) dx.$$

The right-hand side of (100) can be bound by using Hypothesis H3:

$$\int_\Omega f(u(w)) \cdot w \, dx \le C_f \int_\Omega (1 + h(u(w))) dx.$$

Therefore (100) leads to

$$\begin{aligned} \delta(1 - C_f \tau) \int_\Omega h(u(w)) dx + \tau \int_\Omega \nabla w : B(w) \nabla w dx + \varepsilon \tau \|w\|^2_{H^m(\Omega, \mathbb{R}^n)} \\ \le C_f \tau \delta |\Omega| + \delta \int_\Omega h(u(w^{k-1})) dx. \end{aligned}$$

Choosing $\tau < 1/C_f$ and recalling that $0 \le \delta \le 1$ leads to

$$\varepsilon \tau \|w\|^2_{H^m(\Omega, \mathbb{R}^n)} \le |\Omega| + \int_\Omega h(u(w^{k-1})) dx.$$

We have found the desired uniform bound on the fixed points of $S(\cdot, \delta)$. As a consequence, Leray-Schauder's fixed point theorem implies the existence of a fixed point $w^k \in L^\infty(\Omega, \mathbb{R}^n)$ to $S(\cdot, 1)$, that is, a solution to (97). This solution satisfies the discrete entropy inequality

$$(1 - C_f \tau) \int_\Omega h(u(w^k)) dx + \tau \int_\Omega \nabla w^k : B(w^k) \nabla w^k dx + \varepsilon \tau \|w^k\|^2_{H^m(\Omega, \mathbb{R}^n)} \tag{101}$$

$$\leq C_f \tau |\Omega| + \int_\Omega h(u(w^{k-1}))dx.$$

Step 4: uniform estimates. Time to fix a new notation. To be precise, we define piecewise-constant-in-time functions from the sequences w^k, $u(w^k)$. For $0 \leq t \leq T$ let

$$w^{(\tau)}(x,t) = w^0(x)\chi_{\{0\}}(t) + \sum_{k=1}^N w^k(x)\chi_{((k-1)\tau,k\tau]}(t),$$
$$u^{(\tau)}(x,t) = u^0(x)\chi_{\{0\}}(t) + \sum_{k=1}^N u(w^k(x))\chi_{((k-1)\tau,k\tau]}(t).$$

Moreover, let us define the discrete backward time derivative operator D_τ as $(D_\tau f)(x,t) = \tau^{-1}(f(x,t) - f(x,t-\tau))$, for $\tau \leq t \leq T$ and for any function $f = f(x,t)$. Taking the sum of (97) for $k = 1, \dots, N$ yields

$$\int_\Omega \frac{u(w^N) - u(w^0)}{\tau} \cdot \phi \, dx + \sum_{k=1}^N \int_\Omega \nabla\phi : B(w^k)\nabla w^k dx + \sum_{k=1}^N \varepsilon(w^k, \phi)_{H^m}$$
$$= \sum_{k=1}^N \int_\Omega f(u(w^k)) \cdot \phi \, dx.$$

Multiplying the above equality times τ and applying the new notation we deduce

$$\int_0^T \int_\Omega D_\tau u^{(\tau)} \cdot \phi \, dxdt + \int_0^T \int_\Omega \nabla\phi : B(w^{(\tau)})\nabla w^{(\tau)} dxdt + \int_0^T \varepsilon(w^{(\tau)}, \phi)_{H^m} dt$$
$$= \int_0^T \int_\Omega f(u^{(\tau)}) \cdot \phi \, dxdt,$$

which, since $B(w) = A(u(w))(h''(u(w)))^{-1}$ and $w = h'(u)$, becomes

$$\int_0^T \int_\Omega D_\tau u^{(\tau)} \cdot \phi \, dxdt + \int_0^T \int_\Omega \nabla\phi : A(u^{(\tau)})\nabla u^{(\tau)} dxdt + \int_0^T \varepsilon(w^{(\tau)}, \phi)_{H^m} dt \tag{102}$$
$$= \int_0^T \int_\Omega f(u^{(\tau)}) \cdot \phi \, dxdt,$$

for piecewise constant functions $\phi : [0,T] \to H^m(\Omega, \mathbb{R}^n)$. However, a density argument ensures that (102) holds true for all $\phi \in L^2(0,T; H^m(\Omega, \mathbb{R}^n))$. Furthermore. summing the discrete entropy inequality (101) for $k = 1, \dots, j$, $j \leq N$ arbitrary, leads to

$$(1 - C_f\tau) \int_\Omega h(u(w^j))dx + \tau \sum_{k=1}^j \int_\Omega \nabla w^k : B(w^k)\nabla w^k dx + \varepsilon\tau \sum_{k=1}^j \|w^k\|^2_{H^m(\Omega,\mathbb{R}^n)} \tag{103}$$

$$\leq C_f\tau j|\Omega| + C_f\tau\sum_{k=1}^{j-1}\int_\Omega h(u(w^k))dx + \int_\Omega h(u(w^0))dx.$$

At this point, we need the following

Lemma 3.2 (Discrete Gronwall inequality)**.** *Let $a \geq 0$, $b_k \geq 0$, $z_k \in \mathbb{R}$ (for $k \in \mathbb{N}$) such that*

$$z_j \leq a + \sum_{k=1}^{j-1} b_k z_k, \qquad j \geq 1.$$

Then

$$z_j \leq a\exp\left(\sum_{k=1}^{j-1} b_k\right), \qquad j \geq 1.$$

Since $\tau j \leq T$, the above lemma can be applied to $z_k = \int_\Omega h(u(w^k))dx$ to find

$$\int_\Omega h(u(w^j))dx \leq C_f T|\Omega| \exp\left(C_f\tau(j-1)\right) \leq C_f T|\Omega| e^{C_f T},$$

which allows us to control the right-hand side of (103), yielding

$$(1 - C_f\tau)\int_\Omega h(u(w^j))dx + \tau\sum_{k=1}^{j}\int_\Omega \nabla w^k : B(w^k)\nabla w^k dx + \varepsilon\tau\sum_{k=1}^{j}\|w^k\|^2_{H^m(\Omega,\mathbb{R}^n)} \qquad (104)$$

$$\leq C_T + \int_\Omega h(u(w^0))dx,$$

for some positive constant C_T depending on T, $|\Omega|$. Moreover, Hypothesis H2' implies

$$\nabla w^k : B(w^k)\nabla w^k = \nabla u^{(\tau)} : h''(u^{(\tau)})A(u^{(\tau)})\nabla u^{(\tau)} \geq \sum_{i=1}^{n}(u_i^{(\tau)})^{2a}|\nabla u_i^{(\tau)}|^2$$

$$= \frac{1}{(a+1)^2}\sum_{i=1}^{n}|\nabla(u_i^{(\tau)})^{a+1}|^2.$$

Therefore, (104) and the above estimate lead to

$$\int_\Omega h(u^{(\tau)})dx + \int_0^T\int_\Omega\sum_{i=1}^{n}|\nabla(u_i^{(\tau)})^{a+1}|^2 dxdt + \varepsilon\int_0^T\|w^{(\tau)}\|^2_{H^m(\Omega,\mathbb{R}^n)}dt \qquad (105)$$

$$\leq C_T + \int_\Omega h(u^0)dx.$$

The above inequality and the L^∞ bounds for u imply

$$\|u^{(\tau)}\|_{L^\infty(\Omega\times(0,T))} + \sum_{i=1}^{n}\|(u_i^{(\tau)})^{a+1}\|_{L^2(0,T;H^1(\Omega))} + \sqrt{\varepsilon}\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \leq C_T. \qquad (106)$$

Now we find a uniform estimate for the discrete time derivative $D_\tau u^{(\tau)}$ of $u^{(\tau)}$. Let $\phi \in L^2(0,T;H^m(\Omega,\mathbb{R}^n))$. It holds (just use Cauchy-Schwartz)

$$\left|\int_0^T\int_\Omega D_\tau u^{(\tau)}\cdot\phi dxdt\right| \le \|A(u^{(\tau)})\nabla u^{(\tau)}\|_{L^2(\Omega\times(0,T))}\|\nabla\phi\|_{L^2(\Omega\times(0,T))} + \varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))}\|\phi\|_{L^2(0,T;H^m(\Omega))} + \|f(u^{(\tau)})\|_{L^2(\Omega\times(0,T))}\|\phi\|_{L^2(\Omega\times(0,T))}.$$

Clearly $\varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \le C_T$ from (106), while $\|f(u^{(\tau)})\|_{L^2(\Omega\times(0,T))} \le C$ thanks to Hypothesis H3. Concerning the remaining term,

$$\begin{aligned}\|A(u^{(\tau)})\nabla u^{(\tau)}\|^2_{L^2(\Omega\times(0,T))} &= \sum_{i=1}^n \|(A(u^{(\tau)})\nabla u^{(\tau)})_i\|^2_{L^2(\Omega\times(0,T))}\\ &\le C\sum_{i,j=1}^n \|A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)}\|^2_{L^2(\Omega\times(0,T))}\\ &= C\sum_{i,j=1}^n \left\|\frac{A_{ij}(u^{(\tau)})}{(u_j^{(\tau)})^a}\nabla(u_j^{(\tau)})^{a+1}\right\|^2_{L^2(\Omega\times(0,T))}\\ &\le C\sum_{i,j=1}^n \left\|\frac{A_{ij}(u^{(\tau)})}{(u_j^{(\tau)})^a}\right\|^2_{L^\infty(\Omega\times(0,T))}\left\|\nabla(u_j^{(\tau)})^{a+1}\right\|^2_{L^2(\Omega\times(0,T))}.\end{aligned}$$

Hypothesis H2" and (106) lead to

$$\|A(u^{(\tau)})\nabla u^{(\tau)}\|^2_{L^2(\Omega\times(0,T))} \le C\sum_{j=1}^n \left\|\nabla(u_j^{(\tau)})^{a+1}\right\|^2_{L^2(\Omega\times(0,T))} \le C_T. \tag{107}$$

Therefore

$$\left|\int_0^T\int_\Omega D_\tau u^{(\tau)}\cdot\phi dxdt\right| \le C_T\|\phi\|_{L^2(0,T;H^m(\Omega))} \qquad \phi\in L^2(0,T;H^m(\Omega,\mathbb{R}^n)),$$

that is,

$$\|D_\tau u^{(\tau)}\|_{L^2(0,T;H^m(\Omega)')} \le C_T. \tag{108}$$

Step 5: limit $(\varepsilon,\tau)\to 0$. The uniform estimates (106), (108) allow us to apply the following generalization of the well-known Aubin's Lemma:

Lemma 3.3 (Nonlinear Aubin's Lemma). *Let $\Omega\subset\mathbb{R}^d$ ($d\ge 1$) be a bounded domain with Lipschitz boundary. Let $(u^{(\tau)})_{\tau>0}$ be a family of nonnegative, piecewise constant in time functions with uniform time step size $\tau>0$. Furthermore, let $\alpha\ge 1/2$, $m\ge 0$, and assume there exists $C>0$ such that for all $\tau>0$,*

$$\|(u^{(\tau)})^\alpha\|_{L^2(0,T;H^1(\Omega))} + \|D_\tau u^{(\tau)}\|_{L^2(\tau,T;H^m(\Omega)')} \le C.$$

Finally, assume that $p\ge 1$ is such that the embedding $H^1(\Omega)\hookrightarrow L^p(\Omega)$ is continuous. Then, up to subsequences, $u^{(\tau)}$ is strongly convergent in $L^{2\alpha}(0,T;L^{p\alpha}(\Omega))$.

The above Lemma implies that, up to a subsequence, $u^{(\tau)} \to u$ strongly in $L^1(\Omega\times[0,T])$ as $(\tau,\varepsilon)\to 0$. Notice that the restriction $a \geq -1/2$ comes from the hypothesis $\alpha \geq 1/2$ of the above lemma: in our case $\alpha = a+1$. Since $u^{(\tau)}$ is uniformely bounded in $L^\infty(\Omega\times[0,T])$, by L^p interpolation we deduce that $u^{(\tau)} \to u$ strongly in $L^p(\Omega\times[0,T])$ as $(\tau,\varepsilon)\to 0$, for any $p\in[1,\infty)$. Moreover, up to a subsequence, $u^{(\tau)}\to u$ a.e. in $\Omega\times[0,T]$.

By the dominated convergence theorem $f(u^{(\tau)}) \to f(u)$ strongly in $L^p(\Omega\times[0,T])$ as $(\tau,\varepsilon)\to 0$, for any $p\in[1,\infty)$. Bounds (106), (108) imply that, up to subsequences,

$$
\begin{aligned}
\varepsilon w^{(\tau)} &\to 0 && \text{strongly in } L^2(0,T;H^m(\Omega)),\\
D_\tau u^{(\tau)} &\rightharpoonup \partial_t u && \text{weakly in } L^2(0,T;H^m(\Omega)').
\end{aligned}
$$

The weak convergence of $\nabla(u_i^{(\tau)})^{a+1}$ and the a.e. convergence of $u^{(\tau)}$ imply that $\nabla(u_i^{(\tau)})^{a+1} \rightharpoonup \nabla u_i^{a+1}$ weakly in $L^2(\Omega\times[0,T])$. Furthermore, from Hypothesis H2" it follows that $A_{ij}(u^{(\tau)})(u_j^{(\tau)})^{-a} \to A_{ij}(u)u_j^{-a}$ strongly in $L^p(\Omega\times[0,T])$, for any $p<\infty$ and $i,j=1,\ldots,n$. Since $(A(u^{(\tau)})\nabla u)_i = (a+1)^{-1}\sum_{j=1}^n A_{ij}(u^{(\tau)})(u_j^{(\tau)})^{-a}\nabla(u_j^{(\tau)})^{a+1}$, this implies that $A(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup A(u)\nabla u$ weakly in $L^p(\Omega\times[0,T])$ for any $1\leq p<2$. This fact and (107) lead to

$$
A(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup A(u)\nabla u \qquad \text{weakly in } L^2(\Omega\times[0,T]).
$$

At this point, by taking the limit $(\tau,\varepsilon)\to 0$ in (102) we find that (96) is satisfied for $\phi\in L^2(0,T;H^m(\Omega))$. However, a standard density argument implies that (96) actually holds for $\phi\in L^2(0,T;H^1(\Omega))$. The initial condition $u(\cdot,0)=u_0$ is satisfied in the sense of $H^1(\Omega)'$ since $H^1(0,T;H^1(\Omega)')\hookrightarrow C^0(0,T;H^1(\Omega)')$. This finishes the proof.

3.4.2 A few examples.

Theorem 3.1 can be applied to some examples of cross-diffusion equations from the previous sections.

Tumor-growth model. Let us begin with the tumor growth model (with no source terms, for the sake of simplicity):

$$
\begin{aligned}
&\partial_t u = \operatorname{div}(A(u)\nabla u) \qquad \text{in } \Omega,\quad t>0,\\
&A(u) = \begin{pmatrix} 2u_1(1-u_1)-\beta\theta u_1u_2^2 & -2\beta u_1u_2(1+\theta u_1)\\ -2u_1u_2+\beta\theta(1-u_2)u_2^2 & 2\beta u_2(1-u_2)(1+\theta u_1)\end{pmatrix}, \qquad u\in\mathcal{D},
\end{aligned}
$$

where the domain $\mathcal{D}$ of the physical variables is defined as

$$
\mathcal{D} = \{(u_1,u_2)\in(0,\infty)^2 \ : \ u_1+u_2<1\}.
$$

We define the entropy density as

$$
h(u) = u_1\log u_1 + u_2\log u_2 + (1-u_1-u_2)\log(1-u_1-u_2), \qquad u\in\mathcal{D}.
$$

The Reader knows already that h is a strictly convex function such that the mapping $h' : \mathcal{D} \to \mathbb{R}^2$ is invertible, and its inverse reads as

$$(h')^{-1}(w)_i = \frac{e^{w_i}}{1 + e^{w_1} + e^{w_2}}, \qquad i = 1, 2, \quad w \in \mathbb{R}^2.$$

This, in particular, means that Hypothesis H1 is fulfilled. What the Reader probably does not know, however, is that

$$z \cdot h''(u)A(u)z = 2z_1^2 + \beta\theta u_2 z_1 z_2 + 2\beta(1 + \theta u_1)z_2^2 \qquad z \in \mathbb{R}^2, \quad u \in \mathcal{D}.$$

It is straightforward to see that the right-hand side of the above quadratic form is positive definite if $\theta < \theta^* \equiv 4\beta^{-1/2}$. This condition guarantees that Hypothesis H2' holds with $a = 0$. The boundedness of $\mathcal{D}$, as well as the fact that $a = 0$ and $f \equiv 0$, imply that Hypotheses H2", H3 are satisfied. Therefore, Theorem 3.1 yields the global-in-time existence of nonnegative, bounded weak solutions $u = u(x, t)$ to the tumor-growth model such that $u(x, t) \in \overline{\mathcal{D}}$ for $x \in \Omega$, $t > 0$.

Maxwell-Stefan model. Remember that the Maxwell-Stefan equations are given by

$$\partial_t u_i + \operatorname{div} J_i = f_i(u), \qquad \nabla u_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{u_i J_k - u_k J_i}{D_{ij}}, \qquad i = 1, \ldots, n.$$

Choosing $n = 3$ and Inverting the relations between ∇u_1, ∇u_2 and j_1, J_2 yields the 2×2 cross-diffusion model

$$\partial_t u = \operatorname{div}(A(u)\nabla u) \qquad \text{in } \Omega, \quad t > 0,$$
$$A(u) = \frac{1}{a(u)} \begin{pmatrix} d_2 + (d_0 d_2)u_1 & (d_0 d_1)u_1 \\ (d_0 d_2)u_2 & d_1 + (d_0 d_1)u_2 \end{pmatrix}, \qquad u \in \mathcal{D},$$

where $d_{i+j-2} = D_{ij}$ and $a(u) = d_1 d_2(1 - u_1 - u_2) + d_0(d_1 u_1 + d_2 u_2)$, and $\mathcal{D}$ is defined as in the tumor-growth model. The entropy density h is also identical to the one from the previous example (in particular, Hypothesis H1 holds true). Moreover, let us compute the quadratic form

$$z \cdot h''(u)A(u)z = \frac{1}{a(u)} \left(d_2 \frac{z_1^2}{u_1} + d_1 \frac{z_2^2}{u_2} + d_0 \frac{(z_1 + z_2)^2}{1 - u_1 - u_2} \right) \qquad z \in \mathbb{R}^2, \quad u \in \mathcal{D}.$$

The definition of $\mathcal{D}$ and $a(u)$ implies that $u \in \mathcal{D} \mapsto a(u) \in \mathbb{R}$ is bounded, and hence a constant $\gamma > 0$ exists such that

$$z \cdot h''(u)A(u)z \geq \gamma(u_1^{-1} z_1^2 + u_2^{-1} z_2^2) \qquad z \in \mathbb{R}^2, \quad u \in \mathcal{D}.$$

This means that Hypothesis H2' is satisfied with $a = -1/2$. Since $a < 0$ and $f \equiv 0$ then Hypothesis H2", H3 are fulfilled, too. Therefore we have global existence of bounded, nonnegative weak solutions $u = u(x, t)$ to the Maxwell-Stefan equations such that $u(x, t) \in \overline{\mathcal{D}}$ for $x \in \Omega$, $t > 0$. We point out that such existence result can be generalized to an arbitrary $n \geq 3$.

3.4.3 Population Models.

We are going to consider a problem to which the previous existence theorem cannot be applied; however, we can still obtain an existence result for such a problem by exploiting similar techniques. We are referring to the SKT model:

$$\partial_t u = \operatorname{div}(A(u)\nabla u) + f(u) \qquad \text{in } \Omega, \quad t > 0, \tag{109}$$

$$\nabla u \cdot \nu = 0 \qquad \text{on } \partial\Omega, \quad t > 0, \tag{110}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega, \tag{111}$$

where

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix} \tag{112}$$

and f is given by the Lotka-Volterra model

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \qquad i = 1, 2. \tag{113}$$

The coefficients a_{ij}, b_{ij} are nonnegative. An entropy density for the system is given by

$$h(u) = a_{12}^{-1}(u_1 \log u_1 - u_1 + 1) + a_{21}^{-1}(u_2 \log u_2 - u_2 + 1), \qquad u_1, u_2 > 0.$$

In fact, if $H[u] \equiv \int_\Omega h(u)dx$, it follows

$$\frac{dH[u(t)]}{dt} + 4\int_\Omega \left(\frac{a_{10}}{a_{12}}|\nabla\sqrt{u_1}|^2 + \frac{a_{20}}{a_{21}}|\nabla\sqrt{u_2}|^2 + \frac{a_{11}}{2a_{12}}|\nabla u_1|^2 + \frac{a_{22}}{2a_{21}}|\nabla u_1|^2 \right) dx$$
$$\leq \max\{a_{12}b_{10}, a_{21}b_{20}\} H[u(t)] + \left(b_{10} + b_{20} + \frac{b_{10}^2}{b_{11}} + \frac{b_{20}^2}{b_{22}} \right) |\Omega|.$$

The above inequality follows from the fact that $h''(u)A(u) = M^I(u) + M^{II}(u) + M^{III}(u)$, where

$$M^I(u) = \begin{pmatrix} \frac{a_{10}}{a_{12}}\frac{1}{u_1} & 0 \\ 0 & \frac{a_{20}}{a_{21}}\frac{1}{u_2} \end{pmatrix}, \quad M^{II}(u) = \begin{pmatrix} \frac{2a_{11}}{a_{12}} & 0 \\ 0 & \frac{2a_{22}}{a_{21}} \end{pmatrix}, \quad M^{III}(u) = \begin{pmatrix} \frac{u_2}{u_1} & 1 \\ 1 & \frac{u_1}{u_2} \end{pmatrix}.$$

the terms containing $|\nabla\sqrt{u_1}|^2$, $|\nabla\sqrt{u_2}|^2$ come from M^I, the terms with $|\nabla u_1|^2$, $|\nabla u_2|^2$ come from M^{II}, while M^{III} is positive semi-definite by Sylvester's criterion. The right-hand side derives from a straightforwad estimate of $\int_\Omega f(u) \cdot h'(u)dx$. Gronwall's Lemma allows us to bound the right-hand side by a suitable constant depending on the final time T.

We will prove the following

Theorem 3.2 (Global existence for SKT). *Let $a_{i0} \geq 0$, $b_{i0} \geq 0$, $a_{ij} > 0$, $b_{ij} \geq 0$ for $i, j = 1, 2$. Let $u^0 : \Omega \to [0, \infty)^2$ be a Lebesgue-measurable function such that $H[u^0] < \infty$. Then there exists a weak solution $u = (u_1, u_2) : \Omega \times [0, \infty) \to [0, \infty)^2$ to the SKT model* (109)–(113) *such that*

$$u \in L^2_{loc}(0, \infty; H^1(\Omega)) \cap W^{1,q'}_{loc}(0, \infty; W^{1,q}(\Omega)'), \qquad q = 2(d + 1).$$

Proof. Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$.We define, as always, the entropy variables as $w_i = \partial h/\partial u_i = \log u_i$, $i = 1, 2$. The transformation $u \mapsto w$ is invertible and its inverse is $u(w) = (e^{w_1}, e^{w_2})$. We also define the matrix $B(w) = A(u(w))(h''(u(w))^{-1}$.

Step 1 of the general existence theorem (Thr. 3.1) holds for the SKT model, too. We obtain the existence of solutions $w^k \in H^m(\Omega)$ to

$$\int_\Omega \left(\frac{u(w^k) - u(w^{k-1})}{\tau} \cdot \phi + \nabla\phi : B(w^k)\nabla w^k \right) dx + \varepsilon(w^k, \phi)_{H^m} = \int_\Omega f(u(w^k)) \cdot \phi \, dx, \quad (114)$$

for all $\phi \in H^m(\Omega)$. Moreover, the following discrete entropy inequality holds, which follows from the positivity of the coefficients a_{ii}, $i = 1, 2$:

$$\int_\Omega h(u(w^k))dx + \tau \sum_{j=1}^k \int_\Omega |\nabla u(w^j)|^2 dx + \varepsilon\tau \sum_{j=1}^k \|w^j\|_{H^m}^2 \le C. \quad (115)$$

We can define, as in the proof of Thr. 3.1, a piecewise-constant-in-time function $u^{(\tau)}(x, t)$ which interpolates in time the sequence $u(w^k)$ and a discrete time derivation operator D_τ ($D_\tau f(t) = (f(t) - f(t - \tau))/\tau$). In this case we do not have L^∞ bounds for u; however, we can exploit other bounds. In fact, from the entropy inequality a uniform bound for the entropy density $h(u^{(\tau)})$ in $L^\infty(0, T; L^1(\Omega))$ follows, which implies (through Csiszar-Kullback inequality) a uniform bound for $u^{(\tau)}$ in $L^\infty(0, T; L^1(\Omega))$. From Poincaré inequality it follows that $u^{(\tau)}$ is uniformely bounded in $L^2(0, T; H^1(\Omega))$. Thus in the end we are left with

$$\|u^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \sqrt{\varepsilon}\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \le C. \quad (116)$$

Let $p = 2+2/d$, $\theta = 2d(p-1)/((d+2)p) \in [0, 1]$. Notice that $\theta p = 2$. Gagliardo-Nirenberg inequality implies

$$\|u^{(\tau)}\|_{L^p(\Omega)} \le C\|u^{(\tau)}\|_{L^1(\Omega)}^{1-\theta}\|u^{(\tau)}\|_{H^1(\Omega)}^{\theta},$$

and so

$$\begin{aligned}
\|u^{(\tau)}\|_{L^p(\Omega\times(0,T))}^p &= \int_0^T \|u^{(\tau)}\|_{L^p(\Omega)}^p dt \le C \int_0^T \|u^{(\tau)}\|_{L^1(\Omega)}^{(1-\theta)p}\|u^{(\tau)}\|_{H^1(\Omega)}^{p\theta} dt \\
&= C \int_0^T \|u^{(\tau)}\|_{L^1(\Omega)}^{p-2}\|u^{(\tau)}\|_{H^1(\Omega)}^2 dt \\
&\le C\|u^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{p-2}\|u^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}^2 \le C.
\end{aligned}$$

This means that $u^{(\tau)}$ is uniformely bounded in $L^{2+2/d}(\Omega \times (0, T))$. Now we have to find a bound for $D_\tau u^{(\tau)}$. Let $\phi \in L^q(0, T; W^{m,q}(\Omega))$ with $q = 2(d + 1)$ be a test function. If $q' = q/(q - 1)$ (as usual) It follows

$$\begin{aligned}
\left| \int_0^T \int_\Omega D_\tau u^{(\tau)} \cdot \phi dx dt \right| &\le \|A(u^{(\tau)})\nabla u^{(\tau)}\|_{L^{q'}(\Omega\times(0,T))}\|\nabla\phi\|_{L^q(\Omega\times(0,T))} \\
&+ \varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))}\|\phi\|_{L^2(0,T;H^m(\Omega))} + \|f(u^{(\tau)})\|_{L^{q'}(\Omega\times(0,T))}\|\phi\|_{L^q(\Omega\times(0,T))}.
\end{aligned}$$

Since each component $A_{ij}(u)$ of $A(u)$ is a first-order polynomial in u_1, u_2, we deduce that $|A(u)\nabla u| \leq C(1+|u|)|\nabla u|$. Since $1/q' = 1/p + 1/2$ Hölder inequality implies

$$\|A(u^{(\tau)})\nabla u^{(\tau)}\|_{L^{q'}(\Omega\times(0,T))} \leq C(1+\|u^{(\tau)}\|_{L^p(\Omega\times(0,T))})\|\nabla u^{(\tau)}\|_{L^2(\Omega\times(0,T))} \leq C.$$

Moreover $\varepsilon\|w^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \leq C\sqrt{\varepsilon} \leq C$. Finally, since $f(u)$ depends at most quadratically on u and $2q' < p$,

$$\|f(u^{(\tau)})\|_{L^{q'}(\Omega\times(0,T))} \leq C(1+\|u^{(\tau)}\|^2_{L^{2q'}(\Omega\times(0,T))}) \leq C(1+\|u^{(\tau)}\|^2_{L^p(\Omega\times(0,T))}) \leq C.$$

By putting together all the previous estimates we conclude

$$\|D_\tau u^{(\tau)}\|_{L^{q'}(0,T;W^{m,q}(\Omega)'} \leq C.$$

We are in the condition to apply the Aubin-Lions lemma: up to a subsequence,

$$u^{(\tau)} \to u \qquad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{117}$$
$$u^{(\tau)} \rightharpoonup u \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{118}$$
$$\varepsilon w^{(\tau)} \to 0 \qquad \text{strongly in } L^2(0,T;H^m(\Omega)), \tag{119}$$
$$D_\tau u^{(\tau)} \rightharpoonup \partial_t u \qquad \text{weakly in } L^{q'}(0,T;W^{m,q}(\Omega)'). \tag{120}$$

Since $u^{(\tau)}$ is strongly convergent in $L^2(\Omega\times(0,T))$ and uniformely bounded in $L^{2+2/d}(\Omega\times(0,T))$ it follows (by L^p interpolation) that

$$u^{(\tau)} \to u \qquad \text{strongly in } L^r(\Omega\times(0,T)) \text{ for any } r < 2+2/d. \tag{121}$$

Since f depends at most quadratically on u and (121) holds,

$$f(u^{(\tau)}) \to f(u) \qquad \text{strongly in } L^r(\Omega\times(0,T)) \text{ for any } r < 1+1/d. \tag{122}$$

Moreover, since A depends at most linearly on u and (118), (121), hold,

$$A(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup A(u)\nabla u \qquad \text{weakly in } L^s(\Omega\times(0,T)) \text{ for any } s < 1+\frac{1}{1+2d}. \tag{123}$$

From (120), (122), (123) it follows that we can pass to the limit $\tau \to 0$ in (114) and find that u satisfies

$$\int_0^T \langle \partial_t u, \phi\rangle dt + \int_0^T\int_\Omega \nabla\phi : A(u)\nabla u dxdt = \int_0^T\int_\Omega f(u)\cdot\phi dxdt, \tag{124}$$

for ϕ smooth enought. However, since $\partial_t u \in L^{q'}(0,T;W^{1,q}(\Omega)')$, a standard density argument yields that (124) actually holds for any $\phi \in L^q(0,T;W^{1,q}(\Omega))$. Again, since $u \in W^{1,q'}(0,T;W^{1,q}(\Omega)') \hookrightarrow C([0,T],W^{1,q}(\Omega)')$, the initial condition is satisfied in the sense of $W^{1,q}(\Omega)'$, i.e. $\lim_{t\to 0} u(\cdot,t) = u_0$ in $W^{1,q}(\Omega)'$. This finishes the proof. □

Nonlinear SKT. The SKT model can be rewritten as

$$\partial_t u_i = \Delta(u_i p_i(u)), \qquad p_i(u) = a_{i0} + \sum_{j=1}^{2} a_{ij} u_j, \qquad i = 1, 2.$$

We can generalize the model by consider nonlinear functions p_i, like e.g.

$$p_i(u) = a_{i0} + \sum_{j=1}^{2} a_{ij} u_j^s, \qquad i = 1, 2.$$

The diffusion matrix A becomes

$$A(u), = \begin{pmatrix} a_{10} + (1+s)a_{11}u_1^s + a_{12}u_2^s & s a_{12} u_1 u_2^{s-1} \\ s a_{21} u_1^{s-1} u_2 & a_{20} + a_{21}u_1^s + (1+s)a_{22}u_2^s \end{pmatrix}.$$

The global existence of weak solutions was shown by Desvilletes et al. [19] for $0 < s < 1$. Jüngel [31] extended the result to the range $1 < s < 4$ under the weak cross-diffusion condition

$$\left(1 - \frac{1}{s}\right)^2 a_{12}a_{21} \le a_{11}a_{22}.$$

The restriction $s < 4$ is needed in the approximation procedure. Desvilletes et al. [20] employed another approximation method removed the restriction on s and relaxed the weak cross-diffusion condition to

$$\left(\frac{s-1}{s+1}\right)^2 a_{12}a_{21} \le a_{11}a_{22}.$$

The function

$$h(u) = \frac{a_{21}u_1^s + a_{12}u_2^s}{s(s-1)}$$

is an entropy density for the system, and it holds

$$v \cdot h''(u)A(u)v \ge a_{21}a_{11}u_1^{2(s-1)}v_1^2 + a_{12}a_{22}u_2^{2(s-1)}v_2^2.$$

If a_{11}, $a_{22} > 0$ the above inequality yields a bound for $\nabla(u_i^s)$ in $L^2(\Omega \times (0,T))$ and for u_i^s in $L^\infty(0,T; L^1(\Omega))$.

***n*-species population model.** Let us consider the linear SKT model for $n \ge 3$ species:

$$\partial_t u_i = \Delta(u_i p_i(u)), \qquad p_i(u) = a_{i0} + \sum_{j=1}^{n} a_{ij} u_j, \qquad i = 1, \dots, n.$$

The analysis of the above model is much more difficult than the case $n = 2$. Chen, Daus and Jüngel [10] showed that under the so-called *detailed balance condition*

$$\exists \pi \in (0,\infty)^n : \quad \pi_i a_{ij} = \pi_j a_{ji} \qquad i, j = 1, \dots, n,$$

the function

$$h(u) = \sum_{i=1}^{n} \pi_i u_i (\log u_i - 1)$$

is an entropy density for the system. In fact, the matrix $h''(u)A(u)$ is symmetric and positive semidefinite (and a suitable lower bound for it is available). As a matter of fact, the detailed balance condition is equivalent to the symmetry of $h''(u)A(u)$. We also wish to point out that such condition imposes constraints on the coefficients of A. For example, for $n = 3$ it must hold

$$a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21}.$$

Without the detailed balance condition h is not an entropy density: in fact, initial data can be chosen such that the functional $H[u] = \int_\Omega h(u)dx$ is increasing for small times. However, the detailed balance condition is not necessary for existence of solutions. In fact, under the weak cross-diffusion hypothesis

$$\frac{s}{2(s+1)} \sum_{j=1}^{n} (\sqrt{a_{ij}} - \sqrt{a_{ji}})^2 < a_{ii} \qquad i = 1, \dots, n,$$

a surrogate entropy inequality can be obtained, which implies the same gradient estimates that follow from the entropy inequality under the detailed balance condition.

3.4.4 Ion-Transport Models.

We will prove the global existence of bounded weak solutions to the two-species ion-transport model:

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = 0 \qquad \text{in } \Omega, \quad t > 0, \tag{125}$$

$$\nu \cdot \nabla u = 0 \qquad \text{on } \partial\Omega, \quad t > 0, \tag{126}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega, \tag{127}$$

and the diffusion matrix A given by

$$A(u) = \begin{pmatrix} D_1(1-u_2) & D_1 u_1 \\ D_2 u_2 & D_2(1-u_1) \end{pmatrix}. \tag{128}$$

Up to exchanging u_1 and u_2, we can assume $D_2 \geq D_1$. Burger et al. [8] have showed that (125)–(128) admits the entropy density:

$$h(u) = \sum_{i=1}^{3} u_i \log u_i, \quad u_3 \equiv 1 - u_1 - u_2, \quad (u_1, u_2) \in \mathcal{D} \equiv \{(u_1, u_2) \in (0,1)^2 \ : \ u_1 + u_2 < 1\}.$$

In fact, it holds

$$v \cdot h''(u)A(u)v$$

$$= D_1 u_3 \left(\frac{v_1^2}{u_1} + \frac{v_2^2}{u_2} \right) + D_1 \frac{1+u_3}{u_3}(v_1+v_2)^2 + (D_2 - D_1)\frac{u_2}{u_3}\left(v_1 + \frac{1-u_1}{u_2} v_2 \right)^2. \quad (129)$$

Since we assumed $D_2 \geq D_1$, it follows that $h''(u)A(u)$ is positive semidefinite for $(u_1, u_2) \in \mathcal{D}$. However, Hypothesis H2' is not satisfied due to the factor u_3 in front of the first term on the right-hand side of (129). There is a "degeneracy", that is, we lose control on the gradient of the solution near $u_3 = 0$. Anyway, even if we cannot apply the general existence result (Thr. 3.1) we can still employ a similar technique to show the following

Theorem 3.3 (Global existence for the ion-transport model)**.** *Let D_1, D_2 be positive constants, and let $u^0 : \Omega \to \overline{\mathcal{D}}$ be a Lebesgue-measurable function such that $H[u^0] < \infty$. Then there exists a weak solution $u : \Omega \times (0,T) \to \overline{\mathcal{D}}$ to* (125)*–*(128) *such that*

$$u_3^{1/2} u_i, u_3^{1/2} \in L^2_{loc}(0,\infty; H^1(\Omega)), \quad \partial_t u_i \in L^2_{loc}(0,\infty; H^1(\Omega)'), \quad (130)$$

with $u_3 = 1 - u_1 - u_2$. The function u satisfies the following weak formulation of (125)*–*(128)*:*

$$\sum_{i=1}^{2} \int_0^T \langle \partial_t u_i, \phi_i \rangle dt + \sum_{i=1}^{2} D_i \int_0^T \int_\Omega \left(u_3^{1/2} \nabla (u_3^{1/2} u_i) - 3 u_3^{1/2} u_i \nabla u_3^{1/2} \right) \cdot \nabla \phi_i dx dt = 0 \quad (131)$$

for all $\phi = (\phi_1, \phi_2) \in L^2_{loc}(0,T; H^1(\Omega))$ and $T > 0$.

Proof. We present here only the main ideas of the proof, since the full proof, which can be found in [8], is rather technical. Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$ (timestep), $w_i = \partial h / \partial u_i = \log(u_i/u_3)$ for $i = 1,2$ (entropy variables), $u_i(w) = e^{w_i}/(e^{w_1} + e^{w_2})$ for $i = 1,2$ (the inverse of the transformation $u \mapsto w$), $B(w) = A(u(w))h''(u(w))^{-1}$. Notice that $B(w)$ is positive semidefinite. Thanks to this fact, by proceeding like in the Step 1 of the proof of Thr. 3.1 we can show the existence of a weak solution to the truncated-regularized problem (97). Moreover, from (129) it follows

$$\int_\Omega h(u(w^{(\tau)}(x,T)))dx + \int_0^T \int_\Omega u_3^{(\tau)} \sum_{i=1}^{2} |\nabla (u_i^{(\tau)})^{1/2}|^2 dx dt$$
$$+ \int_0^T \int_\Omega |\nabla (u_3^{(\tau)})^{1/2}|^2 dx dt + \varepsilon \int_0^T \|w^{(\tau)}\|^2_{H^m} dt \leq C. \quad (132)$$

We cannot get from (132) any bound for ∇u_i alone, since u_3 might vanish. To overcome the troubles caused by this lack of gradient estimates we will exploit a generalized Aubin-Lions Lemma; however, this result requires a bound for the discrete time derivative $D_\tau u^{(\tau)}$ of $u^{(\tau)}$ in $L^2(0,T; H^1(\Omega)')$. As a consequence, we cannot take the limit $(\varepsilon, \tau) \to 0$. Therefore, we will first take the limit $\varepsilon \to 0$, and then $\tau \to 0$.

We do not write down here the details about the limit $\varepsilon \to 0$, since the main difficulties lie in the limit limit $\tau \to 0$. We just state that the limit $\varepsilon \to 0$ leads to the following equation:

$$\int_0^T \int_\Omega D_\tau u^{(\tau)} \cdot \phi dx dt$$

$$+\sum_{i=1}^{2} D_i \int_0^T \int_\Omega (u_3^{(\tau)})^{1/2} \Big(\nabla((u_3^{(\tau)})^{1/2} u_i^{(\tau)}) - 3u_i^{(\tau)} \nabla (u_3^{(\tau)})^{1/2}\Big) \cdot \nabla\phi \, dxdt = 0 \qquad (133)$$

for suitable test functions ϕ. Estimate (132) and the uniform L^∞ bounds for $u^{(\tau)}$ lead to

$$\|(u_3^{(\tau)})^{1/2} u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \|(u_3^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\Omega))} \le C, \quad i = 1, 2. \qquad (134)$$

From (133), (134) it follows

$$\|D_\tau u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le C, \quad i = 1, 2, 3. \qquad (135)$$

Aubin-Lion's Lemma implies that $u_3^{(\tau)} \to u_3$ strongly in $L^2(0,T;L^2(\Omega))$ (up to a subsequence). Furthermore,

$$(u_3^{(\tau)})^{1/2} \to u_3^{1/2} \qquad \text{strongly in } L^4(0,T;L^4(\Omega)), \qquad (136)$$

$$\nabla (u_3^{(\tau)})^{1/2} \rightharpoonup \nabla u_3^{1/2} \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \qquad (137)$$

The L^∞ bounds allows us to write that $u_i^{(\tau)} \rightharpoonup^* u_i$ weakly* in $L^\infty(0,T;L^\infty(\Omega))$, $i = 1, 2$. We apply now [32, Thr. A.6] with $y^{(\tau)} = (u_3^{(\tau)})^{1/2}$ and $z^{(\tau)} = u_i^{(\tau)}$ and deduce

$$(u_3^{(\tau)})^{1/2} u_i^{(\tau)} \to u_3^{1/2} u_i \qquad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

As a consequence

$$3(u_3^{(\tau)})^{1/2} u_i^{(\tau)} \nabla (u_3^{(\tau)})^{1/2} \rightharpoonup 3u_3^{1/2} u_i \nabla u_3^{1/2} \qquad \text{weakly in } L^1(0,T;L^1(\Omega)).$$

Moreover, since $(u_3^{(\tau)})^{1/2} u_i^{(\tau)}$ is bounded in $L^2(0,T;H^1(\Omega))$, we deduce that

$$\nabla((u_3^{(\tau)})^{1/2} u_i^{(\tau)}) \rightharpoonup \nabla(u_3^{1/2} u_i) \qquad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Therefore we can take the limit $\tau \to 0$ in (133) and obtain that (131) holds for ϕ smooth enough. As usual, a density argument allows us to deduce that (131) holds for all $\phi = (\phi_1, \phi_2) \in L^2_{loc}(0,T;H^1(\Omega))$. This finishes the proof. □

Theorem 3.3 can be generalized in various ways.

- Drift terms depending on the electric potential V can be included in the model, and the proof can be adjusted to cover this case. The entropy density will contain an additional term proportional to $(u_1 + u_2)\nabla V$.
- Eq. (128) describes the case of linear transition rates $q_i(u_3) = D_i u_3$. Nonlinear transition rates can be employed; for example, the power-law case $q_i(u_3) = D_i u_3^s$, $s > 0$, was considered in [31, Sect. 4].
- The $n-$species case was considered in [52]. The techniques are similar to those employed in the proof of Thr. 3.3, but the computations are more involved.
- Source terms $f(u)$ on the right-hand side of (125) can be considered, as long as they have the form $f(u) = f^{(0)}(u_3) + u_1 f^{(1)}(u_3) + u_2 f^{(2)}(u_3)$, since we only have weak convergence for $u_1^{(\tau)}$, $u_2^{(\tau)}$.

3.4.5 About uniqueness of weak solutions.

At this point, after having considered the problem of existence of weak solutions to cross-diffusion equations, the Reader might ask: *what are we going to do about uniqueness?* The part that follows is meant as an answer to this fairly legittimate question.

For the sake of simplicity, let us first consider the problem of showing uniqueness of weak solutions for the Fokker-Planck equation

$$\partial_t u - \operatorname{div}(\nabla u + u\nabla V) = 0 \qquad \text{in } \Omega, \quad t > 0, \tag{138}$$

$$(\nabla u + u\nabla V)\cdot\nu = 0 \qquad \text{on } \partial\Omega, \quad t > 0, \qquad u(\cdot,0) = u_0 \qquad \text{in } \Omega, \tag{139}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain and the potential $V = V(x,t)$ is a given function. Our goal is to prove a uniqueness result under minimal assumptions on V. We present two ways of doing it. In what follows, we assume for the sake of simplicity that the considered solutions of the Fokker-Planck are bounded.

First method: naive solution. This is the strategy that immediately comes to mind.

1. Consider two solutions u, v to (138), (139) with the same initial datum u_0;
2. take the difference of the equations satisfied by u, v;
3. test this newly found equation against $u - v$;
4. use integral inequalities and similar stuff to get something that would allow us to apply Gronwall's Lemma;
5. apply Gronwall's Lemma.

It is straightforward to see that after the third step in this strategy what we get is

$$\int_\Omega \frac{(u(x,t)-v(x,t))^2}{2}dx + \int_0^t\int_\Omega |\nabla(u-v)|^2 dxdt = -\int_0^t\int_\Omega (u-v)\nabla V\cdot\nabla(u-v)dxdt. \tag{140}$$

Let us consider the right-hand side of the above inequality:

$$-\int_0^t\int_\Omega (u-v)\nabla V\cdot\nabla(u-v)dxdt \leq \int_0^t \|u-v\|_{L^p(\Omega)}\|\nabla V\|_{L^q(\Omega)}\|\nabla(u-v)\|_{L^2(\Omega)}dt,$$

where $p, q \in (2,\infty)$ are such that $1/p + 1/q = 1/2$. We estimate the right-hand side of the above inequality by replacing $\|\nabla V\|_{L^q(\Omega)}$ with $\|\nabla V\|_{L^\infty(0,t;L^q(\Omega))}$ and by exploiting the following Gagliardo-Nirenberg inequality:

$$\|u-v\|_{L^p(\Omega)} \leq C\|\nabla(u-v)\|_{L^2(\Omega)}^\theta \|u-v\|_{L^2(\Omega)}^{1-\theta}, \qquad \theta = \frac{d(p-2)}{2p}.$$

It follows

$$
\begin{aligned}
&-\int_0^t\int_\Omega (u-v)\nabla V\cdot\nabla(u-v)dxdt \\
&\quad\le C\|\nabla V\|_{L^\infty(0,t;L^q(\Omega))}\int_0^t \|\nabla(u-v)\|_{L^2(\Omega)}^{1+\theta}\|u-v\|_{L^2(\Omega)}^{1-\theta}dt \\
&\quad\le C\|\nabla V\|_{L^\infty(0,t;L^q(\Omega))}\|u-v\|_{L^2(0,t;L^2(\Omega))}^{1-\theta}\cdot\|\nabla(u-v)\|_{L^2(0,t;L^2(\Omega))}^{1+\theta}.
\end{aligned}
$$

At this point we must assume $\theta < 1$ in order to control the right-hand side of (140), otherwise we would need an assumption of smallness on $\|\nabla V\|_{L^\infty(0,t;L^q(\Omega))}$. If $\theta < 1$ then we can apply Young inequality and obtain

$$
\begin{aligned}
&-\int_0^t\int_\Omega (u-v)\nabla V\cdot\nabla(u-v)dxdt \\
&\quad\le C\|\nabla V\|_{L^\infty(0,t;L^q(\Omega))}^{2/(1-\theta)}\|u-v\|_{L^2(0,t;L^2(\Omega))}^2+\frac{1}{2}\|\nabla(u-v)\|_{L^2(0,t;L^2(\Omega))}^2.
\end{aligned}
$$

Therefore we must assume that $\nabla V \in L^\infty_{loc}(0,\infty;L^q(\Omega))$. If this is true, the above inequality and (140) allow us to apply Gronwall's inequality and deduce that $\|u(t)-v(t)\|_{L^2(\Omega)} = 0$ for any $t > 0$. How much is q? Since $1/p + 1/q = 1/2$ and $\theta = \frac{d(p-2)}{2p}$, we deduce $q = d/\theta$. Being $\theta \in (0,1)$, this means $q > d$. Summarizing up, this first method allows us to show uniqueness of weak solutions provided that $\nabla V \in L^\infty_{loc}(0,\infty;L^q(\Omega))$ for some $q > d$.

Second method: entropy method. This idea is due to Gajewski [23]. Recall the Boltzmann entropy density $h(u) = u\log u$. Given two solutions u, v of (138), (139), define the "relative entropy between u, v"

$$
S(u,v) = \int_\Omega \left(h(u)+h(v)-2h\left(\frac{u+v}{2}\right)\right)dx.
$$

Since h is strictly convex it follows that $h(u)+h(v)-2h\left(\frac{u+v}{2}\right) \ge 0$ a.e. in Ω, $t > 0$. Clearly $S(u(t),v(t)) = 0$ if $u(t) = v(t)$ a.e. in Ω. On the other hand, Let $S(u(t),v(t)) = 0$. It follows that $h(u(t)) + h(v(t)) - 2h\left(\frac{u(t)+v(t)}{2}\right) = 0$ a.e. in Ω. Again, the strict convexity of h implies that $u(t) = v(t)$ a.e. in Ω. [6]

We wish to show that $t \mapsto S(u(t),v(t))$ is nonincreasing in time. Being $u(\cdot,0) = v(\cdot,0)$ this will imply uniqueness of solutions. Let us assume, for the sake of simplicity, that $u, v > 0$ a.e. in $\Omega\times(0,\infty)$ (this condition can be removed). Taking the time derivative of $S(u(t),v(t))$ leads to

$$
\begin{aligned}
\frac{d}{dt}S(u(t),v(t)) &= \langle\partial_t u,\log u\rangle + \langle\partial_t v,\log v\rangle - \langle\partial_t(u+v),\log(u+v)\rangle \\
&= -4\int_\Omega(|\nabla\sqrt{u}|^2+|\nabla\sqrt{v}|^2-|\nabla\sqrt{u+v}|^2)dx, \qquad t>0.
\end{aligned}
$$

[6]Furthermore, by means of a Taylor expansion it is possible to see that $S(u(t),v(t)) \ge c\|u(t)-v(t)\|_{L^2(\Omega)}^2$ with c depending on the L^∞ norm of u, v.

It turns out that the right-hand side of the above equality in nonpositive. In fact, define the function $F : [0, 1] \to \mathbb{R}$,

$$F(s) = \int_\Omega (|\nabla\sqrt{u}|^2 + |\nabla\sqrt{sv}|^2 - |\nabla\sqrt{u + sv}|^2)dx \qquad 0 \le s \le 1.$$

Clearly $F(0) = 0$. Moreover, a few straightforward computations lead to

$$F'(s) = \int_\Omega \left|\nabla\sqrt{v} - \sqrt{\frac{v}{u + sv}}\nabla\sqrt{u + sv}\right|^2 dx \ge 0 \qquad 0 \le s \le 1.$$

As a consequence $F(1) \ge F(0) = 0$. In particular $\frac{d}{dt}S(u(t), v(t)) \le 0$ for $t > 0$, thus showing uniqueness.

What kind of condition did we impose on V? Actually, V does not appear in the expression for $\frac{d}{dt}S(u(t), v(t))$, for all the contributions coming from the drift term cancel out. However, since in the computations integrals of the form $\int_\Omega \nabla u \cdot \nabla V dx$ appear, and it is reasonable to expect that $\nabla u \in L^2(0, T; L^2(\Omega))$, so we must assume that $\nabla V \in L^2(0, T; L^2(\Omega))$. This is the only assumption on V, which represents an improvement with respect to the first method of showing uniqueness.

Finally, what happens if u, v are only nonnegative? A possible solution is to replace the function h inside $S(u, v)$ with its regularized version $h_\varepsilon(u) = h(u + \varepsilon)$. The argument presented in the previous case works in this situation, too.

Can we apply this argument to the ion-transport model (125)–(128)? Yes, if we assume that $D_i = 1$ for $i = 1, 2$. Under this assumption the equations take the form

$$\partial_t u_i = \operatorname{div}(u_3\nabla u_i - u_i\nabla u_3) \qquad i = 1, 2. \tag{141}$$

Summing up the equations in (141) we deduce that u_3 satisfies the heat equation: $\partial_t u_3 = \Delta u_3$. Of course, uniqueness for this equation is immediate. The uniqueness for the other components is shown by means of the entropy argument that we have presented just now. The following result is proved in [52].

Theorem 3.4. *Let $D_1 = D_2 = 1$. Then there exists at most one bounded weak solution to* (125)–(128) *in the class of functions satisfying* (130).

Proof. Given two solutions $u = (u_1, u_2)$, $v = (v_1, v_2)$ to (125)–(128) with the same initial datum, let $S(u(t), v(t))$ be the relative entropy between u, v, in the same way as in the previous argument. Moreover define

$$S_\varepsilon(u, v) = \sum_{i=1}^2 \int_\Omega \left(h_\varepsilon(u_i) + h_\varepsilon(v_i) - 2h_\varepsilon\left(\frac{u_i + v_i}{2}\right)\right) dx, \qquad h_\varepsilon(s) = h(s + \varepsilon),$$

and $h(s) = s \log s$, $s > 0$. Computing and estimating the time derivative of $S_\varepsilon(u(t), v(t))$ and then integrating in time leads to

$$S_\varepsilon(u(t), v(t)) \le 2\sum_{i=1}^2 \int_0^t \int_\Omega \left(\frac{u_i}{u_i + \varepsilon} - \frac{u_i + v_i}{u_i - v_i + 2\varepsilon}\right) \sqrt{u_n}\nabla\sqrt{u_n} \cdot \nabla u_i dx$$

$$+ 2\sum_{i=1}^{2}\int_0^t\int_\Omega\left(\frac{v_i}{v_i+\varepsilon}-\frac{u_i+v_i}{u_i-v_i+2\varepsilon}\right)\sqrt{u_n}\nabla\sqrt{u_n}\cdot\nabla v_i dx.$$

The dominated convergence theorem and (130) imply that the right-hand side tends to 0 as $\varepsilon \to 0$. From this fact and Fatou's Lemma we deduce that $S(u(t), v(t)) = 0$ for $t > 0$, i.e. $u = v$ a.e. in Ω, $t > 0$. This finishes the proof. □

The above result can also be proved for an arbitrary number of species by following the same strategy.

3.5 Further examples of cross-diffusion PDEs.

In the part that follows, a couple more examples of cross-diffusion PDEs will be presented: a class of energy-transport equations describing the evolution of charge density and temperature of a fluid of particles in a semiconductor, and a cross-diffusion system with Laplacian structure derived from a Fokker-Planck equation for a probability density associated to some stochastic process. For both models, existence of nonnegative weak solutions will be shown, and the long-time behaviour of solutions will be studied.

3.5.1 Energy-transport models.

We refer to [53] for this part.

We aim to prove the global well-posedness of the energy-transport equations

$$\partial_t n = \Delta(n\theta^{1/2-\beta}), \quad \partial_t(n\theta) = \kappa\Delta(n\theta^{3/2-\beta}) + \frac{n}{\tau}(1-\theta) \quad \text{in } \Omega,\ t > 0, \tag{142}$$

where $-\frac{1}{2} \le \beta < \frac{1}{2}$, $\kappa = \frac{2}{3}(2-\beta)$, and $\Omega \subset \mathbb{R}^d$ with $d \le 3$ is a bounded domain. This system describes the evolution of a fluid of particles with density $n(x,t)$ and temperature $\theta(x,t)$. The parameter $\tau > 0$ is the relaxation time, which is the typical time of the system to relax to the thermal equilibrium state of constant temperature. The system arises in the modeling of semiconductor devices in which the elastic electron-phonon scattering is dominant. The above model is a simplification for vanishing electric fields. The full model was derived from the semiconductor Boltzmann equation in the diffusion limit using a Chapman-Enskog expansion around the equilibrium distribution [6]. The parameter β appears in the elastic scattering rate [30, Section 6.2]. Certain values were used in the physical literarure, for instance $\beta = \frac{1}{2}$ [15], $\beta = 0$ [37], and $\beta = -\frac{1}{2}$ [30, Chapter 9]. The choice $\beta = \frac{1}{2}$ leads in our situation to two uncoupled heat equations for n and $n\theta$ and does not need to be considered. We impose physically motivated mixed Dirichlet-Neumann boundary and initial conditions

$$n = n_D,\ \theta = \theta_D \quad \text{on } \Gamma_D, \quad \nabla(n\theta^{1/2-\beta})\cdot\nu = \nabla(n\theta^{3/2-\beta})\cdot\nu = 0 \quad \text{on } \Gamma_N,\ t > 0, \tag{143}$$

$$n(0) = n_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega, \tag{144}$$

where Γ_D models the contacts, $\Gamma_N = \partial\Omega \backslash \Gamma_D$ the union of insulating boundary segments, and ν is the exterior unit normal to $\partial\Omega$ which is assumed to exist a.e.

The mathematical analysis of (142)-(144) is challenging since the equations are not in the usual divergence form, they are strongly coupled, and they degenerate at $\theta = 0$ (in this regard they are similar to the ion-transport model (125)–(128)). The strong coupling makes impossible to apply maximum principle arguments in order to conclude the nonnegativity of the temperature θ.

On the other hand, this system possesses an interesting mathematical structure. First, it can be written in "symmetric" form by introducing the so-called entropy variables $w_1 = \log(n/\theta^{3/2})$ and $w_2 = -1/\theta$. Then, setting $w = (w_1, w_2)^\top$ and $\rho = (n, \frac{3}{2}n\theta)^\top$, (142) is formally equivalent to

$$\partial_t \rho = \operatorname{div}\left(A(n,\theta)\nabla w\right) + \frac{1}{\tau}\begin{pmatrix} 0 \\ n(1-\theta) \end{pmatrix},$$

where the diffusion matrix

$$A(n,\theta) = n\theta^{1/2-\beta}\begin{pmatrix} 1 & (2-\beta)\theta \\ (2-\beta)\theta & (3-\beta)(2-\beta)\theta^2 \end{pmatrix}$$

is symmetric and positive semi-definite. Second, system (142) possesses the entropy (or free energy)

$$S[n(t),(n\theta)(t)] = \int_\Omega n \log \frac{n}{\theta^{3/2}} dx,$$

which is nonincreasing along smooth solutions to (142). Even more entropy functionals exist; see [34] and below. However, they do not provide a lower bound for θ when n vanishes. We notice that both properties, the symmetrization via entropy variables and the existence of an entropy, are strongly related [17, 30].

Equations (142) resemble the diffusion equation $\partial_t w = \Delta(a(x,t)w)$, which was analyzed by Pierre and Schmitt [40]. By Pierre's duality estimate, an L^2 bound for $\sqrt{a}w$ in terms of the L^2 norm of $\sqrt{a}$ has been derived. In our situation, we obtain even H^1 estimates for $w = n$ and $w = n\theta$.

In spite of the above structure, there are only a few analytical results for (142)-(144).

- Drift-diffusion equations with temperature-dependent mobilities but without temperature gradients [51] (also see [49]) or nonisothermal systems containing simplified thermodynamic forces [1] have been studied.
- Xu included temperature gradients in the model but he truncated the Joule heating to allow for a maximum principle argument [50].
- Later, existence results for the complete energy-transport equations (including electric fields) have been achieved, see [22, 27] for stationary solutions near thermal equilibrium, [11, 12] for transient solutions close to equilibrium, and [16, 18] for systems with **uniformly positive** definite diffusion matrices (this assumption on the diffusion matrix avoids the degeneracy at $\theta = 0$).

- A degenerate energy-transport system was analyzed in [33], but only a simplified (stationary) temperature equation was studied.

All these results give partial answers to the well-posedness problem only. In [53] Jüngel and Zamponi proved for the first time a global-in-time existence result for any data and with physical transport coefficients.

Surprisingly, the logarithmic entropy structure does not help. Our key idea is to use the new variables $u = n\theta^{1/2-\beta}$ and $v = n\theta^{3/2-\beta}$ and nonlogarithmic entropy functionals. Then system (142) becomes

$$\partial_t N(u,v) = \Delta u, \quad \partial_t E(u,v) = \kappa \Delta v + R(u,v),$$

where $N(u,v) = u^{3/2-\beta} v^{\beta-1/2}$, $E(u,v) = u^{1/2-\beta} v^{\beta+1/2}$, and $R(u,v) = \tau^{-1} N(u,v)(1 - v/u)$. Discretizing this system by the implicit Euler method and employing the Stampacchia truncation method and a particular cut-off test function, we are able to prove the nonnegativity of u, v, and θ.

In the following, we detail our main results and explain the ideas of the proofs. Let $\partial\Omega \in C^1$, $\text{meas}(\Gamma_D) > 0$, and Γ_N is relatively open in $\partial\Omega$. Furthermore, let

$$n_D,\ \theta_D \in L^\infty(\Omega) \cap H^1(\Omega), \quad \inf_{\Gamma_D} n_D > 0,\ \inf_{\Gamma_D} \theta_D > 0, \tag{145}$$

$$n_0,\ \theta_0 \in L^\infty(\Omega) \cap H^1(\Omega), \quad \inf_{\Omega} n_0 > 0,\ \inf_{\Omega} \theta_0 > 0. \tag{146}$$

We define the space $H^1_D(\Omega)$ as the closure of $C_0^\infty(\Omega \cup \Gamma_N)$ in the H^1 norm [46, Section 1.7.2]. This space can be characterized by all functions in $H^1(\Omega)$ which vanish on Γ_D in the weak sense. This space is the test function space for the weak formulation of (142). Our first main result reads as follows.

Theorem 3.5 (Global existence)**.** *Let $T > 0$, $d \le 3$, $-\frac{1}{2} \le \beta < \frac{1}{2}$, $\tau > 0$ and let* (145)*-*(146) *hold. Then there exists a weak solution (n,θ) to* (142)*-*(144) *such that $n > 0$, $n\theta > 0$ in Ω, $t > 0$, satisfying*

$$n,\ n\theta,\ n\theta^{1/2-\beta},\ n\theta^{3/2-\beta} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$
$$\partial_t n,\ \partial_t(n\theta) \in L^2(0,T;H^1_D(\Omega)').$$

The idea of the proof is to employ the implicit Euler method with time step $h > 0$ and the new variables $u_j = n_j \theta_j^{1/2-\beta}$ and $v_j = n_j \theta_j^{3/2-\beta}$, which approximate $u = n\theta^{1/2-\beta}$ and $v = n\theta^{3/2-\beta}$ at time $t_j = jh$, respectively. We wish to solve

$$(n_j - n_{j-1}) - h\Delta u_j = 0, \quad \frac{1}{\kappa}(n_j\theta_j - n_{j-1}\theta_{j-1}) - h\Delta v_j = \frac{hn_j}{\kappa\tau}(1-\theta). \tag{147}$$

To simplify the presentation, we ignore the boundary conditions and a necessary truncation of the temperature. A nice feature of this formulation is that we can apply a Stampacchia truncation procedure to prove the strict positivity of u_j and v_j.

We point out a significant difference between this approach and the one employed in the existence proof of the ion-transport model (125)–(128). While in the latter a regularising term of the form $\Delta^m w$ (with w being the entropy variable) was added to the discretized equation, in (147) no such regularization in considered; indeed, u, v are not even entropy variables. As a consequence, the boundedness of u_j, v_j is not guaranteed; in particular, we don't know if the ratio $\theta_j = v_j/u_j$ is strictly positive. As a matter of fact, to show the strict positivity of θ_j is one of the main difficulties of the proof.

We define a nondecreasing smooth cut-off function ϕ such that $\phi(x) = 0$ if $x \le M$ and $\phi(x) > 0$ if $x > M$ for some $M > 0$. We use the test functions $u_j\phi(1/\theta_j)$ and $v_j\phi(1/\theta_j)$ in the weak formulation of (147), respectively, and we subtract both equations to find after a straightforward computation (see Step 3 in the proof of Theorem 3.5) that

$$0 = \int_\Omega \Bigg(\left(1 - \frac{1}{\kappa} - \frac{h}{\kappa\tau}\right) n_j v_j \phi\left(\frac{1}{\theta_j}\right) + \frac{v_j}{\kappa} n_{j-1}\theta_{j-1}\left(\frac{1}{\theta_j} - \frac{\kappa}{\theta_{j-1}}\right)\phi\left(\frac{1}{\theta_j}\right) \\ + \frac{h}{v_j^2}\left|v_j\nabla u_j - u_j\nabla v_j\right|^2 \phi'\left(\frac{1}{\theta_j}\right) + \frac{h n_j\theta_j v_j}{\kappa\tau}\phi\left(\frac{1}{\theta_j}\right)\Bigg)dx.$$

Since $\kappa > 1$, there exists $h > 0$ sufficiently small such that the first summand becomes nonnegative. The third and last summands are nonnegative, too. (Recall that we need to truncate θ_j with positive truncation.) Hence, the integral over the second term is nonpositive. Then, choosing $M \ge \kappa/\theta_{j-1}$,

$$0 \ge \int_\Omega v_j n_{j-1}\theta_{j-1}\left(\frac{1}{\theta_j} - \frac{\kappa}{\theta_{j-1}}\right)\phi\left(\frac{1}{\theta_j}\right)dx \ge \int_\Omega v_j n_{j-1}\theta_{j-1}\left(\frac{1}{\theta_j} - M\right)\phi\left(\frac{1}{\theta_j}\right)dx.$$

Because $\phi(1/\theta_j) = 0$ for $1/\theta_j \le M$, this is only possible if $1/\theta_j - M \le 0$ or $\theta_j \ge 1/M > 0$. Clearly, the bound M depends on j, and in the de-regularization limit $h \to 0$, the limit of θ_j becomes nonnegative only.

A priori estimates which are uniform in the approximation parameter $h > 0$ are obtained by proving an entropy inequality for the an entropy functional having the structure

$$S_{b_1,b_2}[n, n\theta] = \int_\Omega n^2(\theta^{b_1} + \theta^{b_2})dx \tag{148}$$

for suitable exponents $b_1, b_2 \in (-\infty, 2)$, which looks like

$$S_{b_1,b_2}[n_j, n_j\theta_j] + h\int_\Omega (p_1(\theta_j)|\nabla n_j|^2 + n^2 p_2(\theta_j)|\nabla\theta_j|^2)dx \le S_{b_1,b_2}[n_{j-1}, n_{j-1}\theta_{j-1}], \quad j \ge 1,$$

where $p_1(\theta)$, $p_2(\theta)$ are sums of power functions with different exponents. This inequality allows us to derive gradient estimates for n_j, $n_j\theta_j^{1/2-\beta}$, and $n_j\theta_j^{3/2-\beta}$. Together with Aubin's lemma and weak compactness arguments, the limit $h \to 0$ can be performed.

We point out another interesting difference between this energy-transport model and the cross-diffusion systems that we have previously studied. One of the requirement of the general existence result (Thr. 3.1) was the (global) invertibility of the mapping $h' : \mathcal{D} \to \mathbb{R}^n$.

In all the systems we studied before, such hypothesis (H1) was always satisfied. However, in the proof of Thr. 3.5 we have employed the entropy functional (148), which *does not satisfy this assumption.*[7] In light of this fact we can understand why no regularization was used in (147): if a high-order regularizing term was added in (147), algebraic bounds for the entropy variables would have to be showed, and that would be either impossible or quite tricky.

Theorem 3.5 can be generalized in different ways. First, the boundary data may depend on time. We do not consider this case here to avoid technicalities. We refer to [16] for the treatment of time-dependent boundary functions. Second, we may allow for temperature-dependent relaxation times,

$$\tau(\theta) = \tau_0 + \tau_1 \theta^{1/2-\beta}, \tag{149}$$

where $\tau_0 > 0$ and $\tau_1 > 0$. This expression can be derived by using an energy-dependent scattering rate [30, Example 6.8]. For this relaxation time, the conclusion of Theorem 3.5 holds.

Corollary 3.1 (Global existence)**.** *Let the assumptions of Theorem 3.5 hold except that the relaxation time is given by* (149)*. Then there exists a weak solution to* (142)-(144) *with the properties stated in Theorem 3.5.*

However, we have not been able to include electric fields in the model. For instance, in this situation, the first equation in (142) becomes

$$\partial_t n = \operatorname{div}\left(\nabla(n\theta^{1/2-\beta}) + n\theta^{-1/2-\beta}\nabla V\right),$$

where $V(x,t)$ is the electric potential which is a given function or the solution of the Poisson equation [30]. The problem is the treatment of the drift term $n\theta^{-1/2-\beta}\nabla V$ for which the techniques developed for the standard drift-diffusion model (see, e.g., [24]) do not apply.

It is possible to prove the following result about the long-time behavior of the solutions.

Theorem 3.6 (Long-time behavior)**.** *Let $d \leq 3$, $0 \leq \beta < \frac{1}{2}$, $\tau > 0$, and $n_D = const.$, $\theta_D = 1$. Let (n,θ) be the weak solution constructed in Theorem 3.5. Then there exist constants C_1, $C_2 > 0$, which depend only on β, n_D, n_0, and θ_0, such that for all $t > 0$,*

$$\|n(t) - n_D\|^2_{L^2(\Omega)} + \|n(t)\theta(t) - n_D\|^2_{L^2(\Omega)} \leq \frac{C_1}{1 + C_2 t}.$$

The proof of this theorem is based on discrete entropy inequality estimates. The main difficulty is to bound the entropy dissipation. Usually, this is done by employing a convex Sobolev inequality (e.g. the logarithmic Sobolev or Beckner inequality). However, these tools are not available for the cross-diffusion system at hand, and we need to employ another technique. Our idea is to estimate the entropy dissipation by using a power-like

[7] For example, the entropy density $s = s(n, n\theta)$ associated to (148) with $(b_1, b_2) = (\beta - 1/2, 5)$ (as it is used in the existence proof) satisfies $s'((0,\infty)^2) \cap (-\infty, 0)^2 = \emptyset$, therefore $s' : (0,\infty)^2 \to \mathbb{R}^2$ cannot be a bijection.

entropy like (148). Denoting the discrete (nonlogarithmic) entropy at time t_j by $S[n_j, n_j\theta_j]$, we arrive at the inequality

$$S[n_j, n_j\theta_j] - S[n_{j-1}, n_{j-1}\theta_{j-1}] \leq ChS[n_j, n_j\theta_j]^2,$$

where $C > 0$ is independent of the time step size h. A discrete nonlinear Gronwall lemma then shows that $S[n_j, n_j\theta_j]$ behaves like $1/(hj) = 1/t_j$, and in the limit $h \to 0$, we obtain the result.

We will only prove the global existence theorem, i.e. Thr. 3.5; the proof of the above result concerning the long-time behaviour of solutions will be skipped. The curious Reader can find it in [53].

Proof of Thr. 3.5. We prove here the existence theorem. We will skip some technical details; for the full proof see [53].

Step 1: Reformulation. Let $T > 0$, $N \in \mathbb{N}$, and set $h = T/N$. We consider the semi-discrete equations

$$\frac{1}{h}(n_j - n_{j-1}) = \Delta(n_j\theta_j^{1/2-\beta}), \quad j = 1, \ldots, N, \tag{150}$$

$$\frac{1}{h}(n_j\theta_j - n_{j-1}\theta_{j-1}) = \kappa\Delta(n_j\theta_j^{3/2-\beta}) + \frac{1}{\tau}n_j(1-\theta_j) \tag{151}$$

with the boundary conditions (143). The idea is to reformulate the elliptic equations in terms of the new variables

$$u_j = n_j\theta_j^{1/2-\beta}, \quad v_j = n_j\theta_j^{3/2-\beta}.$$

Observing that $n_j = u_j^{3/2-\beta}v_j^{\beta-1/2}$ and $\theta_j = v_j/u_j$, equations (150)-(151) are formally equivalent to

$$u_j^{3/2-\beta}v_j^{\beta-1/2} - h\Delta u_j = u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2}, \tag{152}$$

$$u_j^{1/2-\beta}v_j^{\beta+1/2} - \kappa h\Delta v_j - \frac{h}{\tau}u_j^{1/2-\beta}v_{j-1}^{\beta-1/2}(u_j - v_j) = u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}. \tag{153}$$

The boundary conditions become

$$u_j = u_D := n_D\theta_D^{1/2-\beta}, \quad v_j = v_D := n_D\theta_D^{3/2-\beta} \quad \text{on } \Gamma_D, \tag{154}$$

$$\nabla u_j \cdot \nu = \nabla v_j \cdot \nu = 0 \quad \text{on } \Gamma_N. \tag{155}$$

In order to show the existence of weak solutions to this discretized system, we need to truncate. For this, let $j \geq 1$ and let u_{j-1}, $v_{j-1} \in L^2(\Omega)$ be given such that $\inf_\Omega u_{j-1} > 0$, $\inf_\Omega v_{j-1} > 0$, $\sup_\Omega u_{j-1} < +\infty$, and $\sup_\Omega v_{j-1} < +\infty$. We define

$$M = \max\left\{\kappa \sup_\Omega \frac{u_{j-1}}{v_{j-1}}, \frac{1}{\inf_{\Gamma_D}\theta_D}\right\} \tag{156}$$

and $\varepsilon = 1/M$. The truncated problem reads as

$$u_j\theta_{j,\varepsilon}^{\beta-1/2} - h\Delta u_j = u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2}, \tag{157}$$

$$\left(1+\frac{h}{\tau}\right)v_j\theta_{j,\varepsilon}^{\beta-1/2} - \kappa h\Delta v_j - \frac{h}{\tau}u_j\theta_{j,\varepsilon}^{\beta-1/2} = u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}, \tag{158}$$

where $\theta_{j,\varepsilon} = \max\{\varepsilon, v_j/u_j\}$. Note that if $u_j > 0$ and $v_j/u_j \geq \varepsilon$ in Ω then (157)-(158) are equivalent to (152)-(153).

Step 2: Solution of the truncated semi-discrete problem. We define the operator $F : L^2(\Omega)\times[0,1] \to L^2(\Omega)$ by $F(\theta,\sigma) = v/u$, where $(u,v) \in H^1(\Omega)^2$ is the unique solution to the linear system

$$\sigma u\theta_\varepsilon^{\beta-1/2} - h\Delta u = \sigma u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2} = \sigma u_{j-1}\left(\frac{u_{j-1}}{v_{j-1}}\right)^{1/2-\beta}, \tag{159}$$

$$\sigma\left(1+\frac{h}{\tau}\right)v\theta_\varepsilon^{\beta-1/2} - \kappa h\Delta v - \sigma\frac{h}{\tau}u\theta_\varepsilon^{\beta-1/2} = \sigma u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2} = \sigma v_{j-1}\left(\frac{u_{j-1}}{v_{j-1}}\right)^{1/2-\beta}, \tag{160}$$

where $\theta_\varepsilon = \max\{\varepsilon, \theta\}$, with the boundary conditions

$$u = 1+\sigma(u_D-1),\ v = \sigma v_D \quad \text{on } \Gamma_D, \quad \nabla u\cdot\nu = \nabla v\cdot\nu = 0 \quad \text{on } \Gamma_N. \tag{161}$$

We have to prove that the operator F is well defined.

First, observe that (159) does not depend on v and that the right-hand side is an element of $L^2(\Omega)$. Therefore, by standard theory of elliptic equations, we infer the existence of a unique solution $u \in H^1(\Omega)$ to (159) with the corresponding boundary conditions in (161). With given u, there exists a unique solution $v \in H^1(\Omega)$ to (160) with the corresponding boundary conditions. It remains to show that u and v are strictly positive in Ω such that the quotient v/u is defined and an element of $L^2(\Omega)$.

To this end, we employ the Stampacchia truncation method. Let

$$m_1 = \min\left\{\inf_{\Gamma_D} u_D, \varepsilon^{1/2-\beta}\inf_\Omega u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2}\right\} > 0.$$

Note that $m_1 > 0$ because of our boundedness assumptions on $\inf_\Omega u_{j-1}$ and $\sup_\Omega v_{j-1}$. Then $(u-m_1)_- = \min\{0, u-m_1\} \in H^1_D(\Omega)$ is an admissible test function in the weak formulation of (159) yielding

$$\begin{aligned} h\int_\Omega|\nabla(u-m_1)_-|^2dx + \sigma\int_\Omega \theta_\varepsilon^{\beta-1/2}(u-m_1)_-^2dx \\ = \sigma\int_\Omega\left(u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2} - m_1\theta_\varepsilon^{\beta-1/2}\right)(u-m_1)_-dx \\ \leq \sigma\int_\Omega\left(u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2} - m_1\varepsilon^{\beta-1/2}\right)(u-m_1)_-dx \leq 0, \end{aligned}$$

taking into account $\theta_\varepsilon^{\beta-1/2} \leq \varepsilon^{\beta-1/2}$ (observe that $\beta < 1/2$) and the definition of m_1. This implies that $(u - m_1)_- = 0$ and consequently $u \geq m_1 > 0$ in Ω. Defining

$$m_2 = \min\left\{\inf_{\Gamma_D} v_D, \left(1 + \frac{h}{\tau}\right)^{-1} \varepsilon^{1/2-\beta} \inf_\Omega u_{j-1}^{1/2-\beta} v_{j-1}^{\beta+1/2}\right\} > 0$$

and employing the test function $(v - m_2)_- \in H_D^1(\Omega)$ in the weak formulation of (160), a similar computation as above and $\theta_\varepsilon^{\beta-1/2} \leq \varepsilon^{\beta-1/2}$ yield

$$\begin{aligned} \kappa h \int_\Omega |\nabla(v - m_2)_-|^2 dx + \sigma\left(1 + \frac{h}{\tau}\right) \int_\Omega \theta_\varepsilon^{\beta-1/2}(v - m_2)_-^2 dx - \frac{\sigma h}{\tau}\int_\Omega u\theta_\varepsilon^{\beta-1/2}(v - m_2)_- dx \\ = \sigma \int_\Omega \left((u_{j-1}^{1/2-\beta} v_{j-1}^{\beta+1/2} - \left(1 + \frac{h}{\tau}\right) m_2 \theta_\varepsilon^{\beta-1/2}\right)(v - m_2)_- dx \leq 0. \end{aligned}$$

Since the integrals on the left-hand side are nonnegative, we conclude that $v \geq m_2 > 0$ in Ω. This shows that u and v are strictly positive with a lower bound which depends on ε and j. Because of $1/u \in L^\infty(\Omega)$ and $u, v \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, $v/u \in W^{1,3/2}(\Omega) \hookrightarrow L^2(\Omega)$ for $d \leq 3$. Hence, the operator F is well defined and its image is contained in $W^{1,3/2}(\Omega)$.

Standard arguments and the compact embedding $W^{1,3/2}(\Omega) \hookrightarrow L^2(\Omega)$ ensure that F is continuous and compact. When $\sigma = 0$, it follows that $u = 1$ and $v = 0$ and thus, $F(\theta, 0) = 0$. Let $\theta \in L^2(\Omega)$ be a fixed point of $F(\cdot, \sigma)$. Then $v/u = \theta$. By standard elliptic estimates, we obtain H^1 bounds for u and v independently of σ. Since u is strictly positive, we infer an L^2 bound for θ independently of σ. Thus, we may apply the Leray-Schauder fixed-point theorem to conclude the existence of a fixed point of $F(\cdot, 1)$, i.e. of a solution $(u, v) = (u_j, v_j) \in H^1(\Omega)^2$ to (157)-(158) with boundary conditions (154)-(155).

In order to close the recursion, we need to show that $\sup_\Omega u_j < +\infty$ and $\sup_\Omega v_j < +\infty$. We employ the following result which is due to Stampacchia [44]: Let $w \in H^1(\Omega)$ be the unique solution to $-\Delta w + a(x)w = f$ with mixed Dirichlet-Neumann boundary conditions and let $a \in L^\infty(\Omega)$ be nonnegative and $f \in L^s(\Omega)$ with $s > d/2$. Then $w \in L^\infty(\Omega)$ with a bound which depends only on f, Ω, and the boundary data. Since the right-hand side of (159) is an element of $L^2(\Omega)$ and $d \leq 3$, we find from the above result that the solution u to (159) is bounded. Furthermore, v solves (see (160))

$$\sigma\left(1 + \frac{h}{\tau}\right) v\theta_\varepsilon^{\beta-1/2} - \kappa h \Delta v = \sigma\frac{h}{\tau} u\theta_\varepsilon^{\beta-1/2} + \sigma u_{j-1}^{1/2-\beta} v_{j-1}^{\beta+1/2} \in L^\infty(\Omega),$$

taking advantage of the L^∞ bound for u. By Stampacchia's result, $v \in L^\infty(\Omega)$. This shows the desired bounds.

Step 3: Removing the truncation. We introduce the function

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq M, \\ 1 + \cos(\pi x/M) & \text{if } M \leq x \leq 2M, \\ 2 & \text{if } x \geq 2M, \end{cases}$$

where we recall the definition (156) of M. In particular, $\phi \in C^1(\mathbb{R})$ satisfies $\phi' \geq 0$ in $\mathbb{R}$. Since $M \geq 1/\inf_{\Gamma_D} \theta_D$, we have $\phi(u_j/v_j) = \phi(u_D/v_D) = \phi(1/\theta_D) = 0$ on Γ_D. Because ϕ' vanishes outside of the interval $[M, 2M]$, it holds that $u_j\phi(u_j/v_j)$, $v_j\phi(u_j/v_j) \in H^1(\Omega)$. Consequently, $v_j\phi(u_j/v_j)$ and $\kappa^{-1}u_j\phi(u_j/v_j)$ are admissible test functions in $H^1_D(\Omega)$ for (157) and (158), respectively, which gives the two equations

$$\int_\Omega u_j\theta_{j,\varepsilon}^{\beta-1/2} v_j\phi\left(\frac{u_j}{v_j}\right)dx + h\int_\Omega \nabla u_j\cdot\nabla\left(v_j\phi\left(\frac{u_j}{v_j}\right)\right)dx = \int_\Omega u_{j-1}^{3/2-\beta}v_{j-1}^{\beta-1/2}v_j\phi\left(\frac{u_j}{v_j}\right)dx,$$

$$\begin{aligned}\frac{1}{\kappa}\left(1+\frac{h}{\tau}\right)\int_\Omega v_j\theta_{j,\varepsilon}^{\beta-1/2}u_j\phi\left(\frac{u_j}{v_j}\right)dx + h\int_\Omega \nabla v_j\cdot\nabla\left(u_j\phi\left(\frac{u_j}{v_j}\right)\right)dx \\ - \frac{h}{\kappa\tau}\int_\Omega u_j^2\theta_{j,\varepsilon}^{\beta-1/2}\phi\left(\frac{u_j}{v_j}\right)dx = \frac{1}{\kappa}\int_\Omega u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}u_j\phi\left(\frac{u_j}{v_j}\right)dx.\end{aligned}$$

We take the difference of these equations:

$$\begin{aligned}&\left(1-\frac{1}{\kappa}\left(1+\frac{h}{\tau}\right)\right)\int_\Omega u_jv_j\theta_{j,\varepsilon}^{\beta-1/2}\phi\left(\frac{u_j}{v_j}\right)dx \\ &\quad + h\int_\Omega (v_j\nabla u_j - u_j\nabla v_j)\cdot\nabla\phi\left(\frac{u_j}{v_j}\right)dx + \frac{h}{\kappa\tau}\int_\Omega u_j^2\theta_{j,\varepsilon}^{\beta-1/2}\phi\left(\frac{u_j}{v_j}\right)dx \\ &\quad + \frac{1}{\kappa}\int_\Omega u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}v_j\phi\left(\frac{u_j}{v_j}\right)\left(\frac{u_j}{v_j}-\kappa\frac{u_{j-1}}{v_{j-1}}\right)dx = 0. \end{aligned} \tag{162}$$

Since $\beta < 1/2$, we have $\kappa = \frac{2}{3}(2-\beta) > 1$. Therefore, we can choose $0 < h < (\kappa-1)\tau$ which implies that $1-\kappa^{-1}(1+h/\tau) > 0$, and the first integral is nonnegative. The same conclusion holds for the second integral in (162) since

$$(v_j\nabla u_j - u_j\nabla v_j)\cdot\nabla\phi\left(\frac{u_j}{v_j}\right) = \frac{1}{v_j^2}\phi'\left(\frac{u_j}{v_j}\right)|v_j\nabla u_j - u_j\nabla v_j|^2 \geq 0.$$

Also the third integral in (162) is nonnegative. Hence, the fourth integral is nonpositive, which can be equivalently written as

$$\int_\Omega u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}v_j\phi\left(\frac{u_j}{v_j}\right)\left(\frac{u_j}{v_j}-M\right)dx \leq \int_\Omega u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}v_j\phi\left(\frac{u_j}{v_j}\right)\left(\kappa\frac{u_{j-1}}{v_{j-1}}-M\right)dx.$$

Taking into account definition (156) of M, we infer that the integral on the right-hand side is nonpositive, which shows that

$$\int_\Omega u_{j-1}^{1/2-\beta}v_{j-1}^{\beta+1/2}v_j\phi\left(\frac{u_j}{v_j}\right)\left(\frac{u_j}{v_j}-M\right)_+dx = 0,$$

where $z_+ = \max\{0, z\}$ for $z \in \mathbb{R}$, employing $\phi(u_j/v_j) = 0$ for $u_j/v_j \leq M$. Now, $\phi(u_j/v_j) > 0$ for $u_j/v_j > M$, and we conclude that $(u_j/v_j - M)_+ = 0$ and $u_j/v_j \leq M$ in Ω. Since $\varepsilon = 1/M$, this means that $v_j/u_j \geq \varepsilon$ and $\theta_{j,\varepsilon} = v_j/u_j$. Consequently, we have proven the existence of a weak solution (v_j, u_j) to the discretized problem (152)-(153) with the boundary conditions (154)-(155), which also yields a weak solution (n_j, θ_j) to (150)-(151) with the boundary conditions (143).

Step 4: Entropy estimates. Let $b \in \mathbb{R}$ and define the functional

$$\phi_b[n, n\theta] = \int_\Omega \left(f_b(n, n\theta) - f_{b,D} - \frac{\partial f_{b,D}}{\partial n}(n - n_D) - \frac{\partial f_{b,D}}{\partial (n\theta)}(n\theta - n_D\theta_D) \right) dx, \qquad (163)$$

where $f_b(n, n\theta) = n^{2-b}(n\theta)^b$ and we have employed the abbreviations

$$f_{b,D} = f_b(n_D, n_D\theta_D), \quad \frac{\partial f_{b,D}}{\partial n} = \frac{\partial f_b}{\partial n}(n_D, n_D\theta_D), \quad \frac{\partial f_{b,D}}{\partial (n\theta)} = \frac{\partial f_b}{\partial (n\theta)}(n_D, n_D\theta_D).$$

The function f_b is convex if $b \geq 2$ or $b \leq 0$ since $\det D^2 f_b(n, n\theta) = b(b-2)\theta^{2(\beta-1)}$ and $\mathrm{tr} D^2 f_b(n, n\theta) = (b-1)(b-2)\theta^b + b(b-1)\theta^{b-2}$. We wish to derive a priori estimates from the so-called entropy functionals

$$S_{b_1,b_2}[n, n\theta] = \frac{1}{|b_1|}\phi_{b_1}[n, n\theta] + \frac{1}{|b_2|}\phi_{b_2}[n, n\theta].$$

The parameters (b_1, b_2) are not completely arbitrary, but must be chosen in a smart way. As a matter of fact, it holds:

Lemma 3.4 (Discrete entropy inequality)**.** *A subset $N_\beta \subset \mathbb{R}^2$ exists such that, if $(b_1, b_2) \in N_\beta$, then*

$$\begin{aligned} S_{b_1,b_2}[n_j, n_j\theta_j] &+ C_1 h \int_\Omega \left(\theta_j^{b_1+1/2-\beta} + \theta_j^{b_2+1/2-\beta}\right)|\nabla n_j|^2 dx \\ &+ C_1 h \int_\Omega n_j^2 \left(\theta_j^{b_1-3/2-\beta} + \theta_j^{b_2-3/2-\beta}\right)|\nabla \theta_j|^2 dx \\ &\leq C_2 h + S_{b_1,b_2}[n_{j-1}, n_{j-1}\theta_{j-1}], \end{aligned} \qquad (164)$$

where $C_1 > 0$ depends on b and β and $C_2 > 0$ depends on τ, n_D, and θ_D. The constant C_2 vanishes if $n_D = const.$ and $\theta_D = 1$. Moreover $(\beta - \frac{1}{2}, 5) \in N_\beta$.

The proof of the above lemma is quite technical and will be skipped; the curious Reader can find it in [53].

Step 5: The limit $h \to 0$. We define the piecewise constant functions $n_h(x,t) = n_j(x)$ and $\theta_h(x,t) = \theta_j(x)$ for $x \in \Omega$ and $t \in ((j-1)h, jh]$, where $0 \leq j \leq N = T/h$. The discrete time derivative of an arbitrary function $w(x,t)$ is defined by $(D_h w)(x,t) = h^{-1}(w(x,t) - w(x, t-h))$ for $x \in \Omega$, $t \geq h$. Then (150)-(151) can be written as

$$D_h n_h = \Delta(n_h \theta_h^{1/2-\beta}), \quad D_h(n_h\theta_h) = \kappa\Delta(n_h\theta_h^{3/2-\beta}) + \frac{n_h}{\tau}(1 - \theta_h). \qquad (165)$$

The entropy inequality (164) for $(b_1, b_2) = (\beta - \frac{1}{2}, 5) \in N_\beta$ becomes, after summation over j,

$$S_{b_1,b_2}[n_h(t), n_h(t)\theta_h(t)] + C_1 \int_0^t \int_\Omega \big((1+\theta_h^{11/2-\beta})|\nabla n_h|^2 + n_h^2(\theta_h^{-2} + \theta_h^{7/2-\beta})|\nabla \theta_h|^2\big) dx\, ds \\ \leq C_2 t + S_{b_1,b_2}[n_0, n_0\theta_0]. \tag{166}$$

It is possible to exploit this inequality to derive h-independent estimates for (n_h) and $(n_h\theta_h)$.

Lemma 3.5. *There exists a constant $C > 0$ such that for all $h > 0$,*

$$\|n_h\|_{L^\infty(0,T;L^2(\Omega))} + \|n_h\theta_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{167}$$

$$\|n_h\theta_h^{1/2-\beta}\|_{L^\infty(0,T;L^2(\Omega))} + \|n_h\theta_h^{3/2-\beta}\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{168}$$

$$\|n_h\|_{L^2(0,T;H^1(\Omega))} + \|n_h\theta_h\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{169}$$

$$\|n_h\theta_h^{1/2-\beta}\|_{L^2(0,T;H^1(\Omega))} + \|n_h\theta_h^{3/2-\beta}\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{170}$$

$$\|D_h n_h\|_{L^2(h,T;H^1_D(\Omega)')} + \|D_h(n_h\theta_h)\|_{L^2(h,T;H^1_D(\Omega)')} \leq C. \tag{171}$$

Again, the proof of the above lemma is quite technical and will therefore be skipped. The basic idea is that it is always possible to bound a power of θ by means of the sum of two other powers with different exponents: $\theta^\alpha \leq C(\theta^{\beta_1} + \theta^{\beta_2})$ with $\beta_1 < \alpha < \beta_2$. Therefore one can derive suitable estimates for $n_h\theta_h^\alpha$ from (164) as long as the exponent α is chosen in the right range. The interested Reader can find the proof in [53].

Aubin's Lemma and Lemma 5 imply that, up to subsequences,

$$n_h \to n, \quad n_h\theta_h \to w \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{172}$$

$$n_h \rightharpoonup n, \quad n_h\theta_h \rightharpoonup w, \quad n_h\theta_h^{1/2-\beta} \rightharpoonup y, \quad n_h\theta_h^{3/2-\beta} \rightharpoonup z \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{173}$$

$$D_h n_h \rightharpoonup \partial_t n, \quad D_h(n_h\theta_h) \rightharpoonup \partial_t w \quad \text{weakly in } L^2(0,T;H^1_D(\Omega)'). \tag{174}$$

In order to identify the functions w, y, z appearing in (172)–(174) we show first that $n, w > 0$ a.e. in $\Omega \times (0,T)$. Let us define the discrete entropy functional:

$$\Lambda[n_h, n_h\theta_h] = \int_\Omega \left(-\log n_h - \frac{1}{\kappa}\log(n_h\theta_h) + \frac{n_h}{n_D} + \frac{1}{\kappa}\frac{n_h\theta_h}{n_D\theta_D} \right) dx, \tag{175}$$

where n_D, θ_D are the values of n_h, θ_h (respectively) on Γ_D. We point out that Λ is well defined since n_h, $n_h\theta_h$ are bounded and strictly positive. By finding a suitable upper bound for $D_h\Lambda[n_h, n_h\theta_h]$ and applying a discrete Gronwall argument, it is possible to prove (we do not do it here; see [53] for details) that

$$\sup_{t\in[0,T]} \Lambda[n_h(t), n_h(t)\theta_h(t)] \leq \Lambda[n_0, n_0\theta_0] \\ + C_D\big(1 + \|n_h\theta_h^{1/2-\beta}\|^2_{L^2(0,T;H^1(\Omega))} + \|n_h\theta_h^{3/2-\beta}\|^2_{L^2(0,T;H^1(\Omega))} + \|n_h\|^2_{L^2(0,T;L^2(\Omega))}\big),$$

where n_0, θ_0 are the values of n_h, θ_h at initial time, respectively. The strong convergence (172) and Fatou's Lemma allow us to conclude that, for some $C > 0$,

$$\sup_{t\in[0,T]} \Lambda[n(t), w(t)] \le C. \tag{176}$$

From the definition (175) of Λ and (176), we deduce that

$$-\log n(x,t) - \frac{1}{\kappa}\log w(x,t) < \infty \quad \text{for a.e. } (x,t) \in \Omega \times (0,T). \tag{177}$$

Since $n, w \in L^2(0,T; L^2(\Omega))$, they are a.e. finite. This fact, together with (177), implies that $n > 0$, $w > 0$ a.e. in $\Omega \times (0,T)$.

From the convergence (172) it follows also that $n_h \to n$, $n_h\theta_h \to w$ a.e. in $\Omega \times (0,T)$. The positivity of n implies that $\theta_h = (n_h\theta_h)/n_h \to w/n$ a.e. in $\Omega \times (0,T)$. Let us define $\theta := w/n$. Since n and w are finite and positive a.e. in $\Omega \times (0,T)$, then $0 < \theta < \infty$ a.e. in $\Omega \times (0,T)$. Clearly $n_h\theta_h^{1/2-\beta} \to n\theta^{1/2-\beta}$, $n_h\theta_h^{3/2-\beta} \to n\theta^{3/2-\beta}$ a.e. in $\Omega \times (0,T)$, recalling that $\beta < 1/2$; thus from the weak convergence (173) we obtain $y = n\theta^{1/2-\beta}$, $z = n\theta^{3/2-\beta}$. These relations, together with (172)–(174), allow us to perform the limit $h \to 0$ in the equations for Dn_h, $D(n_h\theta_h)$. This finishes the proof. □

3.5.2 A cross-diffusion system derived from a Fokker-Planck equation.

We refer to [29] for the part that follows.

We are now going to study the following cross-diffusion system

$$\partial_t u_i = \Delta\big(a(u_1/u_2)u_i\big) + \mu_i u_i, \quad t > 0, \quad u_i(0) = u_i^0 \ge 0 \quad \text{in } \mathbb{T}^d,\ i = 1,2, \tag{178}$$

where $\mathbb{T}^d$ is the d-dimensional torus with $d \ge 1$, $a : (0,\infty) \to (0,\infty)$ is a continuously differentiable function, and $\mu_i \in \mathbb{R}$.

Eq. (178) can be rewritten in divergence form:

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad t > 0, \quad u(0) = u^0 \quad \text{in } \mathbb{T}^d,$$

where $f(u) = (\mu_1 u_1, \mu_2 u_2)^\top$ and the diffusion matrix $A(u)$ reads as

$$A(u) = \begin{pmatrix} a(u_1/u_2) + (u_1/u_2)a'(u_1/u_2) & -(u_1/u_2)^2 a'(u_1/u_2) \\ a'(u_1/u_2) & a(u_1/u_2) - (u_1/u_2)a'(u_1/u_2) \end{pmatrix}. \tag{179}$$

This system can be formally derived [36] from a $(d+1)$-dimensional Fokker-Planck equation for the probability density $f(x,y,t)$, where $x \in \mathbb{R}^d$, $y \in \mathbb{R}$. The function u_i is obtained from f by partial averaging,

$$u_i(x,t) = \int_{\mathbb{R}} f(x,y,t)e^{\lambda_i y}dy, \quad i = 1,2,$$

μ_i is a function of λ_i, and $a(u_1/u_2)$ is related to the diffusion coefficients in the Fokker-Planck equation. Strictly speaking, equation (178) holds in $\mathbb{R}^d$ (or on some subset of $\mathbb{R}^d$)

but we consider this equation on the torus for the sake of simplicity (and to avoid possible issues with boundary conditions). The curious Reader can find the derivation of the model in [29].

System (178) has been suggested by P.-L. Lions in [36], and the global-in-time existence of (weak) solutions has been identified as an open problem. In this paper, we solve this problem by applying the entropy method for diffusive equations.

The underlying Fokker-Planck equation for $f(x, y, t)$ models the time evolution of the value of a financial product in an idealized financial market, depending on various underlying assets or economic values. The function u_i is an average with respect to the variable y, which may be interpreted as the value of an economic parameter, and the exponential weight emphasizes large positive or large negative values of y, depending on the sign of λ_i.

We assume that there exist $a_0 > 0$ and $p \geq 0$ such that for all $r > 0$,

$$a(r) \geq r|a'(r)|, \quad a(r) \geq \frac{a_0}{r^p + r^{-p}}. \tag{180}$$

The first condition means that a grows at most linearly. The second condition is a technical assumption needed for the entropy method. Examples are $a(r) = 1$, which leads to uncoupled heat equations for u_1 and u_2, $a(r) = r^\alpha$ with $0 < \alpha \leq 1$, $a(r) = r^\beta/(1 + r^{\beta-1})$ with $\beta > 0$, and $a(r) = 1/r$. The last example gives the equations

$$\partial_t u_1 = \Delta u_2, \quad \partial_t u_2 = \Delta\left(\frac{u_2^2}{u_1}\right). \tag{181}$$

Surprisingly, this system corresponds (up to a factor) to an energy-transport model for semiconductors. Indeed, introducing the electron density $n := u_1$ and the electron temperature $\theta := u_2/u_1$, equations (181) can be written as

$$\partial_t n = \Delta(n\theta), \quad \partial_t(n\theta) = \Delta(n\theta^2),$$

which correspond to (178) with $\beta = -1/2$.

Another class of models which resembles (178) are the equations

$$\partial_t u_i = \Delta(p_i(u)u_i), \quad i = 1, \ldots, m, \tag{182}$$

modeling the time evolution of population densities u_i. These systems are analyzed in, e.g., [20, 31], essentially for $m = 2$. In this application, p_i is often given by the sum $p_{i1}(u_1) + p_{i2}(u_2)$, and consequently, the results of [20, 31] do not apply and we need to develop new ideas.

We will prove the global-in-time existence of weak solutions to (178).

Theorem 3.7 (Existence of weak solutions)**.** *Let* (180) *hold and let* $T > 0$, $\alpha \geq p + 4$, μ_1, $\mu_2 \in \mathbb{R}$, $0 \leq a \in C^1(0, \infty)$, $u^0 = (u_1^0, u_2^0) \in L^2(\mathbb{T}^d)^2$ *with* u_1^0, $u_2^0 \geq 0$ *in* $\mathbb{T}^d$ *and* $H[u^0] < \infty$. *Then there exists a solution* $u = (u_1, u_2)$ *to* (178) *satisfying* $u_i > 0$ *in* $\mathbb{T}^d$, $t > 0$, $i = 1, 2$, *and*

$$u_i,\ a(u_1/u_2)u_i \in L^\infty(0, T; L^2(\mathbb{T}^d)),$$

$$\nabla u_i,\ \nabla\big(a(u_1/u_2)u_i\big) \in L^2(0,T;L^2(\mathbb{T}^d)), \quad \partial_t u_i \in L^2(0,T;H^1(\mathbb{T}^d)'), \quad i=1,2.$$

If additionally $\mu_i \le 0$ for $i=1,2$, we have the uniform bounds

$$u_i,\ a(u_1/u_2)u_i \in L^\infty(0,\infty;L^2(\mathbb{T}^d)), \quad \nabla u_i,\ \nabla\big(a(u_1/u_2)u_i\big) \in L^2(0,\infty;L^2(\mathbb{T}^d)). \tag{183}$$

The key idea of the proof is to employ the entropy functional

$$H[u] = \int_{\mathbb{T}^d} h(u)dx, \quad h(u) = \left(\frac{u_1}{u_2}\right)^\alpha u_1^2 + \left(\frac{u_1}{u_2}\right)^{-\alpha} u_2^2 + u_1 - \log u_1 + u_2 - \log u_2, \tag{184}$$

where $\alpha \ge p+4$ and $u=(u_1,u_2)\in(0,\infty)^2$. We will show that

$$\frac{d}{dt}H[u] + \int_{\mathbb{T}^d}\left(\left(\frac{u_1}{u_2}\right)^{\alpha-p} + \left(\frac{u_1}{u_2}\right)^{p-\alpha}\right)\big(|\nabla u_1|^2+|\nabla u_2|^2\big)dx \le CH[u] \tag{185}$$

for some constant $C>0$ which vanishes if $\mu_1=\mu_2=0$. In this situation, the mapping $t\mapsto H[u(t)]$ is nonincreasing; otherwise, for $\mu_i\ne 0$, $t\mapsto H[u(t)]$ is bounded on finite time intervals. We infer from the inequality $x+x^{-1}\ge 2$ for all $x>0$ uniform bounds for $u_i(t)$ in $H^1(\mathbb{T}^d)$, which are needed for the compactness argument.

Some auxiliary results. We state here some algebraic properties of the matrices $h''(u)$ and $A(u)$ and some estimates related to the entropy density $h(u)$ and the components of $A(u)$. Recall that $h(u)$ is defined in (184) and $A(u)$ in (179).

Lemma 3.6 (Properties of h)**.** *Let $\alpha>0$. The function $h:(0,\infty)^2\to\mathbb{R}^2$, defined in* (184)*, is convex, its derivative h' is invertible, and there exists $C_h>0$ such that for all $u=(u_1,u_2)\in(0,\infty)^2$,*

$$h(u) \ge \frac{1}{2}(u_1^2+u_2^2), \quad \sum_{i=1}^2 \mu_i u_i \partial_i h(u) \le C_h h(u), \tag{186}$$

where we recall that $\partial_i h = \partial h/\partial u_i$.

Lemma 3.7 (Positive semidefiniteness of $h''A$)**.** *Let condition* (180) *hold. If $\alpha(\alpha+2)>1$, the matrix $h''(u)A(u)$ is positive semidefinite in $(0,\infty)^2$. Furthermore, if additionally $\alpha\ge p$, there exists a constant $\kappa=\kappa(\alpha)>0$ such that for all $u=(u_1,u_2)\in(0,\infty)^2$ and $z\in\mathbb{R}^2$,*

$$z^\top h''(u)A(u)z \ge \kappa\left(\left(\frac{u_1}{u_2}\right)^{\alpha-p} + \left(\frac{u_1}{u_2}\right)^{p-\alpha}\right)|z|^2.$$

Lemma 3.8. *Let $\alpha\ge 2$. Then, for all $u_1,\ u_2>0$,*

$$\begin{aligned} a\left(\frac{u_1}{u_2}\right)^2(u_1^2+u_2^2) &\le C_a\left(u_1^2+u_2^2+\frac{u_1^4}{u_2^2}\right) \\ &\le \xi_\alpha C_a\left(\left(\frac{u_1}{u_2}\right)^\alpha u_1^2 + \left(\frac{u_1}{u_2}\right)^{-\alpha}u_2^2\right) \le \xi_\alpha C_a h(u), \end{aligned}$$

where $C_a=a(1)^2$ and $\xi_\alpha>0$ is a suitable constant which only depends on α.

Lemma 3.9. *There exists $C_A > 0$ such that for all u_1, $u_2 > 0$,*

$$|A(u)| \le C_A\bigg(1 + \Big(\frac{u_1}{u_2}\Big)^2 + \Big(\frac{u_1}{u_2}\Big)^{-2}\bigg).$$

Proof of Thr. 3.7. Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, and $m \in \mathbb{N}$ with $m > d/2$. Then the embedding $H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ is compact. Furthermore, let $w^{k-1} = (w_1^{k-1}, w_2^{k-1}) \in L^\infty(\mathbb{T}^d)^2$ be given and let $u^{k-1} = (h')^{-1}(w^{k-1})$. By Lemma 3.6, the pair $u^{k-1} = (u_1^{k-1}, u_2^{k-1})$ is well defined and we have $u^{k-1} \in L^\infty(\mathbb{T}^d)^2$. We wish to find $w_k = (w_1^k, w_2^k) \in H^m(\mathbb{T}^d)^2$ such that for all $\phi = (\phi_1, \phi_2) \in H^m(\mathbb{T}^d)^2$,

$$\begin{aligned}\frac{1}{\tau}\int_{\mathbb{T}^d}(u^k - u^{k-1})\cdot\phi dx + \int_{\mathbb{T}^d}\nabla\phi : B(w^k)\nabla w^k dx \\ + \tau\int_{\mathbb{T}^d}(D^m w^k\cdot D^m\phi + w^k\cdot\phi)dx = \sum_{i=1}^2\mu_i\int_{\mathbb{T}^d}u_i^k\phi_i dx,\end{aligned} \tag{187}$$

where $B(w^k) = A(u^k)h''(u^k)^{-1}$,

$$D^m w^k\cdot D^m\phi := \sum_{|\alpha|=m}\sum_{i=1}^2 D^\alpha u_i^k D^\alpha\phi_i,$$

$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d})$ a partial derivative of order $|\alpha|$.

Step 1: solution of (187). This is a standard argument based on Leray-Schauder's fixed point theorem which has already been used many times in these notes, and therefore we skip it. The interested Reader can find it in [29]. In the end we obtain a solution $w^k \in H^m(\mathbb{T}^d)$ to (187).

Step 2: a priori estimates.

We define the piecewise constant functions in time $w^{(\tau)}(x,t) = w^k(x)$ and $u^{(\tau)}(x,t) = u^k(x)$ for $x \in \mathbb{T}^d$ and $t \in ((k-1)\tau, k\tau]$, $k = 1, \ldots, j$. Furthermore, we introduce the shift operator $\sigma_\tau u^{(\tau)}(x,t) = u^{k-1}(x)$ for $x \in \mathbb{T}^d$, $t \in ((k-1)\tau, k\tau]$. With this notation, we can rewrite (187) as:

$$\begin{aligned}\frac{1}{\tau}\int_0^T\int_{\mathbb{T}^d}(u^{(\tau)} - \sigma_\tau u^{(\tau)})\cdot\phi dxdt + \int_0^T\int_{\mathbb{T}^d}\nabla\phi : A(u^{(\tau)})\nabla u^{(\tau)}dxdt \\ + \tau\int_0^T\int_{\mathbb{T}^d}\big(D^m w^{(\tau)}\cdot D^m\phi + w^{(\tau)}\cdot\phi^{(\tau)}\big)dxdt + \sum_{i=1}^2\mu_i\int_0^T\int_{\mathbb{T}^d}u_i^{(\tau)}\phi_i dxdt,\end{aligned} \tag{188}$$

and the following discrete entropy inequality is deduced by choosing $\phi = w^{(\tau)}$ in (188) and applying Lemma 3.7:

$$\int_{\mathbb{T}^d}h(u^{(\tau)}(t))dx + \int_0^t\int_{\mathbb{T}^d}\bigg(\Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)^{\alpha-p} + \Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)^{p-\alpha}\bigg)|\nabla u^{(\tau)}|^2 dxds$$

$$+\tau\int_0^t\int_{\mathbb{T}^d}\big(|D^m w^{(\tau)}|^2+|w^{(\tau)}|^2\big)dxds\le C,\qquad t\in[0,T]. \tag{189}$$

It follows that

$$\|w^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d))}\le C\tau^{-1/2}. \tag{190}$$

By Lemma 3.8, Lemma 3.6, and estimate (189), we find that

$$\int_{\mathbb{T}^d}\left(\left|a\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)u_1^{(\tau)}\right|^2+\left|a\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)u_2^{(\tau)}\right|^2\right)dx\le 3C_a\int_{\mathbb{T}^d}h(u^{(\tau)})dx\le C, \tag{191}$$

$$\int_{\mathbb{T}^d}\big((u_1^{(\tau)})^2+(u_2^{(\tau)})^2\big)dx\le\int_{\mathbb{T}^d}h(u^{(\tau)})dx\le C. \tag{192}$$

Moreover, using Lemma 3.9 and (189),

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\Big(\big|\nabla\big(a(u_1^{(\tau)}/u_2^{(\tau)})u_1^{(\tau)}\big)\big|^2+\big|\nabla\big(a(u_1^{(\tau)}/u_2^{(\tau)})u_2^{(\tau)}\big)\big|^2\Big)dxdt\\
&\quad=\int_{\mathbb{T}^d}|A(u^{(\tau)})\nabla u^{(\tau)}|^2dx\le\int_0^T\int_{\mathbb{T}^d}|A(u^{(\tau)})|^2|\nabla u^{(\tau)}|^2dxdt\\
&\quad\le C_A\int_0^T\int_{\mathbb{T}^d}\left(1+\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^4+\left(\frac{u_2^{(\tau)}}{u_1^{(\tau)}}\right)^4\right)|\nabla u^{(\tau)}|^2dxdt\\
&\quad\le C\int_0^T\int_{\mathbb{T}^d}\left(\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{\alpha-p}+\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{p-\alpha}\right)|\nabla u^{(\tau)}|^2dxdt\le C.
\end{aligned} \tag{193}$$

The last but one inequality follows from the elementary estimate $1+y^4\le y^{\alpha-p}+y^{p-\alpha}$ for $y>0$ which holds because of the assumption $\alpha-p\ge 4$. Estimates (191)-(193) yield for $i=1,2$,

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}+\|\nabla u_i^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\le C, \tag{194}$$

$$\|a(u_1^{(\tau)}/u_2^{(\tau)})u_i\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}+\|\nabla(a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)})\|_{L^2(0,T;L^2(\mathbb{T}^d))}\le C. \tag{195}$$

These estimates are uniform in $T>0$ if $\mu_i\le 0$.

Next, we derive a uniform estimate for the discrete time derivative $(u^{(\tau)}-\sigma_\tau u^{(\tau)})/\tau$. For $\phi\in L^2(0,T;H^m(\mathbb{T}^d))$, we estimate

$$\begin{aligned}
\frac{1}{\tau}\left|\int_0^T(u^{(\tau)}-\sigma_\tau u^{(\tau)})\cdot\phi dxdt\right|&\le\|A(u^{(\tau)})\nabla u^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|\nabla\phi\|_{L^2(0,T;L^2(\mathbb{T}^d))}\\
&\quad+\tau\|w^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d))}\|\phi\|_{L^2(0,T;H^m(\mathbb{T}^d))}\\
&\quad+\max\{\mu_1,\mu_2\}\|u^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|\phi\|_{L^2(0,T;L^2(\mathbb{T}^d))}\\
&\le C\|\phi\|_{L^2(0,T;H^m(\mathbb{T}^d))},
\end{aligned}$$

taking into account the bounds (190), (193), and (194). Therefore,

$$\tau^{-1}\|u^{(\tau)}-\sigma_\tau u^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d)')}\le C. \tag{196}$$

Step 3: limit $\tau \to 0$. Estimates (194) and (196) allow us to apply the Aubin-Lions lemma in the discrete version of [21] to obtain the existence of a subsequence, which is not relabeled, such that, as $\tau \to 0$,

$$u_i^{(\tau)} \to u_i \quad \text{strongly in } L^2(0,T;L^2(\mathbb{T}^d)) \text{ and a.e.}, \ i = 1,2.$$

Moreover, by (190), (194), and (196), for the same subsequence and $i = 1,2$,

$$\begin{aligned} \tau w_i^{(\tau)} &\to 0 \quad \text{strongly in } L^2(0,T;H^m(\mathbb{T}^d)), \\ \nabla u_i^{(\tau)} &\rightharpoonup \nabla u_i \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)), \\ \tau^{-1}(u_i^{(\tau)} - \sigma_\tau u_i^{(\tau)}) &\rightharpoonup \partial_t u_i \quad \text{weakly in } L^2(0,T;H^m(\mathbb{T}^d)'). \end{aligned}$$

The pointwise convergence of $(u_i^{(\tau)})$, Fatou's lemma, and estimate (189) imply that, for a.e. $t \in (0,T)$,

$$\begin{aligned} \sum_{i=1}^2 \int_{\mathbb{T}^d} (u_i(t) - \log u_i(t))dx &\le \liminf_{\tau\to 0} \sum_{i=1}^2 \int_{\mathbb{T}^d} \big(u_i^{(\tau)}(t) - \log u_i^{(\tau)}(t)\big)dx \\ &\le \liminf_{\tau\to 0} \int_{\mathbb{T}^d} h(u^{(\tau)}(t))dx \le C. \end{aligned}$$

This means that $u_i > 0$ a.e. in $\mathbb{T}^d \times (0,T)$.

Estimate (191) and (193) show that, up to a subsequence,

$$a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)} \rightharpoonup q_i \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)), \ i = 1,2,$$

where $q_i \in L^2(0,T;H^1(\mathbb{T}^d))$. We wish to identify q_i. To this end, let us define $\chi_\varepsilon^{(\tau)} = \mathbf{1}_{\{u_1^{(\tau)}\ge\varepsilon,\, u_2^{(\tau)}\ge\varepsilon\}}$ and $\chi_\varepsilon = \mathbf{1}_{\{u_1\ge\varepsilon,\, u_2\ge\varepsilon\}}$, where $\mathbf{1}_A$ denotes the characteristic function on the set A. Clearly, $\chi_\varepsilon^{(\tau)} \to \chi_\varepsilon$ strongly in $L^s(0,T;L^s(\mathbb{T}^d))$ for all $1 \le s < \infty$. We infer that

$$\chi_\varepsilon^{(\tau)} a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)} \rightharpoonup \chi_\varepsilon a(u_1/u_2)u_i \quad \text{weakly in } L^s(0,T;L^s(\mathbb{T}^d)), \ 1 \le s < 2.$$

We deduce that $q_i = a(u_1/u_2)u_i$ on the set $\{u_1 \ge \varepsilon,\, u_2 \ge \varepsilon\}$. Since $\varepsilon > 0$ is arbitrary and $u_i > 0$ a.e. in $\mathbb{T}^d \times (0,T)$, this identification holds, in fact, a.e. in $\mathbb{T}^d \times (0,T)$.

Consequently, we may perform the limit $\tau \to 0$ in (188) to deduce that u is a weak solution to (178) with test functions $L^2(0,T;H^m(\mathbb{T}^d)')$. However, since $a(u_1/u_2)u_i \in L^2(0,T;H^1(\mathbb{T}^d))$, we can employ a standard density argument to infer that (178) also holds in $L^2(0,T;H^1(\mathbb{T}^d)')$. Since $u_i \in L^2(0,T;H^1(\mathbb{T}^d))$ and $\partial_t u_i \in L^2(0,T;H^1(\mathbb{T}^d)')$, it follows that $u_i \in C^0([0,T];L^2(\mathbb{T}^d))$, so the initial datum is satisfied in $L^2(\mathbb{T}^d)$. Finally, since the bounds are uniform in T if $\mu_i \le 0$, the statement (183) follows.

□

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