From the Boltzmann equation to hydrodynamic models

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Contents

1 Introduction

This topic course is concerned with the derivation of hydrodynamic models (Euler and Navier-Stokes equation) from the Boltzmann equation. The course is divided into three parts. In the first part, we will give a short introduction of kinetic theory and the Boltzmann equation (in short, BE), its physical meaning and mathematical properties (conservation laws, entropy decay). In the second part we will see how it is possible to obtain fluid models by taking formal hydrodynamic limits in suitable a-dimensional formulations of the BE, under compressible or incompressible regimes. In the third part of the course we will show how to derive, in an analytically rigorous way, the incompressible Navier-Stokes equations on the torus in the mathematical framework of Sobolev spaces; we will then prove existence and exponential decay of solutions to the linearized Boltzmann equation by means of so-called "hypocoercivity" estimates; finally, we will show how such solutions converge (with explicitly computable rates) to the corresponding solutions of the incompressible Navier-Stokes equations.

The main reference for these Lecture Notes is M. Briant's seminal work [2]. Other references are C. Cercignani et al.'s book on the mathematics of dilute gases [3] and F. Golse's Lectures series about derivation of macroscopic models from kinetic equations [5].

2 Kinetic theory and the Boltzmann equation

The starting point of our discussion is a fundamental equation of kinetic theory, that is, the Boltzmann equation. We shall first introduce briefly the field of kinetic theory, then we will have a few words about the Boltzmann equation.

2.1 Generalities about kinetic theory.

A system of N particles moving in a domain $\Omega \subset \mathbb{R}^d$ can be described by Newton's laws of motion $m\ddot{x} = F$, or equivalently, Hamilton's equations $\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$, $\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$. This mathematical description of the system, which leads to a set of $6N$ Hamiltonian evolution equations in the phase space $\Omega \times \mathbb{R}^d$, is called *microscopic* description, since in this formalism we are trying to compute the evolution of *each* single particle.

From a purely theoretical standpoint, this looks satisfactory. However, solving the motion of N particles when N is large (as is expected to be in realistic situations) is very difficult. Fortunately there is a workaround to this difficulty: we do not need to compute the motion of each particle; it is enough to find out how some average quantities (particle/mass/charge density, velocity/momentum, temperature/energy) evolve with time. Treat your system of particles like a continuum, apply Newton's laws to an infinitesimal volume inside the domain Ω, add some constitutive relations, integrate in space and you get the equations of fluid mechanics (e.g. Euler, Navier-Stokes). This is called *macroscopic* description of the system. It leads to simpler equations, but at the cost of a complete loss of information regarding the microscopic behaviour of the system.

Kinetic theory lies in the middle of these two opposite ways. It describes the so-called *mesoscopic* scale. It employs a probabilistic approach to simplify the description of the system without losing interesting information about microscopic dynamics.

The core idea of kinetic theory is to model a system constituted by a large number of particles by means of a distribution function $f = f(x, v, t)$ (x, v being position and velocity, respectively), i.e. a function on the phase space $\Omega \times \mathbb{R}^d$ plus the time t. The intuitive definition of f is:

$$\begin{aligned} f(x, v, t)dxdv = &\text{ Number of particles with position in the ball } B(x, dx) \\ &\text{ and velocity in the ball } B(v, dv) \text{ at time } t. \end{aligned}$$

Equivalently, given an arbitrary $\mathcal{B} \subset \Omega \times \mathbb{R}^d$,

$$\begin{aligned} \iint_{\mathcal{B}} f(x, v, t)dxdv = &\text{ Number of particles with position } x \text{ and velocity } v \\ &\text{ such that } (x, v) \in \mathcal{B} \text{ at time } t. \end{aligned}$$

It is therefore quite natural to require that f satisfies some integrability properties, that is $f(\cdot, \cdot, t) \in L^1_{x,loc}(\Omega, L^1_v(\mathbb{R}^d))$, for $t \geq 0$. This formalism allows us to express observable macroscopic quantities as "moments" of f, i.e. as integrals of functions of v (usually polynomials) with respect to the measure $f(x, v, t)dxdv$. The most important examples of such objects are

$$\rho(x, t) = \int_{\mathbb{R}^d} f(x, v, t)dxdv \qquad \text{local density,}$$

$$\rho u(x, t) = \int_{\mathbb{R}^d} v f(x, v, t)dxdv \qquad \text{local momentum,}$$

$$u(x, t) = \frac{1}{\rho(x, t)} \int_{\mathbb{R}^d} v f(x, v, t)dxdv \qquad \text{local velocity,}$$

$$\theta(x, t) = \frac{1}{d\rho(x, t)} \int_{\mathbb{R}^d} |v - u|^2 f(x, v, t)dxdv \qquad \text{local temperature,}$$

$$E(x, t) = \int \frac{|v|^2}{2} f(x, v, t)dxdv = \rho(x, t)\frac{|u|^2}{2} + \frac{d}{2}\rho(x, t)\theta(x, t) \qquad \text{local energy.}$$

Clearly $\int_\Omega \rho(x, t)dx$, $\int_\Omega \rho u(x, t)dx$, $\int_\Omega E(x, t)dx$ represent the total mass (or number of particles), total momentum, and total energy of the system.

Now that we know how to use the distribution function f to compute macroscopic quantities, we need to write an equation which describes its time evolution. This equation should take into account the free motion of the particles, the action of external forces on them and the interactions between different particles (e.g. collisions).

Let us consider a system of noninteracting particles subject ot an external positional force $F = F(x)$. It is (intuitively) clear by the definition of f that the distribution function remains constant along the trajectories of the particles. If $x(t)$, $v(t)$ are the position and velocity of a particle at time t (respectively), then we require the function $t \mapsto f(x(t), v(t), t)$ to be constant, i.e.

$$\frac{d}{dt} f(x(t), v(t), t) = \partial_t f + v(t) \cdot \nabla_x f + \frac{F(x(t))}{m} \cdot \nabla_v f = 0 \qquad t > 0.$$

If the above relation must hold for all particles of the system, then f has to satisfy:

$$\partial_t f + v \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_v f = 0 \qquad x \in \Omega, \quad v \in \mathbb{R}^d, \quad t > 0. \tag{1}$$

Eq. (1) is called (linear) *Vlasov equation.* In absence of external force F it reduces to the free transport equation:

$$\partial_t f + v \cdot \nabla_x f = 0 \qquad x \in \Omega, \quad v \in \mathbb{R}^d, \quad t > 0. \tag{2}$$

However, interactions between particles do exist, and we need to take care of them. We can do this in (at least) two ways.

If the interactions are long range (e.g. electromagnetic) then we can describe this situation with a “mean field equation”, that is, an equation which takes into account the interactions between particles by means of a so-called self-consistent potential:

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \Psi(x,t) \cdot \nabla_v f = 0, \qquad \Psi(x,t) = -\int_{\mathbb{R}^d} \psi(x-y)\rho(y,t)dy. \tag{3}$$

Choosing for example $\psi(z) = \frac{q^2}{4\pi\varepsilon_0 |z|}$ (the Poisson kernel), where q is the electric charge and ε_0 is the vacuum permittivity, leads to the Vlasov-Poisson equation, which is used to describe systems of electrically charged particles (e.g. plasmas).

In the case of small range interactions (collisions) this description cannot be used. Instead, such a situation will be modeled through an equation with the structure

$$\partial_t f + v \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_v f = Q(f),$$

where $Q = Q(f)$ is an (often nonlinear) operator which contains the physics of the collision processes. The celebrated Boltzmann equation, which describes the dynamics of rarefied gases, falls in this category.

2.2 The Boltzmann equation.

2.2.1 About collisions.

So, let’s talk about the collision operator $Q(f)$. We consider a system of monoatomic particles and make the following assumptions.

1. Collisions between particles are binary. This means that we neglect collisions involving three or more particles. This assumption makes sense for a dilute gas. Such a situation can be modeled through the so-called Boltzmann-Grad limit: $N \sim r^{-2}$ where N is the number of particles and r is the diameter of each particle.

2. Collisions are localised in space and time, as if the particles were billiard spheres. We can therefore state that a collision happens at a position x and at a time t.

3. Collisions are elastic, that is, total momentum and total kinetic energy are conserved by the collision process. More precisely, if two particles with velocities v', v'_* (respectively) collide, then after the collision their respective new velocities v, v_* will satisfy
$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$
We point out that the particles mass does not appear in the above equations since all particles are assumed to be undistinguishable (and in particular have the same mass).

4. Collisions processes are (micro)reversible, i.e. the microscopic dynamics is reversible in time.

5. We assume the system is chaotic (in the Boltzmann sense), that is, before two particles collide they evolve indipendently from each other.

We can easily derive the precollisional velocities v', v'_* in terms of the postcollisional velocities v, v_* (and conversely):

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \tag{4}$$

where
$$\sigma = \frac{v'-v'_*}{|v'-v'_*|} \in \mathbb{S}^{d-1}.$$

Thanks to the above relations we can obtain an expression for the collision operator Q of the Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{5}$$
$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta)(f'f'_* - ff_*)dv_* d\sigma. \tag{6}$$

In (6) we adopted the shorthand $f' \equiv f(x,v',t)$, $f'_* \equiv f(x,v'_*,t)\ldots$, (v',v'_*) are linked to (v,v_*,σ) through (4), and θ is the angle between $v-v_*$ and σ, and B is the collision kernel. We point out that we are considering no external forces in (5).

We observe that the collision operator $Q(f,f)$ is bilinear in f. This comes naturally from the fact that we are considering only binary collisions, and due to the chaos assumption (the particles are independent before a collision) the distribution function of two particles with velocity v, v_* (respectively) decouples as product ff_* of the single-particle distribution functions. Moreover, we can easily recognize two contribution in the expression of $Q(f,f)$:

- a gain term, equal to the number of particles per unit time acquiring velocity v after a collision with another particle:

$$(\partial_t f)_{coll,+} = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta) f' f'_* dv_* d\sigma;$$

- a loss term, equal to the number of particles per unit time with velocity v experiencing a collision (and consequently changing their velocity):

$$(\partial_t f)_{coll,-} = -\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta) f f_* dv_* d\sigma.$$

The above written representation is called "$\sigma-$representation". Another commonly used expression for Q is referred to with the name "$\omega-$representation" and employs the following relations between the pre- and postcollisional velocities:

$$v' = v - ((v-v_*)\cdot\omega)\omega, \qquad v'_* = v_* + ((v-v_*)\cdot\omega)\omega, \tag{7}$$

with $\omega \in \mathbb{S}^{d-1}$ given by

$$\omega = \frac{v-v'}{|v-v'|}.$$

The collision operator in this representation reads as

$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \tilde{B}(v-v_*, \omega)(f'f'_* - ff_*) dv_* d\omega. \tag{8}$$

The new collision kernel $\tilde{B}$ is related to B by

$$\tilde{B}(z,\omega) = 2^{d-1} \left| \frac{z}{|z|} \cdot \omega \right|^{d-2} B(|z|, \sigma).$$

2.2.2 On the scattering kernel.

Let us now speak a few words about the collision kernel B (in the σ representation). The mathematical properties of B reflect the physics of the collisional process. First of all, we assume that the kernel decouples as a product of a "radial" part (i.e. a part depending on the modulus of the relative postcollisional velocity $|v-v_*|$) and an "angular" part (depending and the angle between the two postcollisional velocities):

$$B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|) b(\cos\theta),$$

where Φ, b are assumed to be nonnegative locally integrable functions. We make this assumption in order to simplify subsequent computations; anyway, it is satisfied in many physically relevant cases.

One of such situations is the so-called *hard sphere case*, i.e. the particles behave like billiard balls bouncing on each other. In this case it holds

$$B(|v - v_*|, \cos\theta) = b(\cos\theta)\Phi(|v - v_*|).$$

In general, we assume that Φ satisfies

$$c_\Phi z^\gamma \leq \Phi(z) \leq C_\Phi z^\gamma \qquad z > 0,$$

for some positive constant c_Φ, C_Φ and $-d < \gamma \leq 1$. We distinguish three cases according to γ:

- $\gamma < 0$: soft potential;
- $\gamma = 0$: Maxwellian molecules;
- $\gamma > 0$: hard potential.

Concerning b we will assume:

$$\begin{aligned}
&\theta \mapsto b(\cos\theta) \ \text{ is continuous in } (0,\pi], \qquad b(\cos\theta) > 0 \ \text{ near } \theta = \frac{\pi}{2}, \\
&b(\cos\theta)(\sin\theta)^{d-2} \sim b_0\theta^{-(1+\nu)} \ \text{ as } \theta \to 0^+,
\end{aligned}$$

for $b_0 > 0$ and $\nu < 2$. The assumption $\nu < 0$ implies that b is locally integrable, and is the so-called Grad's cutoff assumption (in this case B is called a Grad cutoff kernel). This assumption plays a crucial role in the theory of the Boltzmann equation.

A particularly physically relevant case is:

$$d = 3, \qquad \Phi(z) = C_\Phi z^\gamma, \qquad \gamma = \frac{s-5}{s-1}, \qquad \nu = \frac{2}{s-1},$$

with $s \geq 2$. This is the case of Φ being and inverse power law. For $s = 2$ we have the Coulomb interaction case, where b is explicitly given as a function of θ:

$$b(\cos\theta) = \frac{b_0}{(\sin\theta)^4}.$$

2.2.3 Initial and boundary conditions.

If we want to solve a PDE (in this case the Boltzmann equation) we need to prescribe initial and boundary conditions, right? Well, the initial condition for the BE is reads simply

$$f(x, v, 0) = f_{in}(x, v) \qquad x \in \Omega, \quad v \in \mathbb{R}^d.$$

What are the properties that the initial datum f_{in} is expected to fulfill? It should be nonnegative (of course) and have finite mass and finite energy, at least when integrated on bounded domains. We therefore impose

$$f_{in} \geq 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^d,$$
$$\int_K \int_{\mathbb{R}^d} (1+|v|^2) f_{in}(x,v)\,dxdv < \infty \quad \text{for every compact set } K \subset \Omega.$$

Now we need to speak about the boundary conditions. Let us assume that $\Omega \neq \mathbb{R}^d$ so that Ω has a boundary. The most common choices for boundary conditions are as follows.

- The bounce-back condition:

  $$f(x,v,t) = f(x,-v,t) \qquad x \in \partial\Omega, \quad v \in \mathbb{R}^d, \quad t > 0.$$

  This condition says that the number of particles colliding with $\partial\Omega$ with velocity v equals the number of particles reflected back by $\partial\Omega$ with velocity $-v$.

- The specular reflection condition (provided that $\partial\Omega$ is smooth enough):

  $$f(x,v,t) = f(x, v - 2(v \cdot n(x))n(x), t) \qquad x \in \partial\Omega, \quad v \in \mathbb{R}^d, \quad t > 0,$$

  where $n(x)$ is the unit normal to $\partial\Omega$ at x. This conveys the idea that the particles colliding with the border are reflected like billiard balls.

- The Maxwellian diffusion boundary condition (again, provided that $\partial\Omega$ is smooth enough):

  $$f(x,v,t) = M_{\partial\Omega}(v) \int_{\{v \cdot n(x) > 0\}} f(x,v,t) v \cdot n(x)\,dv \qquad x \in \partial\Omega, \quad v \in \mathbb{R}^d, \quad t > 0,$$

  where

  $$M_{\partial\Omega}(v) = (2\pi T_{\partial\Omega})^{-d/2} e^{-|v|^2/2T_{\partial\Omega}}$$

  is the thermodynamical equilibrium distribution between the wall and the gas, $T_{\partial\Omega}$ being the temperature of $\partial\Omega$. This corresponds to the idea that the particles are absorbed by the boundary and reemitted according to the thermodynamic equilibrium distribution $M_{\partial\Omega}$.

- Periodic boundary conditions, i.e. $\Omega = \mathbb{T}^d$. This is clearly the easiest case from a mathematical viewpoint. It has been proven that it is equivalent to the case when Ω is a box with specular reflection boundary conditions.

2.2.4 Conservation laws and entropy dissipation.

We know that elastic collisions preserve mass, momentum, and energy. Since the collision operator is build upon the assumption of elastic collision, one may expect that these conservation properties translate from the microscopic to the macroscopic level. This is indeed the case, as we will see shortly. The starting point is noticing that the Boltzmann collision kernel $B(|v-v_*|, \cos\theta)$ is invariant under the changes of variables $(v, v_*) \leftrightarrow (v', v'_*)$ and $(v, v_*) \leftrightarrow (v_*, v)$. As a consequence, given any function $\phi = \phi(v)$,

$$\begin{aligned}\int_{\mathbb{R}^d} Q(f,f)(v)\phi(v)dx &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B[f'f'_* - ff_*]\phi(v)dvdv_*d\sigma \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B[f'f'_* - ff_*]\phi(v')dvdv_*d\sigma \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B[f'f'_* - ff_*]\phi(v_*)dvdv_*d\sigma \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B[f'f'_* - ff_*]\phi(v'_*)dvdv_*d\sigma.\end{aligned}$$

Summing up the above relations and dividing by 4 we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} Q(f,f)(v)\phi(v)dv \\ &\quad = -\frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B[f'f'_* - ff_*](\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*))dvdv_*d\sigma. \qquad (9)\end{aligned}$$

Replacing $\phi(v)$ in (9) with 1, v, $|v|^2$ yields

$$\int_{\mathbb{R}^d} Q(f,f)(v)dx = \int_{\mathbb{R}^d} Q(f,f)(v)v_i dx = \int_{\mathbb{R}^d} Q(f,f)(v)|v|^2 dx = 0 \qquad i = 1,2,3. \qquad (10)$$

This means that, if f satisfies the Boltzmann equation:

$$\partial_t f + v\cdot\nabla_x f = Q(f,f),$$

then the total mass and total energy are conserved if Ω has no boundary (i.e. $\Omega = \mathbb{R}^d$), or if periodic, bounce-back of specular reflection conditions are imposed (exercise for the willful Reader):

$$\frac{d}{dt}\int_\Omega \rho(x,t)dx = \frac{d}{dt}\int_\Omega E(x,t)dx = 0,$$

while the total momentum is conserved in the case $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$:

$$\frac{d}{dt}\int_\Omega \rho(x,t)u(x,t)dx = 0.$$

There is another very interesting consequence of (9). Let us choose $\phi = \log f$. The monotonicity of the logarithm function implies

$$D(f) \equiv -\int_{\mathbb{R}^d} Q(f,f)(v)\log f(v)dv$$

$$= \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B[f' f'_* - f f_*][\log(f' f'_*) - \log(f f_*)] dv dv_* d\sigma \geq 0. \tag{11}$$

Define the functional

$$S(f) = \int_{\Omega \times \mathbb{R}^d} f \log f dx dv.$$

It follows that any solution f of the Boltzmann equation with no boundary or periodic, bounce-back or specular reflection boundary conditions satisfies

$$\frac{d}{dt} S(f) = - \int_{\Omega} D(f) dx \leq 0.$$

The functional S is called *Boltzmann entropy*, while $D(f)$ is the *entropy dissipation*. The statement that the entropy of a solution to the Boltzmann equation, under the aforementioned boundary conditions, is nonincreasing in time is known as the *Boltzmann H theorem*. It might not seems so, but this is a big deal. Indeed, such result implies the time irreversibility of the Boltzmann equation, which seems to clash with the reversibility of the collision process.

2.2.5 Local and global equilibria.

By definition, a *local thermodynamic equilibrium* of the Boltzmann equation is any solution $f = f(x, v, t)$ such that $D(f) \equiv 0$. It is possible to show that this only happens if

$$\log f(x, v, t) = \alpha_0(x, t) + \sum_{i=1}^{d} \alpha_i(x, t) v_i + \alpha_{d+1}(x, t) |v|^2$$

which corresponds to

$$f(x, v, t) = M_{\rho(x,t), u(x,t), \theta(x,t)}(v) \equiv \frac{\rho(x, t)}{(2\pi\theta(x, t))^{d/2}} e^{-\frac{|v - u(x,t)|^2}{2\theta(x,t)}}.$$

The function $M_{\rho,u,\theta}(v)$ is called the Maxwellian distribution with density ρ, momentum ρu, energy E.

A *global equilibrium* is a local equilibrium f such that $v \cdot \nabla_x f \equiv 0$. In the case of a torus there exists a unique global equilibrium $M_{\rho,u,\theta}(v)$ which is independent of x, t with moments ρ, ρu, θ equal to the corresponding moments of the initial distribution f_{in}. In such a case we can consider w.l.o.g.

$$M(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.$$

3 Formal hydrodynamical limits

This section is devoted to the formal derivation of the most relevant equations of fluid dynamics (Navier-Stokes, Euler) in compressible and incompressible regimes. The starting point is the Boltzmann equation in adimensional form. A scaling of the Boltzmann equation is needed since the macroscopic properties of the fluid arise from the mesoscopic dynamics described by the Boltzmann equation at different time and space scales. Simply put, one needs to specify what is big and what is small in order to get physically meaningful fluid equations.

3.1 Adimensional formulation of the Boltzmann equation

The macroscopic dynamics can be observed at a larger scale than the microscopic one. Let us consider a macroscopic length scale l_0 (typically the diameter of the domain containing the fluid) and a macroscopic time scale t_0. We also choose a reference (thermal) speed c_0 which is related to a reference temperature θ_0 and which equals the speed of sound in the case of a monoatomic gas. Let N_0 be the average number of particles in a volume of measure l_0^3 (so that $\rho_0 \equiv N_0 l_0^{-3}$ is the average particle density of the gas). We define adimensional variables

$$t' = \frac{t}{t_0}, \qquad x' = \frac{x}{l_0}, \qquad v' = \frac{v}{c_0},$$

and an adimensional distribution function

$$f'(x', v', t') = \frac{l_0^3 c_0^3}{N_0} f(x, v, t).$$

We define the *mean free time* τ_0 as follows:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B(v, v_*, \sigma) M_{(\rho_0,0,\theta_0)}(v) M_{(\rho_0,0,\theta_0)}(v_*) dv dv_* d\sigma = \frac{\rho_0}{\tau_0}.$$

τ_0 is the mean time between two consecutive collisions of a particle at equilibrium $M_{(\rho_0,0,\theta_0)}(v)$. A related quantity is the *mean free path*

$$\lambda_0 = c_0 \tau_0,$$

which equals the mean distance traveled by a particle between two consecutive collisions at equilibrium $M_{(\rho_0,0,\theta_0)}(v)$. Finally we define an adimensional collision kernel $B'(v', v'_*, \sigma) = \rho_0 \tau_0 B(v, v_*, \sigma)$ and we obtain an adimensional form of the Boltzmann equation (we omit the primes for the sake of a shorter notation):

$$\mathrm{Ma} \partial_t f + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f, f), \tag{12}$$

where the Mach number Ma and the Knudsen number Kn are defined as

$$\mathrm{Ma} = \frac{l_0}{c_0 t_0}, \qquad \mathrm{Kn} = \frac{c_0 \tau_0}{l_0} = \frac{\lambda_0}{l_0}.$$

The hydrodynamic models are obtained from the scaled Boltzmann equation (12) by taking the limit $\mathrm{Kn} \to 0$ and exploiting the conservation laws and entropy inequality satisfied by it.

3.2 The compressible Euler limit

As we are going to consider $\mathrm{Kn} \to 0$, let us denote the Knudsen number by ε, and consider $0 < \varepsilon \leq 1$. We stress the dependency of the solution to (12) on ε by writing f_ε in place of f. Eq. (12) becomes

$$\varepsilon \mathrm{Ma} \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon). \tag{13}$$

It we take the formal limit $\varepsilon \to 0$ in (13) and assume that $f_\varepsilon \to f$ than f satisfies $Q(f, f) = 0$, which implies

$$f(x, v, t) = M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{d/2}} \exp\left(-\frac{|v-u(x,t)|^2}{2\theta(x,t)}\right).$$

By integrating (13) in $\mathbb{R}^d$ and exploiting the conservation laws (10) we get

$$\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} f_\varepsilon dv + \nabla_x \cdot \int_{\mathbb{R}^d} v f_\varepsilon dv = 0, \tag{14}$$

$$\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} v f_\varepsilon dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v f_\varepsilon dv = 0, \tag{15}$$

$$\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_\varepsilon dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \frac{|v|^2}{2} f_\varepsilon dv = 0. \tag{16}$$

Taking the limit $\varepsilon \to 0$ in the above relations leads to

$$\begin{aligned}
\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv + \nabla_x \cdot \int_{\mathbb{R}^d} v M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv = 0,\\
\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} v M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv = 0,\\
\mathrm{Ma}\, \partial_t \int_{\mathbb{R}^d} \frac{|v|^2}{2} M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \frac{|v|^2}{2} M_{(\rho(x,t),u(x,t),\theta(x,t))}(v) dv = 0.
\end{aligned}$$

It holds

$$\begin{aligned}
&\int_{\mathbb{R}^d} M_{(\rho,u,\theta)}(v) dv = \rho,\\
&\int_{\mathbb{R}^d} v M_{(\rho,u,\theta)}(v) dv = \int_{\mathbb{R}^d} (v-u) M_{(\rho,u,\theta)}(v) dv + \int_{\mathbb{R}^d} u M_{(\rho,u,\theta)}(v) dv = \rho u,\\
&\int_{\mathbb{R}^d} v \otimes v M_{(\rho,u,\theta)}(v) dv = \int_{\mathbb{R}^d} (v-u) \otimes (v-u) M_{(\rho,u,\theta)}(v) dv + \int_{\mathbb{R}^d} u \otimes u M_{(\rho,u,\theta)}(v) dv\\
&\qquad = \frac{1}{d} \int_{\mathbb{R}^d} |v-u|^2 M_{(\rho,u,\theta)}(v) dv\, \mathbb{I} + \rho u \otimes u = \rho\theta\, \mathbb{I} + \rho u \otimes u,\\
&\int_{\mathbb{R}^d} \frac{|v|^2}{2} M_{(\rho,u,\theta)}(v) dv = \frac{1}{2} \mathrm{tr} \int_{\mathbb{R}^d} v \otimes v M_{(\rho,u,\theta)}(v) dv = \frac{d}{2}\rho\theta + \frac{\rho|u|^2}{2},
\end{aligned}$$

$$\int_{\mathbb{R}^d} v\frac{|v|^2}{2}M_{(\rho,u,\theta)}(v)dv = \int_{\mathbb{R}^d}(v-u)\otimes(v-u)M_{(\rho,u,\theta)}(v)dv\, u$$
$$+\frac{1}{2}\int_{\mathbb{R}^d}|v-u|^2M_{(\rho,u,\theta)}(v)dv\, u+\frac{1}{2}\int_{\mathbb{R}^d}M_{(\rho,u,\theta)}(v)dv\,|u|^2u$$
$$=\rho\theta u+\frac{d}{2}\rho\theta u+\frac{\rho}{2}|u|^2u=\rho u\left(\frac{d+2}{2}\theta+\frac{1}{2}|u|^2\right).$$

Therefore we are left with

$$\mathrm{Ma}\,\partial_t\rho+\nabla_x\cdot(\rho u)=0, \tag{17}$$
$$\mathrm{Ma}\,\partial_t(\rho u)+\nabla_x\cdot(\rho u\otimes u)+\nabla_x(\rho\theta)=0, \tag{18}$$
$$\mathrm{Ma}\,\partial_t\left(\rho\left(\frac{1}{2}|u|^2+\frac{d}{2}\theta\right)\right)+\nabla_x\cdot\left(\rho u\left(\frac{d+2}{2}\theta+\frac{1}{2}|u|^2\right)\right)=0. \tag{19}$$

Eqs. (17)–(19) are the compressible Euler equations for a perfect monoatomic gas with pressure equal to $\rho\theta$ and internal energy $e=\frac{1}{2}|u|^2+\frac{d}{2}\theta$.

We have used the conservation laws thus far. Let us now apply the H-theorem. By multiplying (13) times $\log f_\varepsilon$, integrating in $\mathbb{R}^d$ and applying (11) we deduce

$$\mathrm{Ma}\partial_t\int_{\mathbb{R}^d} f_\varepsilon\log f_\varepsilon dv+\nabla_x\cdot\int_{\mathbb{R}^d} v f_\varepsilon\log f_\varepsilon dv\le 0.$$

Again, by taking the limit $\varepsilon\to 0$ in the above inequality and computing some Gaussian integral we obtain

$$\mathrm{Ma}\partial_t\left(\rho\log(\rho\theta^{-d/2})\right)+\nabla_x\cdot\left(\rho u\log(\rho\theta^{-d/2})\right)\le 0. \tag{20}$$

Inequality (20) is known as the Lax admissibility condition: it singles out among the solutions to the Euler equation the physically relevant ones.

3.3 The compressible Navier-Stokes limit

The idea behind the compressible Navier-Stokes limit is to write the solution f_ε to the Boltzmann equation (13) as sum of the Maxwellian equilibrium $M_{(\rho,u,\theta)}$ and higher order perturbations:

$$f_\varepsilon(x,v,t)=\sum_{k=0}^{\infty}\varepsilon^k g_k(x,v,t),\qquad g_0\equiv M_{(\rho,u,\theta)}.$$

This is called *Hilbert expansion* (or Chapman-Enskog expansion). If we plug this expansion inside the adimensional Boltzmann equation we obtain (obviously) a superposition of terms of different order. Let us consider the zeroth order term. Since $Q(M_{(\rho,u,\theta)},M_{(\rho,u,\theta)})=0$ it holds

$$L(g_1)\equiv Q(M_{(\rho,u,\theta)},g_1)+Q(g_1,M_{(\rho,u,\theta)})=\mathrm{Ma}\partial_t(M_{(\rho,u,\theta)})+v\cdot\nabla_x(M_{(\rho,u,\theta)}). \tag{21}$$

Eq. (21) provides us with with an (implicit) expression for g_1. The operator L is the linearized Boltzmann operator (around the equilibrium M). Being (21) a linear inhomogeneous equation in g_1, its solution reads as $g_1 = g^* + \phi$, where g^* is a fixed solution to (21) while ϕ is an arbitrary solution to the homogeneous problem $L(\phi) = 0$. Under the right assumptions on Q we can solve (21). Let us first compute the right-hand side of (21). It holds

$$\begin{aligned}
&\mathrm{Ma}\partial_t(M_{(\rho,u,\theta)}) + v\cdot\nabla_x(M_{(\rho,u,\theta)}) = -\left(\frac{1}{2}\xi(V) : D(u) + \eta(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}}\right)M_{(\rho,u,\theta)},\\
&V = \frac{v-u}{\sqrt{\theta}}, \qquad D(u) = \frac{1}{2}(\nabla_x u + (\nabla_x u)^\top) - \frac{1}{d}(\nabla_x\cdot u)\mathbb{I},\\
&\xi(v) = v\otimes v - \frac{1}{d}|v|^2\mathbb{I}, \qquad \eta(v) = \frac{1}{2}v\left(|v|^2 - (d+2)\right).
\end{aligned} \tag{22}$$

L is a linear operator. Moreover $L(\lambda g) = \lambda L(g)$ if $\lambda = \lambda(x,t)$ does not depend on v. Therefore we try the following ansatz:

$$g_1 = -\frac{1}{2}D(u) : g_{1,\xi} - \frac{\nabla_x\theta}{\sqrt{\theta}}\cdot g_{1,\eta} + \phi_1,$$

with $g_{1,\xi}$, $g_{1,\eta}$ solving

$$L(g_{1,\xi}) = \xi(V)g_0, \quad L(g_{1,\eta}) = \eta(V)g_0, \quad \int_{\mathbb{R}^d}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} g_{1,\xi}dv = \int_{\mathbb{R}^d}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} g_{1,\eta}dv = 0,$$

while ϕ_1 solves

$$L(\phi_1) = 0, \qquad \int_{\mathbb{R}^d}\begin{pmatrix}1\\ v\\ |v|^2/2\end{pmatrix}\phi_1 dv = \begin{pmatrix}\rho_1\\ \rho_1 u_1\\ \rho_1 e_1\end{pmatrix},$$

with ρ_1, $\rho_1 u_1$, $\rho_1 e_1$ being the moments of g_1, i.e. the first order corrections to the moments of g_0. It is possible to prove [5] that a solution to the above problem is given by

$$g_{1,\xi} = \frac{\theta^{\gamma/2}}{\rho}\alpha(|V|)\xi(V)g_0, \qquad g_{1,\eta} = \frac{\theta^{\gamma/2}}{\rho}\beta(|V|)\eta(V)g_0,$$

for suitable functions $\alpha, \beta : [0,\infty) \to [0,\infty)$. It follows

$$g_1 = -\frac{\theta^{\gamma/2}}{\rho}\left(\frac{1}{2}\alpha(|V|)\xi(V) : D(u) + \beta(|V|)\eta(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}}\right)g_0 + \phi_1.$$

All that remains is to find ϕ_1. We first notice that L can be rewritten as

$$L(\phi) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} BM'M'_*[\psi' + \psi'_*]dv_*d\sigma - \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} BMM_*[\psi + \psi_*]dv_*d\sigma,$$

where $\psi \equiv \phi/M$. However, the fact that collisions preserve momentum and energy implies that $M'M'_* = MM_*$, therefore

$$L(\phi) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} BMM_*[\psi' + \psi'_* - \psi - \psi_*] dv_* d\sigma.$$

Let $L(\phi) = 0$. Standard symmetry arguments (remember the discussion in Section 2.2.4) allow us to write

$$0 = \int_{\mathbb{R}^d} \psi L(\phi) dv = -\frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} BMM_*[\psi' + \psi'_* - \psi - \psi_*]^2 dv dv_* d\sigma.$$

This implies that

$$\psi' + \psi'_* - \psi - \psi_* \equiv 0 \qquad v, v_* \in \mathbb{R}^d, \quad \sigma \in \mathbb{S}^{d-1}. \tag{23}$$

We wish to solve (23). The argument we are going to employ can be found in [3]. Let us define $U = v' - v$, $V = v'_* - v$. It follows that

$$4U \cdot V = (v_* - v + |v - v_*|\sigma) \cdot (v_* - v - |v - v_*|\sigma) = |v - v_*|^2 - |v - v_*|^2 = 0.$$

Therefore solving (23) is equivalent to solving

$$\psi(v + U + V) + \psi(v) = \psi(v + U) + \psi(v + V) \equiv 0 \qquad v, U, V \in \mathbb{R}^d, \quad U \cdot V = 0. \tag{24}$$

Fix $v \in \mathbb{R}^d$ in an arbitrary way and define $\Psi(z) \equiv \psi(v + z) - \psi(v)$, $z \in \mathbb{R}^d$. Then (24) implies in particular

$$\Psi(U + V) = \Psi(U) + \Psi(V) \qquad U, V \in \mathbb{R}^d, \quad U \cdot V = 0. \tag{25}$$

At this point we need a result by Cauchy (which the willful Reader can try to prove, it's not so difficult) stating that, if $\chi : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\chi(x) + \chi(y) = \chi(x + y)$ for $x, y \in \mathbb{R}$ (or $\mathbb{R}_+$), then $\chi_0 \in \mathbb{R}$ exists such that $\chi(x) = \chi_0 x$ for $x \in \mathbb{R}$ (resp. $\mathbb{R}_+$).

Let us define $k(U) \equiv (\Psi(U) + \Psi(-U))/2$, $h(U) \equiv (\Psi(U) - \Psi(-U))/2$. Clearly k, h satisfy (25). Let us consider k. Let $p_1, p_2 \in \mathbb{R}^d$ such that $|p_1| = |p_2|$, and define $U = (p_1 + p_2)/2$, $V = (p_1 - p_2)/2$. Since $U \cdot V = 0$, it holds

$$k(p_1) = k(U + V) = k(U) + k(V) = k(U) + k(-V) = k(U - V) = k(p_2).$$

This means that k is a radial function: $k(U) = \tilde{k}(|U|^2)$, $U \in \mathbb{R}^d$, for some $\tilde{k} : \mathbb{R}_+ \to \mathbb{R}$. It follows

$$\tilde{k}(|U|^2) + \tilde{k}(|V|^2) = k(U) + k(V) = k(U + V) = \tilde{k}(|U + V|^2) = \tilde{k}(|U|^2 + |V|^2).$$

Cauchy's result implies

$$k(U) = \tilde{k}(|U|^2) = k_0 |U|^2, \qquad U \in \mathbb{R}^d. \tag{26}$$

Let us now focus our attention on h. Let $(e^{(i)})_{i=1,\dots,d}$ be an arbitrary orthonormal basis of $\mathbb{R}^d$. Let $U, V \in \mathbb{R}^d$ such that $U \cdot V = 0$, and write $U = \sum_{j=1}^d U_j e^{(j)}$, $V = \sum_{j=1}^d V_j e^{(j)}$. Since $e^{(i)} \cdot e^{(j)} = 0$ for $i \neq j$ then

$$\begin{aligned}\sum_{j=1}^d (h(U_j e^{(j)}) + h(V_j e^{(j)})) &= \left(h\left(\sum_{j=1}^d U_j e^{(j)}\right) + h\left(\sum_{j=1}^d V_j e^{(j)}\right)\right) = h(U) + h(V) \\ &= h(U+V) = h\left(\sum_{j=1}^d (U_j + V_j) e^{(j)}\right) = \sum_{j=1}^d h\left((U_j + V_j)e^{(j)}\right),\end{aligned}$$

and so

$$h(U_i e^{(i)}) + h(V_i e^{(i)}) - h\left((U_i + V_i)e^{(i)}\right) = \sum_{j \neq i} \left(h\left((U_j + V_j)e^{(j)}\right) - h(U_j e^{(j)}) - h(V_j e^{(j)}) \right),$$

for any $i = 1, \dots, d$. Since h is odd then the right-hand side of the above inequality is odd w.r.t. $(U_j, V_j)_{j \neq i}$, while the left-hand side is constant w.r.t. the same variables. This means that

$$h(U_i e^{(i)}) + h(V_i e^{(i)}) - h\left((U_i + V_i)e^{(i)}\right) = 0 \qquad i = 1, \dots, d.$$

Again, Cauchy's result implies that

$$h(U_i e^{(i)}) = \tilde{h}_i U_i \qquad U_i \in \mathbb{R}, \quad i = 1, \dots, d.$$

Summing the above relations and exploiting again the orthogonality of the $(e^{(i)})_{i=1,\dots,d}$ yields

$$h(U) = \tilde{h} \cdot U \qquad U \in \mathbb{R}^d, \tag{27}$$

for a suitable $\tilde{h} \in \mathbb{R}^d$. Since $\Psi = h + k$ and $\Psi(z) = \psi(v+z) - \psi(v)$, $z \in \mathbb{R}^d$, from (26), (27) we conclude

$$\psi(v + U) - \psi(v) = h(U) + k(U) = k_0 |U|^2 + \tilde{h} \cdot U \qquad U \in \mathbb{R}^d.$$

As a consequence we have an explicit expression for ϕ_1:

$$\phi_1(x,v,t) = M(x,v,t)\psi(x,v,t) = (\psi_0(x,t) + \psi_1(x,t) \cdot v + \psi_2(x,t)|v|^2) g_0(x,v,t).$$

The quantities $\psi_i(x,t)$, $i = 0,1,2$, can be explicitly computed in terms of the moments $\rho_1(x,t)$, $\rho_1(x,t)u(x,t)$, $\rho_1(x,t)e_1(x,t)$.

At this point we have all the pieces of the puzzle. Let

$$\rho_\varepsilon = \rho + \varepsilon \rho_1, \qquad u_\varepsilon = u + \varepsilon u_1, \qquad e_\varepsilon = e + \varepsilon e_1.$$

By replacing $f_\varepsilon = g_0 + \varepsilon g_1$ in (14)–(16) and carrying out some straightforward computations we conclude that $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ satisfy the Navier-Stokes equations:

$$\mathrm{Ma}\partial_t \rho_\varepsilon + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0, \tag{28}$$

$$\text{Ma}\partial_t(\rho_\varepsilon u_\varepsilon) + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x(\rho_\varepsilon \theta_\varepsilon) = \varepsilon \nabla_x[\nu(\rho_\varepsilon, \theta_\varepsilon) D(u_\varepsilon)], \tag{29}$$

$$\text{Ma}\partial_t E_\varepsilon + \nabla_x \cdot (u_\varepsilon(E_\varepsilon + \rho_\varepsilon \theta_\varepsilon)) = \frac{\varepsilon}{2}\nu(\rho_\varepsilon, \theta_\varepsilon)\text{tr}(D(u_\varepsilon)^2) + \varepsilon \nabla_x \cdot [\kappa(\rho_\varepsilon, \theta_\varepsilon)\nabla_x \theta_\varepsilon], \tag{30}$$

where the energy E_ε equals the one of a monoatomic gas:

$$E_\varepsilon = \rho_\varepsilon \left(\frac{1}{2}|u_\varepsilon|^2 + \frac{d}{2}\theta_\varepsilon\right),$$

while the dynamic viscosity ν and the thermal conductivity κ are given by

$$\nu(\rho_\varepsilon, \theta_\varepsilon) = \rho_\varepsilon \theta_\varepsilon^{\gamma/2} \int_{\mathbb{R}^d} \alpha(|V|)\text{tr}(\xi(V)^2)\frac{e^{-|V|^2/2}}{(2\pi)^{d/2}}dV,$$

$$\kappa(\rho_\varepsilon, \theta_\varepsilon) = \rho_\varepsilon \theta_\varepsilon^{\gamma/2} \int_{\mathbb{R}^d} \beta(|V|)|\eta(V)|^2\frac{e^{-|V|^2/2}}{(2\pi)^{d/2}}dV.$$

3.4 Incompressible fluid limits

For this section we refer to [5].

The compressibility of a fluid is measured by the Mach number Ma. Therefore, in order to derive incompressible fluid equations we should consider Ma $\to 0$. But how small is Ma compared to $\varepsilon = \text{Kn}$? To answer this question we consider the von Karman relation:

$$\text{Re} = \frac{\text{Ma}}{\text{Kn}}.$$

Here Re is the Reynolds number, and it described the viscosity of a fluid. Physics demands the viscosity of a fluid to be a finite quantity, therefore Ma should tend to zero at least like ε when $\varepsilon \to 0$. The simplest choice for Ma is clearly $\text{Ma} = \varepsilon^p$ for some $p \geq 1$. It turns out that choosing $p > 1$ yields the incompressible Euler equations, while the choice $p = 1$ leads to the incompressible Navier-Stokes equations. Let us consider the latter case and define $\text{Ma} = \text{Kn} = \varepsilon$. The scaled Boltzmann equation (13) becomes

$$\varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon). \tag{31}$$

We consider the system to be close to equilibrium, and the distribution function f_ε to be equal to a small perturbation of a global Maxwellian:

$$f_\varepsilon = M(1 + \varepsilon g_\varepsilon), \qquad M = M_{(1,0,1)}.$$

This is indeed the consitutive assumption in the derivation of incompressible fluid models. Inserting the above relation inside (31) leads to

$$\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon} L_M(g_\varepsilon) + Q_M(g_\varepsilon, g_\varepsilon), \tag{32}$$

where L_M, Q_M are defined as

$$L_M(g) = \frac{1}{M}(Q(M, Mg) + Q(Mg, M)), \qquad Q_M(g, g) = \frac{1}{M}Q(Mg, Mg).$$

By taking the limit $\varepsilon \to 0$ in (32) we deduce $L_M(g) = 0$, where $g \equiv \lim_{\varepsilon\to 0} g_\varepsilon$ (which we assume it exists). It can be shown that this implies

$$g = \rho + v \cdot u + \frac{|v|^2 - d}{2}\theta. \tag{33}$$

The zeroth and first order moments of (32) are as follows

$$\varepsilon\partial_t \int_{\mathbb{R}^d} g_\varepsilon dv + \nabla_x \cdot \int_{\mathbb{R}^d} v g_\varepsilon M dv = 0, \qquad \varepsilon\partial_t \int_{\mathbb{R}^d} v g_\varepsilon dv + \nabla_x \cdot \int_{\mathbb{R}^d} (v \otimes v) g_\varepsilon M dv = 0. \tag{34}$$

By taking the limit $\varepsilon \to 0$ in the above relations we deduce

$$\nabla_x \cdot u = 0, \qquad \nabla_x(\rho + \theta) = 0. \tag{35}$$

In (35) we recognize the incompressibility condition and the Boussinesq relation. Repeating this argument while considering the energy conservation equation leads again to the condition $\nabla \cdot u = 0$.

Now it is time to derive the evolution equations for u, θ. We start by noticing that the first relation in (34) is equivalent to

$$\partial_t \int_{\mathbb{R}^d} g_\varepsilon dv + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^d} \left(v \otimes v - \frac{|v|^2}{d}\right) g_\varepsilon M dv + \frac{1}{\varepsilon}\nabla \int_{\mathbb{R}^d} \frac{|v|^2}{d} g_\varepsilon M dv = 0,$$

which can also be restated as

$$\partial_t \int_{\mathbb{R}^d} g_\varepsilon dv + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^d} \xi(v) g_\varepsilon M dv + \frac{1}{\varepsilon}\nabla p_\varepsilon = 0, \tag{36}$$

where p_ε is the pressure:

$$p_\varepsilon = \int_{\mathbb{R}^d} \frac{|v|^2}{d} g_\varepsilon M dv,$$

while $\xi(v)$ is as in the previous section (see eq. (22)). In the same way, from the mass and energy conservation it follows

$$\partial_t \int_{\mathbb{R}^d} \frac{|v|^2 - (d+2)}{2} g_\varepsilon M dv + \frac{1}{\varepsilon}\nabla_x \int_{\mathbb{R}^d} \eta(v) g_\varepsilon M dv = 0. \tag{37}$$

It holds (formally)

$$\lim_{\varepsilon\to 0} \partial_t \int_{\mathbb{R}^d} v g_\varepsilon M dv = \partial_t \int_{\mathbb{R}^d} v g M dv = \partial_t u,$$

$$\lim_{\varepsilon\to 0}\partial_t \int_{\mathbb{R}^d} \frac{|v|^2-(d+2)}{2} g_\varepsilon M dv = \partial_t \int_{\mathbb{R}^d} \frac{|v|^2-(d+2)}{2} g M dv = \frac{d+2}{2}\partial_t\theta.$$

We must now estimate the limits of the terms involving $\eta(v)$ and $\xi(v)$ in (36), (37). We do this by exploiting the self-adjointness of L_M in the space $L^2(\mathbb{R}^d, Mdv)$. Let $R(L_M)$, $N(L_M)$ be the image and null space of L_M, respectively. Being L_M self-adjoint then $R(L_M) = N(L_M)^\perp$. We define the operator $L_M^{-1} : N(L_M)^\perp \to N(L_M)^\perp$ ("pseudo-inverse" of L_M) as the inverse of $L_M|_{N(L_M)^\perp} : N(L_M)^\perp \to N(L_M)^\perp$, i.e. the restriction of L_M to $N(L_M)^\perp$. With this setting in mind, we can write

$$\int_{\mathbb{R}^d} \xi(v) g_\varepsilon M dv = \int_{\mathbb{R}^d} L_M^{-1}(\xi(v)) L_M(g_\varepsilon) M dv, \quad \int_{\mathbb{R}^d} \eta(v) g_\varepsilon M dv = \int_{\mathbb{R}^d} L_M^{-1}(\eta(v)) L_M(g_\varepsilon) M dv.$$

From (32) we deduce that

$$\frac{1}{\varepsilon} L_M(g_\varepsilon) = v\cdot\nabla_x g_\varepsilon - Q_M(g_\varepsilon, g_\varepsilon) + \varepsilon\partial_t g_\varepsilon \to v\cdot\nabla_x g - Q_M(g,g) \quad \text{as } \varepsilon\to 0.$$

Therefore we are left with the following evolution equations for u, θ:

$$\partial_t u + \nabla_x \cdot \int_{\mathbb{R}^d} L_M^{-1}(\xi(v))[v\cdot\nabla_x g - Q_M(g,g)] M dv = 0, \tag{38}$$

$$\frac{d+2}{2}\partial_t\theta + \nabla_x \cdot \int_{\mathbb{R}^d} L_M^{-1}(\eta(v))[v\cdot\nabla_x g - Q_M(g,g)] M dv = 0. \tag{39}$$

The attentive Reader has surely notice that something is missing in (38): namely, the pressure. As a matter of fact, given the incompressibility constraint $\nabla_x \cdot u$, the equation for u is usually considered in the weak formulation as tested against a divergence-free vector-valued test function; for this reason any gradient field added to the equation does not change its weak formulation. Therefore we will neglect from now on any gradient field in eq. (38).

From (33) and the Boussinesq relation (35) it follows

$$\begin{aligned}\nabla_x \int_{\mathbb{R}^d} v\otimes L_M^{-1}(\eta(v)) g M dv &= \int_{\mathbb{R}^d} L_M^{-1}(\eta(v))\otimes v \frac{|v|^2-(d+2)}{2} M dv\cdot\nabla_x\theta \\ &= -\int_{\mathbb{R}^d} \beta(|v|)\eta(V)\otimes\eta(V) M dv\cdot\nabla_x\theta,\end{aligned}$$

which yields the thermal diffusion term in the equation for the temperature:

$$\nabla_x^2 : \int_{\mathbb{R}^d} v\otimes L_M^{-1}(\eta(v)) g M dv = \kappa\Delta\theta. \tag{40}$$

Moreover, the term involving ξ reads as

$$\nabla_x \cdot \int_{\mathbb{R}^d} v\otimes L_M^{-1}(\xi(v)) g M dv = \int_{\mathbb{R}^d} L_M^{-1}(\xi(v))\otimes(v\otimes v) M dv : \nabla_x u$$

$$= -\int_{\mathbb{R}^d} \alpha(|v|)\xi(v) \otimes \xi(v) M dv : \nabla_x u.$$

The above expression gives rise to the viscous term in the equation for u:

$$\nabla_x^2 : \int_{\mathbb{R}^d} v \otimes L_M^{-1}(\xi(v)) g M dv = \mu \Delta u. \tag{41}$$

Now we need to deal with the terms containing $Q_M(g, g)$. This is done by exploiting the fact [1] that

$$Q_M(g, g) = -L_M(g^2) \qquad \text{for all } g \in N(L_M).$$

See [5] for more details. From the above relation and the self-adjointness of L_M it follows

$$\int_{\mathbb{R}^d} L_M^{-1}(\eta(v)) Q_M(g, g) M dv = -\int_{\mathbb{R}^d} L_M^{-1}(\eta(v)) L_M(g^2) M dv$$
$$= -\int_{\mathbb{R}^d} \eta(v) g^2 M dv = -(d+2) u\theta, \tag{42}$$
$$\int_{\mathbb{R}^d} L_M^{-1}(\xi(v)) Q_M(g, g) M dv = -\int_{\mathbb{R}^d} L_M^{-1}(\xi(v)) L_M(g^2) M dv$$
$$= -\int_{\mathbb{R}^d} \xi(v) g^2 M dv = -2\xi(u). \tag{43}$$

Putting relations (38)–(43) together yields the following incompressible Navier-Stokes equations:

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u, \qquad \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta \theta, \tag{44}$$

where κ, μ are suitable positive constants, representing thermal and cinematic viscosity (respectively).

4 Analytical study of the Boltzmann equation

In this part we will perform an analytically rigorous study of the Boltzmann equation in a perturbative setting (that is, we consider the linearized Boltzmann equation) as the Knudsen number tends to zero, and prove that the moments of the solution to the Boltzmann equation converge to a solution to the incompressible Navier-Stokes equation.

We consider the equation in the $d-$dimensional torus $\mathbb{T}^d$ ($d \geq 2$), which amounts to assume periodic boundary conditions (for the sake of simplicity). Recall the a-dimensional Boltzmann equation

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \qquad \text{on } \mathbb{T}^d \times \mathbb{R}^d, \tag{45}$$

[1] It can be proved by differentiating the relation

$$Q(M_{(\rho,u,\theta)}, M_{(\rho,u,\theta)}) = 0$$

twice with respect to (ρ, u, θ) and carrying out some computations.

where $\varepsilon =$ Mach number $=$ Knudsen number and the collision operator Q is defined as

$$Q(f,g) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v-v_*|) b(\cos\theta)[f'g'_* - fg_*] dv_* d\sigma. \tag{46}$$

Also remember the shortened notation $f' \equiv f(v')$, $f_* \equiv f(v_*)$ etc. and the definition of the pre-collisional velocities

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \qquad \cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma.$$

We already know that the global equilibria are normalized Maxwellians, which we will denote from now on with $\mu(v)$:

$$\mu(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}.$$

Notation. Let $j, l \in \mathbb{R}^d$ arbitrary multi-indexes. We define

- $\partial_l^j = \partial_{v_j} \partial_{x_l}$,
- $c_i(j) \equiv j_i$ is the $i-$th component of j,
- $|j| \equiv \sum_{i=1}^d c_i(j)$ is the length of j,

Moreover, δ_{i_0} is the Kronecker delta, i.e. $c_i(\delta_{i_0}) = 1$ if $i = i_0$ and zero otherwise; $L^p_{x,v} \equiv L^p(\mathbb{T}^d \times \mathbb{R}^d)$, $L^p_x \equiv L^p(\mathbb{T}^d)$, $L^p_v \equiv L^p(\mathbb{R}^d)$ are the stardard Lebesgue spaces, while the standard Sobolev spaces $H^s_{x,v}$, H^s_x, H^s_v are defined in an analogue way; finally, the standard Sobolev norms $\|\cdot\|_{H^s_{x,v}}$, $\|\cdot\|_{H^s_x}$, $\|\cdot\|_{H^s_v}$ are defined in the natural way (e.g. $\|f\|^2_{H^s_{x,v}} = \sum_{|j|+|l| \leq s} \|\partial_l^j f\|^2_{L^2_{x,v}}$).

4.1 Motivation and goals

We have already seen that hydrodynamic models can be obtained from the Boltzmann equation by taking the limit Kn $\to 0$. Physically speaking, this makes sense since the Knudsen number is the inverse of the average number of collisions for each particle per unit time. We are going to prove this convergence in a rigorous way and specify in what sense we have convergence.

We will obtain the incompressible Navier-Stokes equations by considering the Boltzmann distribution f as a perturbation (of order ε) of a global Maxwellian

$$f(x,v,t) = f_\varepsilon(x,v,t) = \mu(v) + \varepsilon \mu(v)^{1/2} h_\varepsilon(x,v,t).$$

The perturbation h_ε satisfies the following linearized Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon}\Gamma(h_\varepsilon, h_\varepsilon), \tag{47}$$

$$L(h) = \mu^{-1/2}[Q(\mu, \mu^{1/2}h) + Q(\mu^{1/2}h, \mu)], \tag{48}$$

$$\Gamma(g, h) = \frac{1}{2}[Q(\mu^{1/2}g, \mu^{1/2}h) + Q(\mu^{1/2}h, \mu^{1/2}g)]. \tag{49}$$

We consider the case of hard or Maxwellian potential, i.e.

$$\exists \gamma \in [0, 1], \quad \exists C_\Phi > 0 \quad : \quad \Phi(z) = C_\Phi z^\gamma \qquad z > 0. \tag{50}$$

and the following strong form of Grad's angular cutoff:

$$\exists C_b > 0 \quad : \quad b(z),\ b'(z) \leq C_b \qquad z \in \mathbb{R}. \tag{51}$$

Our first goal is to show existence and exponential decay for solutions to the linearized Boltzmann equation (47) by means of a constructive method. The result we will obtain will be uniform in the Knudsen number. We will then employ this result to derive explicit rates of convergence for $(h_\varepsilon)_{\varepsilon>0}$ towards its limit as $\varepsilon \to 0$. As a consequence, we will be able to prove and quantify the convergence from the Boltzmann equation to the incompressible Navier-Stokes equations.

In order to achieve our goal we will build a norm on Sobolev spaces, which is topologically equivalent to the standard Sobolev norm, and such that the Boltzmann distribution satisfies a Gronwall type inequality with respect to the new norm. This latter will have the form of a linear combination of quadratic terms like $(\partial^{j_1}_{l_1} f, \partial^{j_2}_{l_2} f)_{L^2_{x,v}}$. Such mixed terms must be considered in the norm in order to control $\|f\|$, due to the hypocoercivity property of the linearized Boltzmann equation. The coefficients of the linear combination in the norm must be chosen in a careful way in order to have an energy estimate and, at the same time, a norm equivalent to the standard $H^s_{x,v}$ norm.

We first apply this method to the linear case (i.e. we neglect the quadratic contribution $\Gamma(h, h)$), and we prove it generates a strong semigroup with a spectral gap, which leads to exponential decay. We will then consider the nonlinear case and show existence of solutions to the initial-boundary value problem for (47) for small initial data. The smallness condition will be independent of ε. This fact allows us to study the limit of the sequence $(h_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$. We will see that it is weakly convergent in $L^\infty_t H^s_x L^2_v$ for $s \geq s_0 > d/2$. Its weak limit h has the form

$$h(x, v, t) = \left[\rho(x, t) + v \cdot u(x, t) + \frac{1}{2}(|v|^2 - N)\theta(x, t)\right] \mu(v)^{1/2}.$$

The physical observables associated to h are weak solutions (in the Leray sense [7]) of the incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla_x u - \nu \Delta u + \nabla p = 0, \tag{52}$$

$$\nabla_x \cdot u = 0, \tag{53}$$

$$\partial_t \theta + u \cdot \nabla_x \theta - \kappa \Delta \theta = 0, \tag{54}$$

where p is the pressure and the constants ν (viscosity), κ (thermal diffusivity) are determined by L. The Boussinesq relation

$$\nabla_x(\rho + \theta) = 0 \tag{55}$$

is also satisfied.

The final step in the derivation is the study of the initial data of the moments u, θ. This is carried out by considering the Fourier transform on the torus of the linear operator and use Duhamel's formula. From this study we obtain strong convergence for the time average of h_ε with an explicit decay rate. From this result and the exponential decay of h and h_ε we conclude a strong convergence of h_ε towards h for all times.

4.2 Main results

In this section we state the main results about the linearized Boltzmann equation and the convergence of its solutions to the corresponding solutions of the incompressible Navier-Stokes equations. We start by stating the assumptions.

These are the assumptions on the linear operator L defined in (48).

H1 (Coercivity and controls) $L : L^2_v \to L^2_v$ is a closed, self-adjoint linear operator. Moreover $L = K - \Lambda$ where $\Lambda, K : L^2_v \to L^2_v$ are linear operator with the following properties.

- There exists a norm $\|\cdot\|_{\Lambda_v}$ on L^2_v and positive constants $\nu_0^\Lambda \ldots \nu_4^\Lambda$ such that

$$\nu_0^\Lambda \|h\|^2_{L^2_v} \le \nu_1^\Lambda \|h\|^2_{\Lambda_v} \le (h, \Lambda(h))_{L^2_v} \le \nu_2^\Lambda \|h\|^2_{\Lambda_v}, \qquad h \in L^2_v,$$

$$(\nabla_v \Lambda(h), \nabla_v h)_{L^2_v} \ge \nu_3^\Lambda \|\nabla_v h\|^2_{\Lambda_v} - \nu_4^\Lambda \|h\|^2_{\Lambda_v}, \qquad h \in H^1_v.$$

- There exists a positive constant C^L such that

$$(L(h), g)_{L^2_v} \le C^L \|h\|_{\Lambda_v} \|g\|_{\Lambda_v} \qquad h, g \in L^2_v.$$

H2 (Mixing property in velocity) For any $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$(\nabla_v K(h), \nabla_v h) \le C(\delta)\|h\|^2_{L^2_v} + \delta \|\nabla_v h\|^2_{L^2_v} \qquad h \in H^1_v.$$

H3 (Relaxation to equilibrium) The null space $N(L)$ of L has finite dimension N. We assume that there exists $(\phi_i(v))_{i=1,\ldots,N}$ orthonormal (w.r.t. the L^2_v scalar product) basis of $N(L)$ with the form $\phi_i(v) = P_i(v)^{-|v|^2/4}$, where $P_i(v)$ is a polynomial. We denote with π_L the prthogonal projection on $N(L)$. We assume that there exists a constant $\lambda > 0$ such that

$$(L(h), h)_{L^2_v} \le -\lambda \|(I - \pi_L)h\|^2_{\Lambda_v} \qquad v \in L^2_v.$$

H1' Assume (H1) holds. Moreover, assume that positive constants ν_5^Λ, ν_6^Λ such that for all $s \geq 1$ and all multi-indexes j, l with $|j| + |l| = s$, $|j| \geq 1$,

$$(\partial_l^j \Lambda(h), \partial_l^j h)_{L_v^2} \geq \nu_5^\Lambda \|\partial_l^j h\|_\Lambda^2 - \nu_6^\Lambda \|h\|_{H_{x,v}^{s-1}} \qquad v \in H_{x,v}^s.$$

H2' (Higher order mixing property) Assume (H2) holds. Moreover assume that for any $\delta > 0$ there exists a constant $C(\delta) > 0$ such that: for all $s \geq 1$ and all multi-indexes j, l with $|j| + |l| = s$, $|j| \geq 1$,

$$(\partial_l^j K(h), \partial_l^j h)_{L_{x,v}^2} \leq C(\delta)\|h\|_{H_{x,v}^{s-1}}^2 + \delta \|\partial_l^j h\|_{L_{x,v}^2}^2, \qquad v \in H_{x,v}^s.$$

Now it's time to make some hypothesis on the quadratic operator Γ.

H4 $\Gamma : L_v^2 \times L_v^2 \to L_v^2$ is a bilinear symmetric operator such that for all $s \geq 0$ and all multi-indexes j, l with $|j| + |l| = s$,

$$|(\partial_l^j \Gamma(g,h), f)_{L_{x,v}^2}| \leq \begin{cases} \mathscr{G}_{x,v}^s(g,h)\|f\|_\Lambda & \text{if } j \neq 0, \\ \mathscr{G}_x^s(g,h)\|f\|_\Lambda & \text{if } j = 0, \end{cases}$$

where $\mathscr{G}_{x,v}^s$, $\mathscr{G}_x^s$ are suitable operators such that $\mathscr{G}_{x,v}^s \leq \mathscr{G}_{x,v}^{s+1}$, $\mathscr{G}_x^s \leq \mathscr{G}_x^{s+1}$, and

$$\exists s_0 \in \mathbb{N} : \ \forall s \geq s_0, \ \exists C_\Gamma > 0 : \quad \begin{cases} \mathscr{G}_{x,v}^s(g,h) \leq C_\Gamma(\|g\|_{H_{x,v}^s}\|h\|_{H_\Lambda^s} + \|h\|_{H_{x,v}^s}\|g\|_{H_\Lambda^s}) \\ \mathscr{G}_x^s(g,h) \leq C_\Gamma(\|g\|_{H_x^s L_v^2}\|h\|_{H_\Lambda^s} + \|h\|_{H_x^s L_v^2}\|g\|_{H_\Lambda^s}) \end{cases}$$

for all $v \in H_{x,v}^s$, and $\|\cdot\|_{H_\Lambda^s}$ is defined as

$$\|f\|_{H_\Lambda^s}^2 = \sum_{|j|+|l| \leq s} \|\partial_l^j f\|_\Lambda^2, \qquad f \in H_{x,v}^s.$$

H5 For all $g, h \in \mathrm{Dom}(\Gamma) \cap L_v^2$ it holds $\Gamma(g,h) \in N(L)^\perp$.

We now state the main results.

The first result concerns the linear Boltzmann equation, i.e. (47) without the term involving Γ. We define preliminarily a functional on $H_{x,v}^s$ $(s \geq 0)$:

$$\|\cdot\|_{\mathcal{H}_\varepsilon^s} = \left[\sum_{\substack{|j|+|l| \leq s \\ |j| \geq 1}} b_{j,l}^{(s)} \varepsilon^2 \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 + \sum_{|l| \leq s} \alpha_l^{(s)} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \sum_{i=1}^d \sum_{\substack{|l| \leq s \\ c_i(l) > 0}} a_{i,l}^{(s)} \varepsilon (\partial_{l-\delta_i}^{\delta_i} \cdot, \partial_l^0 \cdot)_{L_{x,v}^2} \right]^{\frac{1}{2}}.$$

The constant $b_{j,l}^{(s)}$, $\alpha_l^{(s)}$ and $a_{i,l}^{(s)}$ are assumed to be positive.

Theorem 4.1 (Semigroup property for L)**.** *If L is a linear operator satisfying (H1'), (H2'), (H3) then there exists $\varepsilon_d \in (0,1]$ such that, for all $s \in \mathbb{Z} \cap (0, \infty)$,*

1. *for all* $\varepsilon \in (0, \varepsilon_d]$, $G_\varepsilon = \varepsilon^{-2}L - \varepsilon^{-1}v \cdot \nabla_x$ *generates a* C^0-*semigroup on* $H^s_{x,v}$;

2. *there exist constants* $C_G^{(s)}, b_{j,l}^{(s)}, \alpha_l^{(s)}, a_{i,l}^{(s)} > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_d]$

$$\|\cdot\|_{\mathcal{H}^s_\varepsilon} \sim \left[\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l| \leq s} \|\partial^0_l \cdot\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|j|+|l| \leq s \\ |j| \geq 1}} \|\partial^j_l \cdot\|^2_{L^2_{x,v}} \right]^{\frac{1}{2}},$$

and

$$(G_\varepsilon(h), h)_{\mathcal{H}^s_\varepsilon} \leq -C_G^{(s)} \|h - \pi_{G_\varepsilon}(h)\|^2_{H^s_\Lambda} \qquad h \in H^s_{x,v},$$

where π_{G_ε} *denotes the orthogonal projection on the null space* $N(G_\varepsilon)$ *of* G_ε.

The above theorem states that the equation $\partial_t f = G_\varepsilon(f)$ has a unique solution $f \in C^1(0, \infty; H^s_{x,v})$ whose component in $N(L)^\perp$ decades to 0 as $t \to \infty$ exponentially in $H^s_{x,v}$ with rate equal to $C_G^{(s)}$.

Now we state some results about the linearized Boltzmann equation (47), that is, we consider also the quadratic perturbation $\Gamma(h_\varepsilon, h_\varepsilon)$.

Proposition 4.1. *Assume the linear operator* L *satisfies (H1'), (H2'), (H3), while the bilinear operator* Γ *satisfies (H4), (H5). Then there exist positive constants* $K_0^{(s)}, K_1^{(s)}, K_2^{(s)}$, *which do not depend on* Γ *nor* ε, *such that, for any* $h_{in} \in H^s_{x,v} \cap (N(G_\varepsilon)^\perp)$ *and* $g \in Dom(\Gamma) \cap H^s_{x,v}$, *if* $h \in H^s_{x,v}$ *solves*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(g, h),$$

then the following inequality holds

$$\frac{d}{dt} \|h\|^2_{\mathcal{H}^s_\varepsilon} \leq -K_0^{(s)} \|h\|^2_{H^s_\Lambda} + K_1^{(s)} (\mathcal{G}^s_x(g, h))^2 + \varepsilon^2 K_2^{(s)} (\mathcal{G}^s_{x,v}(g, h))^2.$$

Theorem 4.2 (Global existence of solutions to (47) for small data)**.** *Let* Q *be a bilinear operator such that*

- *the BE* (45) *admits a global equilibrium* $\mu \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$, $\mu \geq 0$;
- *the linearized operator* L *defined in* (48) *satisfies (H1'), (H2'), (H3);*
- *the bilinear perturbation* Γ *defined in* (49) *satisfies (H4), (H5).*

Finally, let $\varepsilon_d \in (0, 1]$ *as in Thr. 4.1 and* s_0 *as in (H4). Then for any* $s \geq s_0$ *there exist positive constants* δ_s, C_s, τ_s *such that, for any* $\varepsilon \in (0, \varepsilon_d]$, $f_{in} \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$, $f_{in} = \mu + \varepsilon \mu^{1/2} h_{in} \geq 0$, $h_{in} \in N(G_\varepsilon)^\perp$ *and*

$$\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leq \delta_s,$$

there exists a unique global smooth solution $f_\varepsilon \in C^0(0, \infty; H^s_{x,v})$ *to* (45) *such that* $f_\varepsilon = \mu + \varepsilon \mu^{1/2} h_\varepsilon \geq 0$ *and*

$$\|h_\varepsilon\|_{\mathcal{H}^s_\varepsilon} \leq \|h_{in}\|_{\mathcal{H}^s_\varepsilon} e^{-\tau_s t}.$$

We point out that we ask h_{in} to be orthogonal to the null space $N(G_\varepsilon)$ of G_ε because we want f_{in} to have the same macroscopic density, velocity and energy as the global equilibrium μ. This property has to holds since the moments

$$\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} = \int_{\mathbb{T}^d \times \mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} f dx dv$$

are conserved by (45).

As a corollary of Thr. 4.2 we have that

$$\|h_\varepsilon(\cdot,\cdot,t)\|_{H^s_{x,v}} \leq \frac{\delta_s}{\varepsilon} e^{-\tau_s t}.$$

The factor $1/\varepsilon$ seems to jeopardize our attempt at controlling the $v-$derivatives of h_ε when $\varepsilon \to 0$. However, Guo [6] showed that we can actually control the $H^s_{x,v}$ norm of h_ε in the limit $\varepsilon \to 0$ as long as we control the both the fluid part and the macroscopic part of the solution (we are going to see what this means). In order to put this idea into practice we define a new seminorm which dominates the microscopic part of the solution (and only it). This new seminorm reads as

$$\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}} = \left[\sum_{\substack{|j|+|l|\leq s \\ |j|\geq 1}} b^{(s)}_{j,l} \|\partial^j_l (I-\pi_L)\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leq s} \alpha^{(s)}_l \|\partial^0_l \cdot\|^2_{L^2_{x,v}} + \sum_{i=1}^d \sum_{\substack{|l|\leq s \\ c_i(l)>0}} a^{(s)}_{i,l} \varepsilon (\partial^{\delta_i}_{l-\delta_i}\cdot, \partial^0_l \cdot)_{L^2_{x,v}} \right]^{\frac{1}{2}}.$$

Theorem 4.3. *Under the same conditions as in Thr. 4.2, for all $s \geq s_0$, there exist $(b^{(s)}_{j,l}), (\alpha^{(s)}_l, (a^{(s)}_{i,l}) > 0$ and $\varepsilon_d \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon_d]$:*

1. *$\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}} \sim \|\cdot\|_{\mathcal{H}^s_{x,v}}$ independently of ε;*
2. *positive constants δ'_s, τ'_s exist, which do not depend on ε, such that, for any h_{in} with $\|h_{in}\|_{\mathcal{H}^s_\varepsilon,\perp} \leq \delta'_s$, then*
$$\|h_\varepsilon\|_{\mathcal{H}^s_\varepsilon,\perp} \leq \delta'_s e^{-\tau'_s t}.$$

So, we can have exponential decay for the $v-$derivatives of h_ε with constants which do not depend on ε. The drawback is, we can only bound the microscopic part of h_ε.

Now we can start discussing the hydrodynamic limit. The previous results suggest that we can indeed expect a convergence of solutions to the LBE towards a solution of fluid dynamic equations, since the estimates we get are independent of ε.

Let us recall the definition of the macroscopic quantities:

$$\rho_\varepsilon(x,t) = \int_{\mathbb{R}^d} h_\varepsilon(x,v,t)\mu(v)^{1/2} dv \qquad \text{particle density},$$
$$u_\varepsilon(x,t) = \int_{\mathbb{R}^d} v h_\varepsilon(x,v,t)\mu(v)^{1/2} dv \qquad \text{mean velocity},$$

$$\theta_\varepsilon(x,t) = \int_{\mathbb{R}^d} \frac{|v|^2-d}{d} h_\varepsilon(x,v,t)\mu(v)^{1/2}dv \qquad \text{temperature.}$$

Thr. 4.2 states that, for $s \geq s_0$,

$$h_\varepsilon \rightharpoonup^* h \qquad \text{weakly-* in } L^\infty_t H^s_l L^2_v.$$

We are in the position to apply [1, Thr. 1.1] and obtain

1. $h \in N(L)$. In particular $h(x,v,t) = \left[\rho(x,t) + v\cdot u(x,t) + \frac{1}{2}(|v|^2-d)\theta(x,t)\right]\mu(v)^{1/2}$;
2. $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) \rightharpoonup^* (\rho, u, \theta)$ weakly* in $L^\infty_t H^s_x$;
3. (ρ, u, θ) satisfies the incompressible Navier-Stokes equations (52)–(54) and the Boussinesq relation (55).

This is all well and good: we know now that we the incompressible Navier-Stokes can be derived from the Boltzmann equation in a rigorous way. Unfortunately, the above result has a few shortcomings:

1. the convergences $h_\varepsilon \rightharpoonup^* h$, $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) \rightharpoonup^* (\rho, u, \theta)$ are only weak;
2. we don't have a convergence rate;
3. most importantly, we have no information about what initial conditions are satisfied by (ρ, u, θ).

The last result we present is meant to address these issues.

Theorem 4.4. *Let $s \geq s_0$, $h_{in} \in H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leq \delta_s$.*

1. *The sequence $(h_\varepsilon)_{\varepsilon>0}$ exists for $\varepsilon \in (0, \varepsilon_d]$ and is weakly* convergent in $L^\infty_t H^s_x L^2_v$ towards a function $h \in N(L)$ whose corresponding moments (ρ, u, θ) satisfy the incompressibility condition* (53) *and $\rho + \theta \equiv 0$.*
2. *Given any $T > 0$, the function $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d \mapsto \int_0^T h(x,v,t)dt \in \mathbb{R}$ belongs to $H^s_x L^2_v$ and there exists a positive constant C such that*
$$\left\| \int_0^\infty h_\varepsilon(\cdot,\cdot,t)dt - \int_0^\infty h(\cdot,\cdot,t)dt \right\|_{H^s_x L^2_v} \leq C\sqrt{\varepsilon|\log\varepsilon|}.$$
3. *The convergence of h_ε towards h is strong if and only if $L(h_{in}) = 0$ and the corresponding moments $(\rho_{in}, u_{in}, \theta_{in})$ satisfy the incompressibility condition* (53) *as well as $\rho_{in} + \theta_{in} \equiv 0$ (called* initial layer conditions*). In such a case*
$$\|h - h_\varepsilon\|_{L^2(0,\infty;H^s_x L^2_v)} \leq C\sqrt{\varepsilon|\log\varepsilon|},$$
and, if $\delta \in (0,1]$ exists such that $h_{in} \in H^{s+\delta}_x L^2_v$, then
$$\|h - h_\varepsilon\|_{L^\infty(0,\infty;H^s_x L^2_v)} \leq C\varepsilon^{\min\{\delta,1/2\}}.$$

The above result tells us that the convergence $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) \to (\rho, u, \theta)$ is strong, and the limit (ρ, u, θ) satisfies the Navier-Stokes equations (52)–(54) together with the Boussinesq condition (55), *as well as the initial conditions*

$$u(\cdot, 0) = \text{projection of } u_{in} \text{ onto the space of divergence-free functions in } H^s_x,$$
$$\rho(\cdot, 0) = -\theta(\theta, 0) = \frac{1}{2}(\rho_{in} - \theta_{in}).$$

However, this comes at a little price: in order to have strong convergence in time we need to assume that the initial data are slightly more regular than H^s_x.

4.3 A few useful properties

Before we go on with the proof of the results stated in the previous section, we are going to need a few estimates and properties. First we will state some properties of π_L, i.e. the orthogonal projection in L^2_v onto $N(L)$. Then we will state some upper bounds for the time derivatives of (the terms appearing in) $\|h_\varepsilon\|^2_{H^s_{x,v}}$ because we want to estimate the $\mathcal{H}^s_\varepsilon-$ and $\mathcal{H}^s_{\varepsilon,\perp}-$ norms of h_ε. We will state all the properties without proof to avoid technicalities (see [2] for details). We assume that L satisfies (H1'), (H2'), (H3), while Γ satisfies (H4), (H5). We also assume that $0 < \varepsilon \leq 1$.

About the fluid projection π_L. It is quite straightforward to obtain the following representation for π_L:

$$\pi_L(h) = \sum_{i=1}^{N} \left(\int_{\mathbb{R}^d} h\phi_i dv \right) \phi_i \qquad \forall h \in L^2_v, \tag{56}$$

where $\phi_1, \ldots, \phi_N$ are an orthogonal (w.r.t. the L^2_v scalar product) basis for $N(L)$, i.e. $N(L) = \text{Span}(\phi_1, \ldots, \phi_N)$. Furthermore (H3) states that $\phi_i = P_i(v)e^{-|v|^2/4}$ for some polynomial $P_i(v)$. Therefore one can easily see, by means of direct computations and the Cauchy-Schwartz inequality, that π_L is continuous as a mapping $H^s_{x,v} \to H^s_{x,v}$. More precisely, the following estimate holds:

$$\forall s \in \mathbb{N},\ \forall |j| + |l| = s,\ \exists C_{\pi s} > 0 :\ \|\partial^j_l \pi_L(h)\|^2_{L^2_{x,v}} \leq C_{\pi s} \|\partial^0_l h\|^2_{L^2_{x,v}} \quad \forall h \in H^s_{x,v}. \tag{57}$$

It also straightforward to understant that the $\Lambda-$norm of $\pi_L(h)$ can be controlled by the $L^2_{x,v}-$norm of h:

$$\exists C_\pi > 0 :\ \|\pi_L(h)\|^2_\Lambda \leq C_\pi \|h\|^2_{L^2_{x,v}} \quad \forall h \in L^2_{x,v}. \tag{58}$$

Now, let's talk about the kernel of the linear part of the Boltzmann operator.

Proposition 4.2. *Let $a, b \in \mathbb{R}\backslash\{0\}$, and define the operator $G \equiv aL - bv \cdot \nabla_x$ acting on $H^1_{x,v}$. Moreover assume that L satisfies (H1) and (H3). Then, for every $h \in H^1_{x,v}$, $G(h) = 0$ if and only if h is linear combination of $\phi_1, \ldots, \phi_N$ with constant coefficients.*

Prop. 4.2 states that, in a suitable sense, $N(G) = N(L)$. This identity is not to be taken literary since a generic element of $N(L)$ is a linear combination of $\phi_1, \dots, \phi_N$ with $x-$dependent coefficients.

A consequence of Prop. 4.2 is as follows. For $0 < \varepsilon \le 1$ define

$$G_\varepsilon = \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v \cdot \nabla_x.$$

The following identity holds

$$\pi_{G_\varepsilon}(h) = \sum_{i=1}^N \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \phi_i dx dv \right) \phi_i = \int_{\mathbb{T}^d} \pi_L(h) dx \qquad \forall h \in L^2_{x,v}.$$

Moreover, if $h \in N(G_\varepsilon)^\perp$, then $\pi_L(h)$ has zero average on the torus. Since π_L does not depend on x, Poincaré inequality yields

$$\|\pi_L(h)\|^2_{L^2_{x,v}} \le C_P \|\nabla_x \pi_L(h)\|^2_{L^2_{x,v}} \le C_P \|\nabla_x h\|^2_{L^2_{x,v}} \qquad \forall h \in N(G_\varepsilon)^\perp. \tag{59}$$

A priori energy estimates. In this part we will study the time evolution of $\|h_\varepsilon\|^2_{\mathcal{H}^s_\varepsilon}$, $\|h_\varepsilon\|^2_{\mathcal{H}^s_{\varepsilon,\perp}}$. Since the estimates for the linear case are quite similar to the linearized one, we will consider the former but we will put in evidence the additional contributions which come from the quadratic term $\Gamma(h,h)$ by writing them inside a (red) framed box.

For a generic but fixed $g \in H^s_{x,v}$ and some $s \in \mathbb{Z} \cap [1, \infty)$ consider a solution $h : [0, \infty) \to N(G_\varepsilon)^\perp \cap H^s_{x,v}$ to the linearized Boltzmann equation

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) \boxed{+ \frac{1}{\varepsilon}\Gamma(g,h)}.$$

Also recall the definition $h^\perp = h - \pi_L(h)$ of $h^\perp$.

We study now how the $H^1_{x,v}-$norm of some quantities evolve in time. It is known [8] that the linearized Boltzmann operator L admits a spectral gap, i.e.

$$\exists \lambda_0 > 0 : \quad -(h, L(h))_{L^2_v} \ge \lambda_0 \|h^\perp\|^2_\Lambda \qquad \forall h \in L^2_v.$$

As a consequence,

$$\frac{d}{dt}\|h\|^2_{L^2_{x,v}} \le -\frac{\lambda}{\varepsilon^2}\|h^\perp\|^2_\Lambda + \boxed{\frac{1}{\lambda}\mathcal{G}^0_x(g,h)^2}, \tag{60}$$

$$\frac{d}{dt}\|\nabla_x h\|^2_{L^2_{x,v}} \le -\frac{\lambda}{\varepsilon^2}\|\nabla_x h^\perp\|^2_\Lambda + \boxed{\frac{1}{\lambda}\mathcal{G}^1_x(g,h)^2}, \tag{61}$$

$$\frac{d}{dt}\|\nabla_v h\|^2_{L^2_{x,v}} \le \frac{K_1}{\varepsilon^2}\|h^\perp\|^2_\Lambda + \frac{K_{dx}}{\varepsilon^2}\|\nabla_x h\|^2_{L^2_{x,v}} - \frac{\nu_3^\Lambda}{\varepsilon^2}\|\nabla_v h\|^2_\Lambda + \boxed{\frac{1}{\lambda}\mathcal{G}^1_{x,v}(g,h)^2}. \tag{62}$$

We also need to control the scalar product $(\nabla_x h, \nabla_v h)_{L^2_{x,v}}$. For any $e > 0$ it holds

$$\frac{d}{dt}(\nabla_x h, \nabla_v h)^2_{L^2_{x,v}} \leq \frac{C^L e}{\varepsilon^3}\|\nabla_x h^\perp\|^2_\Lambda - \frac{1}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} + \frac{2C^L}{e\varepsilon}\|\nabla_v h\|^2_\Lambda + \boxed{\frac{e}{C^L\varepsilon}\mathcal{G}^1_x(g,h)^2}. \quad (63)$$

The next step is to study how the $H^s_{x,v}-$norm of some quantities evolve in time. Let j, l multi-indexes such that $|j| + |l| = s$. As before, a control on the pure $x-$derivatives of h is straightforward to obtain from the spectral gap estimate and hypothesis (H4):

$$\frac{d}{dt}\|\partial^0_l h\|^2_{L^2_{x,v}} \leq -\frac{\lambda}{\varepsilon^2}\|h^\perp\|^2_\Lambda + \boxed{\frac{1}{\lambda}\mathcal{G}^s_x(g,h)^2}, \quad (64)$$

On the other hand, if $|j| \geq 1$, we get the following estimate

$$\frac{d}{dt}\|\partial^j_l h\|^2_{L^2_{x,v}} \leq -\frac{\nu_5^\Lambda}{\varepsilon^2}\|\partial^j_l h\|^2_\Lambda + \frac{3(\nu_1^\Lambda)^2 d}{\nu_5^\Lambda \nu_0^\Lambda}\sum_{i, c_i(j)>0}\|\partial^{j-\delta_i}_{l+\delta_i} h\|^2_\Lambda + \frac{K_{s-1}}{\varepsilon^2}\|h\|^2_{H^{s-1}_{x,v}} + \boxed{\frac{3}{\nu_5^\Lambda}\mathcal{G}^s_{x,v}(g,h)^2}. \quad (65)$$

The case $|j| = 1$ (i.e. $j = \delta_i$ for some $i = 1, \ldots, N$) is particularly interesting:

$$\frac{d}{dt}\|\partial^{\delta_i}_{l-\delta_i} h\|^2_{L^2_{x,v}} \leq -\frac{\nu_5^\Lambda}{\varepsilon^2}\|\partial^{\delta_i}_{l-\delta_i} h\|^2_\Lambda + \frac{3(\nu_1^\Lambda)^2 d}{\nu_5^\Lambda \nu_0^\Lambda}\sum_{i, c_i(j)>0}\|\partial^0_l h\|^2_\Lambda + \frac{K_{s-1}}{\varepsilon^2}\|h\|^2_{H^{s-1}_{x,v}} + \boxed{\frac{3}{\nu_5^\Lambda}\mathcal{G}^s_{x,v}(g,h)^2}. \quad (66)$$

We also need to control the following scalar product $(\nabla_x h, \nabla_v h)^2_{L^2_{x,v}}$. For any $e > 0$ it holds

$$\frac{d}{dt}(\partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h)^2_{L^2_{x,v}} \leq \frac{C^L e}{\varepsilon^3}\|\partial^0_l h^\perp\|^2_\Lambda - \frac{1}{\varepsilon}\|\partial^0_l h\|^2_{L^2_{x,v}} + \frac{2C^L}{e\varepsilon}\|\partial^{\delta_i}_{l-\delta_i} h\|^2_\Lambda + \boxed{\frac{e}{C^L\varepsilon}\mathcal{G}^1_x(g,h)^2}. \quad (67)$$

Let us stress the fact that, in the previous estimates, we put in evidence the dependence of the various terms with respect to ε. This is (of course) crucial in the limit $\varepsilon \to 0$.

The final step in this section is to study the evolution of $h^\perp$ and its derivatives in the $L^2_{x,v}-$norm. We consider here $g = h$. We also omit to single out the terms coming from Γ since we will only use the linearized equation.

The $v-$derivative of $h^\perp$ can be controlled as follows:

$$\frac{d}{dt}\|\nabla_v h^\perp\|^2_{L^2_{x,v}} \leq \frac{K_1^\perp}{\varepsilon^2}\|h^\perp\|^2_\Lambda + K^\perp_{dx}\|\nabla_x h\|^2_{L^2_{x,v}} - \frac{\nu_3^\Lambda}{2\varepsilon^2}\|\nabla_v h^\perp\|^2_\Lambda + \frac{3}{\nu_3^\Lambda}\mathcal{G}^1_{x,v}(h,h)^2. \quad (68)$$

Furthermore another bound for the scalar product $(\nabla_x h, \nabla_v h)_{L^2_{x,v}}$ can be found, which involves the $\Lambda-$norm of $\nabla_v h^\perp$ in place of $\nabla_v h$:

$$\frac{d}{dt}(\nabla_x h, \nabla_v h)_{L^2_{x,v}} \leq \frac{K^\perp e}{\varepsilon^3}\|\nabla_x h^\perp\|^2_{L^2_{x,v}} + \frac{\|\nabla_v h^\perp\|^2_\Lambda}{4C_{\pi 1}C_\pi C_p e\varepsilon} - \frac{1}{2\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} + \frac{4C_\pi}{\varepsilon}\mathcal{G}^1_{x,v}(h,h)^2, \quad (69)$$

for any $e > 0$.

Again, we need similar estimates for higher order derivatives. Let j, l be multi-indexes such that $|j| + |l| = s$. A before, we consider two cases: $|j| = 1$, $|j| \geq 2$.

When $|j| \geq 2$,

$$\frac{d}{dt}\|\partial_l^j h^\perp\|^2_{L^2_{x,v}} \leq -\frac{\nu_5^\Lambda}{\varepsilon^2}\|\partial_l^j h^\perp\|^2_\Lambda + \frac{9d(\nu_1^\Lambda)^2}{2\nu_5^\Lambda(\nu_0^\Lambda)^2}\sum_{i,c_i(j)>0}\|\partial_{l+\delta_i}^{j-\delta_i} h^\perp\|^2_\Lambda$$
$$+ K_{dl}^\perp \sum_{|l'|\leq s-1}\|\partial_{l'}^0 h\|^2_{L^2_{x,v}} + \frac{K_{s-1}^\perp}{\varepsilon^2}\|h^\perp\|^2_{H^{s-1}_{x,v}} + \frac{3}{\nu_5^\Lambda}\mathcal{G}^s_{x,v}(h,h)^2. \tag{70}$$

On the other hand, when $|j| = 1$,

$$\frac{d}{dt}\|\partial_{l-\delta_i}^{\delta_i} h^\perp\|^2_{L^2_{x,v}} \leq -\frac{\nu_5^\Lambda}{\varepsilon^2}\|\partial_{l-\delta_i}^{\delta_i} h^\perp\|^2_{L^2_{x,v}} + K_{dl}^\perp \sum_{|l'|=s}\|\partial_{l'}^0 h\|^2_{L^2_{x,v}} + \frac{K_{s-1}^\perp}{\varepsilon^2}\|h^\perp\|^2_{H^{s-1}_{x,v}}$$
$$+ \frac{3}{\nu_5^\Lambda}\mathcal{G}^s_{x,v}(h,h)^2. \tag{71}$$

Finally, we state another bound for the scalar product $(\partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h)_{L^2_{x,v}}$:

$$\frac{d}{dt}(\partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h)_{L^2_{x,v}} \leq e\frac{\tilde{K}^\perp}{\varepsilon^3}\|\partial_l^0 h^\perp\|^2_{L^2_{x,v}} + \frac{\|\partial_{l-\delta_i}^{\delta_i} h^\perp\|^2_\Lambda}{4C_{\pi s}C_\pi C_p d e\varepsilon} - \frac{1}{2\varepsilon}\|\partial_l^0 h\|^2_{L^2_{x,v}}$$
$$+ \frac{1}{4d\varepsilon}\sum_{|l'|\leq s-1}\|\partial_{l'}^0 h\|^2_{L^2_{x,v}} + \frac{2C_\pi}{\varepsilon}\mathcal{G}^s_{x,v}(h,h)^2, \tag{72}$$

for any $e \geq 1$.

4.4 Semigroup property of the linear operator: proof of Thr. 4.1

It's time to have a closer look at the linear equation

$$\partial_t h = G_\varepsilon(h) \qquad \text{in } \mathbb{T}^d \times \mathbb{R}^d.$$

Let $h_{in} \in H^s_{x,v} \cap \mathrm{G}_\varepsilon$, and let h be a solution to $\partial_t h = G_\varepsilon(h)$ in $\mathbb{T}^d \times \mathbb{R}^d$ such that $h(\cdot,\cdot,0) = h_{in}$ in $\mathbb{T}^d \times \mathbb{R}^d$.

We point out that in $G_\varepsilon(h_{in}) = 0$ then $h(\cdot,\cdot,t) \equiv h_{in}$ for $t > 0$, so the semigroup e^{tG_ε} is the identity on $N(G_\varepsilon)$. On the other hand, the equation $\partial_t h = G_\varepsilon(h)$, due to the self-adjointness property of L, preserves the condition $h \perp N(G_\varepsilon)$, i.e. if $h_{in} \in N(G_\varepsilon)^\perp$ then $h(t) \in N(G_\varepsilon)$ for $t > 0$. Therefore we consider from now on $h_{in} \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$.

We will need the following version of Poincaré Lemma:

Lemma 4.1. *There exists a constant $C_P > 0$ such that*

$$\|\pi_L(h)\|^2_{L^2_{x,v}} \leq \|\nabla_x \pi_L(h)\|^2_{L^2_{x,v}} \leq \|\nabla_s h\|^2_{L^2_{x,v}}, \qquad h \in N(G_\varepsilon)^\perp.$$

The proof of Thr. 4.1 works by induction on s. First one proves the theorem for $s = 1$, then shows it holds for $s > 1$ under the assumption that it is true for any Sobolev index $\leq s - 1$.

Step 1: $s = 1$**.** We assume the operator L satisfies (H1), (H2), (H3) and $\varepsilon \in (0, 1]$. Assumption (H3) implies that $\varepsilon^{-2}L$ is a nonpositive self-adjoint operator on $L^2_{x,v}$. On the other end $\varepsilon^{-1}v \cdot \nabla_x$ is skew-symmetric on $L^2_{x,v}$. It follows that $\|h(t)\|_{L^2_{x,v}}$ is nonincreasing in time. It can be showed [9] that G_ε generates a C^0-semigroup on $L^2_{x,v}$ for all $\varepsilon > 0$.

Let us now turn our attention to the evolution of h in the $\mathcal{H}^1_\varepsilon$-norm. We define $h^\perp \equiv (I - \pi_L)h$ the orthogonal projection of h on $N(L)^\perp$. Since $h(t) \in N(G_\varepsilon)^\perp$ for all $t > 0$ we can use the results in section 4.3. Multiply (60) times A, (61) times α, (62) times $b\varepsilon^2$ and (63) times $a\varepsilon$ and then sum everything. We get

$$\begin{aligned} \frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \leq \frac{1}{\varepsilon^2}(bK_1 - \lambda A)\|h^\perp\|^2_\Lambda + \frac{1}{\varepsilon^2}(C^L ea - \lambda\alpha)\|\nabla_x h^\perp\|^2_\Lambda \\ + \left(\frac{2C^L a}{e} - b\nu^\Lambda_3\right)\|\nabla_v h\|^2_\Lambda + (bK_{dx} - a)\|\nabla_x h\|^2_{L^2_{x,v}}. \end{aligned} \tag{73}$$

Now it is time to choose the free parameters.

1. Fix b such that $-\nu^\Lambda_3 b < -1$.
2. Fix A such that $bK_1 - \lambda A \leq -1$.
3. Fix a such that $bK_{dx} - a \leq -1$.
4. Fix e such that $\frac{2C^L a}{e} - b\nu^\Lambda_3 \leq -1$.
5. Fix α such that $C^L ea - \lambda\alpha$, $a^2 \leq \alpha b$, $b \leq \alpha$.

It follows (recall that $0 < \varepsilon \leq 1$):

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \leq -\left(\|h^\perp\|^2_\Lambda + \|\nabla_x h^\perp\|^2_\Lambda + \|\nabla_v h\|^2_\Lambda + \|\nabla_x h\|^2_{L^2_{x,v}}\right). \tag{74}$$

Now we apply the equivalence of the $L^2_{x,v}-$norm and the $\Lambda-$norm (58) and the Poincaré inequality (59) we obtain

$$\|h\|^2_\Lambda \leq C\left(\|h^\perp\|^2_\Lambda + \frac{1}{2}\|\nabla_x h\|^2_{L^2_{x,v}}\right),$$
$$\|\nabla_x h\|^2_\Lambda \leq C'\left(\|\nabla_x h^\perp\|^2_\Lambda + \frac{1}{2}\|\nabla_x h\|^2_{L^2_{x,v}}\right).$$

From the above relations and (74) we deduce that a constant $K > 0$ exists such that

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \leq -C^{(1)}_G\left(\|h\|^2_\Lambda + \|\nabla_x h\|^2_\Lambda + \|\nabla_v h\|^2_\Lambda\right), \qquad 0 < \varepsilon \leq 1.$$

The choice of parameters (in particular, relations $a^2 \leq \alpha b$, $b \leq \alpha$) implies

$$A\|h\|^2_{L^2_{x,v}} + \frac{b}{2}(\|\nabla_x h\|^2_{L^2_{x,v}} + \varepsilon^2\|\nabla_v h\|^2_{L^2_{x,v}}) \leq \|h\|^2_{\mathcal{H}^1_\varepsilon},$$
$$A\|h\|^2_{L^2_{x,v}} + \frac{3\alpha}{2}(\|\nabla_x h\|^2_{L^2_{x,v}} + \varepsilon^2\|\nabla_v h\|^2_{L^2_{x,v}}) \geq \|h\|^2_{\mathcal{H}^1_\varepsilon},$$

which means that the following equivalence between norms hold:

$$\|\cdot\|_{\mathcal{H}^1_\varepsilon} \sim \left(\|\cdot\|^2_{L^2_{x,v}} + \|\nabla_x \cdot\|^2_{L^2_{x,v}} + \varepsilon^2\|\nabla_v \cdot\|^2_{L^2_{x,v}}\right)^{1/2}.$$

This is the statement of Thr. 4.1 for $s = 1$.

Step 2: $s > 1$. We assume the theorem's statement is true for any Sobolev index $\leq s-1$, and prove the statement for s. Moreover assume that L satisfies (H1'), (H2'), (H3), and let $\varepsilon \in (0,1]$ and $h_{in} \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$. Finally let h be the solution of $\partial_t h = G_\varepsilon(h)$ on $\mathbb{T}^d \times \mathbb{R}^d$ such that $h(0) = h_{in}$. As a consequence $h(t) \in N(G_\varepsilon)^\perp$ for $t > 0$. From the previous step we know how to handle the case where the number of derivatives in x and v only differs by one. Therefore we can work with a seminorm involving only some of the terms on $\|\cdot\|_{H^s_{x,v}}$. We define

$$F_s(t) = B \sum_{\substack{|j|+|\ell|=s \\ |j|\geq 2}} \varepsilon^2\|\partial^j_\ell h\|^2_{L^2_{x,v}} + B' \sum_{\substack{|l|=s \\ i, c_i(l)>0}} Q_{l,i}(t),$$
$$Q_{l,i}(t) = \alpha\|\partial^0_l h\|^2_{L^2_{x,v}} + b\varepsilon^2\|\partial^{\delta_i}_{l-\delta_i} h\|^2_{L^2_{x,v}} + a\varepsilon(\partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h)_{L^2_{x,v}}$$

We are going to study the time evolution of every term appearing in F_s in order to find an upper bound for $\frac{dF_s}{dt}$.

Let us begin by studying the evolution of $Q_{l,i}$, with $|j| + |l| = s$. By taking a linear combination of (64), (66), (67) (with coefficients α, $b\varepsilon^2$, $a\varepsilon$ respectively) we obtain

$$\frac{d}{dt}Q_{l,i}(t) \leq \frac{1}{\varepsilon^2}(C^L ea - \lambda\alpha)\|\partial^0_l h^\perp\|^2_\Lambda + \left(\frac{2C^L a}{\varepsilon} - \nu^\Lambda_5 b\right)\|\partial^{\delta_i}_{l-\delta_i} h\|^2_\Lambda$$
$$+ \left(\frac{3\nu^\Lambda_1}{\nu^\Lambda_5 \nu^\Lambda_0} b - a\right)\|\partial^0_l h\|^2_{L^2_{x,v}} + K_{s-1} b\|h\|^2_{H^{s-1}_{x,v}}.$$

The above inequality resembles (73) closely, with the only exception being the term $K_{s-1}b\|h\|^2_{H^{s-1}_{x,v}}$. We can proceed in a similar way to the case $s = 1$ and choose the free parameters such that positive constants $s_Q > 0$, $C_{s-1} > 0$ exist such that, for all $0 < \varepsilon \leq 1$,

$$Q_{l,i}(t) \sim \|\partial^0_l h\|^2_{L^2_{x,v}} + \varepsilon^2\|\partial^{\delta_i}_{l-\delta_i} h\|^2_{L^2_{x,v}},$$
$$\frac{d}{dt}Q_{l,i}(t) \leq -K_Q(\|\partial^0_l h\|^2_\Lambda + \|\partial^{\delta_i}_{l-\delta_i} h\|^2_\Lambda) + C_{s-1}\|h\|^2_{H^{s-1}_{x,v}}.$$

We point out that in the derivation of the above estimates we used the inequality

$$\|\partial_l^0 h\|_\Lambda^2 \leq C'(\|\partial_l^0 h^\perp\|_\Lambda^2 + \|\partial_l^0 h\|_{L^2_{x,v}}^2),$$

which comes from the equivalence of the $\Lambda-$ and $L^2_{x,v}-$norm, i.e. (58).

The first relation invoving $Q_{l,i}(t)$ yields the following estimate for F_s:

$$F_s(t) \sim \sum_{|l|=s} \|\partial_l^0 h\|_{L^2_{x,v}}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|=s \\ |j|\geq 1}} \|\partial_l^j h\|_{L^2_{x,v}}^2.$$

The time evolution of F_s is obtained by combining the bounds for $\frac{d}{dt}Q_{l,i}$ and $\frac{d}{dt}\|\partial_l^j h\|_{L^2_{x,v}}^2$ (given by (65)). Again, by choosing the parameters in a careful way we obtain

$$\forall \varepsilon \in (0, \varepsilon_d], \quad \frac{d}{dt}F_s(t) \leq C_+^{(s-1)} \|h\|_{H^{s-1}_{x,v}}^2 - \sum_{|j|+|l|=s} \|\partial_l^j h\|_\Lambda^2.$$

Since the $L^2_{x,v}$ norm is controlled by the Λ norm, it follows

$$\forall \varepsilon \in (0, \varepsilon_d], \quad \frac{d}{dt}F_s(t) \leq C_+^{(s)} \sum_{|j|+|l|\leq s-1} \|\partial_l^j h\|_\Lambda^2 - \sum_{|j|+|l|=s} \|\partial_l^j h\|_\Lambda^2.$$

Since the above inequality holds true for all s, we can take a linear combination of $F_1, \dots, F_s$ to obtain

$$\forall \varepsilon \in (0, \varepsilon_d], \quad \frac{d}{dt}\sum_{p=1}^{s} C_p F_p(t) \leq -C_G^{(s)} \sum_{|j|+|l|\leq s} \|\partial_l^j h\|_\Lambda^2.$$

By induction assumption

$$\sum_{p=1}^{s} C_p F_p \sim \|h\|_{L^2_{x,v}}^2 + \sum_{|l|\leq s} \|\partial_l^0 h\|_{L^2_{x,v}}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|\leq s \\ |j|\geq 1}} \|\partial_l^j h\|_{L^2_{x,v}}^2.$$

This finishes the proof of Thr. 4.1.

Remark 4.1. The proof of Prop. 4.1 is similar in the philosophy to the proof of Thr. 4.1. To avoid technicalities, we omit it. The curious Reader can find it in [2].

4.5 Perturbative result for the BE: proof of Thr. 4.2

We deal here with the linearized Boltzmann equation (47), which is equivalent to the full Boltzmann equation (45) under the considered scaling $f = \mu + \varepsilon \mu^{1/2} h$.

The proof is based upon an iteration scheme. The a-priori estimates provided by Prop. 4.1 will yield first the existence of solutions and then the exponential decay of those solutions for small enough initial data.

Step 1: construction of solutions to the linearized equation. We plan to approximate the solution to the nonlinear problem with a sequence of solutions to a linearization of the problem itself. Then we will bound such sequence in a Sobolev space equipped with an ad-hoc norm, so that we will extract a subsequence out of it.

The starting point of this construction process is a function $h_0 \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$ which we will define later. For any $n \geq 0$ and $h_n \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$, let $h_{n+1} \in H^s_{x,v}$ be the unique solution to

$$\partial_t h_{n+1} + \frac{1}{\varepsilon} v \cdot \nabla_x h_{n+1} = \frac{1}{\varepsilon^2} L(h_{n+1}) + \frac{1}{\varepsilon}\Gamma(h_n, h_{n+1}) \qquad \text{in } \mathbb{T}^d \times \mathbb{R}^d, \quad t > 0, \tag{75}$$

$$h_{n+1}(\cdot,\cdot,0) = h_{in} \qquad \text{in } \mathbb{T}^d \times \mathbb{R}^d. \tag{76}$$

The sequence h_n is well defined. Indeed, it is possible to show the following

Lemma 4.2. *Let L satisfy (H1'), (H2'), (H3), and let Γ fulfill (H4), (H5). There exists $\varepsilon_d \in (0,1]$ such that for all $s \geq s_0$ (defined in (H4)) there exists $\delta_s > 0$ such that, for all $\varepsilon \in (0, \varepsilon_d]$ and initial data h_{in} satisfying $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leq \delta_s$, the sequence $(h_n)_{n\in\mathbb{N}}$ is well defined and $h_n \in C(\mathbb{R}^+, H^s_{x,v}) \cap N(G_\varepsilon)^\perp$, $n \in \mathbb{N}$.*

We omit the proof, which can be found in [2].

The next step is to show that h_n is uniformely bounded in a suitable norm. Let us define the following functional on $H^s_{x,v}$:

$$E(h) = \sup_{t>0} \left(\|h(t)\|^2_{\mathcal{H}^s_\varepsilon} + \int_0^t \|h(\sigma)\|^2_{H^s_\Lambda} d\sigma \right). \tag{77}$$

We are going to prove the following

Lemma 4.3. *Let L satisfy (H1'), (H2'), (H3), and let Γ satisfy (H4), (H5). There exists $\varepsilon_d \in (0,1]$ such that for all $s \geq s_0$ (defined in (H4)) there exists $\delta_s > 0$ such that, for all $\varepsilon \in (0, \varepsilon_d]$ and initial data h_{in} satisfying $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leq \delta_s$,*

$$E(h_n) \leq \delta_s \qquad \Rightarrow \qquad E(h_{n+1}) \leq \delta_s.$$

Proof. We know that $h_{in} \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$. Moreover, Lemma 4.2 implies that $h_n \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$ since $s \geq s_0$. Furthermore, Γ satisfies (H5). Therefore from Prop. 4.1 we deduce the estimate (which holds for $0 < \varepsilon \leq \varepsilon_d$):

$$\frac{d}{dt}\|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon} \leq -K_0^{(s)} \|h_{n+1}\|^2_{H^s_\Lambda} + K_1^{(s)} (\mathcal{G}^s_x(h_n, h_{n+1}))^2 + \varepsilon^2 K_2^{(s)} (\mathcal{G}^s_{x,v}(h_n, h_{n+1}))^2.$$

Hypothesis (H4) and relation

$$C_m \left(\|\cdot\|^2_{L^2_{x.v}} + \sum_{|l|\leq s} \|\cdot\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leq s \\ |j|\geq 1}} \|\cdot\|^2_{L^2_{x,v}} \right) \leq \|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \leq C_M \|\cdot\|^2_{H^s_{x,v}}$$

lead to the following upper bounds

$$\mathcal{G}_x^s(h_n, h_{n+1})^2 \le \frac{C_\Gamma^2}{C_m} \left(\|h_n\|_{\mathcal{H}_\varepsilon^s}^2 \|h_{n+1}\|_{H_\Lambda^s}^2 + \|h_{n+1}\|_{\mathcal{H}_\varepsilon^s}^2 \|h_n\|_{H_\Lambda^s}^2 \right),$$

$$\mathcal{G}_{x,v}^s(h_n, h_{n+1})^2 \le \frac{C_\Gamma^2}{C_m \varepsilon^2} \left(\|h_n\|_{\mathcal{H}_\varepsilon^s}^2 \|h_{n+1}\|_{H_\Lambda^s}^2 + \|h_{n+1}\|_{\mathcal{H}_\varepsilon^s}^2 \|h_n\|_{H_\Lambda^s}^2 \right)$$

from which we deduce

$$\begin{aligned}\frac{d}{dt}\|h_{n+1}\|_{\mathcal{H}_\varepsilon^s}^2 &\le -K_0^{(s)}\|h_{n+1}\|_{H_\Lambda^s}^2 + K_1\|h_n\|_{\mathcal{H}_\varepsilon^s}^2\|h_{n+1}\|_{H_\Lambda^s}^2 + K_2\|h_{n+1}\|_{\mathcal{H}_\varepsilon^s}^2\|h_n\|_{H_\Lambda^s}^2 \\ &\le [K_1 E(h_n) - K_0^{(s)}]\|h_{n+1}\|_{H_\Lambda^s}^2 + K_2 E(h_{n+1})\|h_n\|_{H_\Lambda^s}^2.\end{aligned}$$

Let us assume now $E(h_n) \le K_0^{(s)}/2K_1$ and integrate the above inequality in the time interval $[0,t]$. We obtain:

$$\|h_{n+1}(t)\|_{\mathcal{H}_\varepsilon^s}^2 + \frac{1}{2}K_0^{(s)}\int_0^t \|h_{n+1}(\sigma)\|_{H_\Lambda^s}^2 d\sigma \le \|h_0\|_{\mathcal{H}_\varepsilon^s}^2 + KE(h_{n+1})E(h_n), \qquad t > 0.$$

Define $C \equiv \{1, K_0^{(s)}/2\}$ and assume that $E(h_n) \le C/2K$. It follows

$$E(h_{n+1}) \le \frac{2}{C}\|h_0\|_{\mathcal{H}_\varepsilon^s}^2.$$

Choosing $\delta_s \le \min\{\frac{C}{2}, \frac{C}{2K}, \frac{K_0^{(s)}}{2K_1}\}$ yields the statement. This finishes the proof of the lemma. □

We are finally in the position to prove the global existence result. We are going to show that

Theorem 4.5. *Let L satisfy (H1'), (H2'), (H3), and let Γ satisfy (H4), (H5). There exists $\varepsilon_d \in (0,1]$ such that for all $s \ge s_0$ (defined in (H4)) there exists $\delta_s > 0$ such that, for all $\varepsilon \in (0, \varepsilon_d]$ and initial data h_{in} satisfying $\|h_{in}\|_{\mathcal{H}_\varepsilon^s} \le \delta_s$, the linearized Boltzmann equation* (47) *has a solution $h \in C(\mathbb{R}^+, H_{x,v}^s)$ such that $E(h) \le C\|h_{in}\|_{\mathcal{H}_\varepsilon^s}^2$, where $C > 0$ is a suitable constant.*

Proof. Lemma 4.3 provides us with a uniform bound for h_n in the E norm under the assumption that $E(h_0) \le \delta_s$ (defined in the Lemma). Define

$$h_0(\cdot,\cdot,t) = \begin{cases} h_{in} & t = 0 \\ 0 & t > 0 \end{cases}.$$

It follows $E(h_0) = \|h_{in}\|_{\mathcal{H}_\varepsilon^s} \le \delta_s$. The sequence h_n is therefore bounded in $L_t^\infty H_{x,v}^s \cap L_t^1 H_\Lambda^s$. From this bound and (75) we also deduce a uniform bound for $\partial_t h_n$ in a suitable norm. Standard compact Sobolev embeddings and Aubin-Lions Lemma yields strong convergence for the sequence h_n towards a function $h \in C(\mathbb{R}^+, H_{x,v}^s)$ satisfying

$$\partial_t h + \frac{1}{\varepsilon}v\cdot\nabla_x h = \frac{1}{\varepsilon^2}L(h) + \frac{1}{\varepsilon}\Gamma(h,h), \qquad h(\cdot,\cdot,0) = h_{in}.$$

This finishes the proof. □

At this point, we have a solution h to

$$\partial_t h = G_\varepsilon(h) + \frac{1}{\varepsilon}\Gamma(h,h)$$

such that $h \in N(G_\varepsilon)^\perp$ for all $\varepsilon \in (0,1]$. Moreover, by (more or less) repeating the computations done in the proof of Lemma 4.3 we can obtain

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} \le (K\|h\|^2_{\mathcal{H}^s_\varepsilon} - K_0)\|h\|^2_{H^s_\Lambda}.$$

As a consequence, if $\|h_{in}\|^2_{\mathcal{H}^s_\varepsilon} \le K_0/2K$ then $\|h\|_{\mathcal{H}^s_\varepsilon}$ is decreasing in time. Since the Λ-norm controls the H^s-norm which in turn controls the $\mathcal{H}^s_\varepsilon$-norm:

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} \le -\frac{K_0}{2}\|h\|^2_{H^s_\Lambda} \le -\frac{K_0}{2}\frac{\nu_0^\Lambda}{\nu_1^\Lambda C_M}\|h\|^2_{\mathcal{H}^s_\varepsilon}.$$

By Gronwall's lemma we conclude

$$\|h(t)\|_{\mathcal{H}^s_\varepsilon} \le \|h_{in}\|_{\mathcal{H}^s_\varepsilon} e^{-\tau_s t} \qquad t > 0,$$

where $\tau_s \equiv \frac{K_0\nu_0^\Lambda}{4\nu_1^\Lambda C_M}$, as long as $\|h_{in}\|^2_{\mathcal{H}^s_\varepsilon} \le \frac{K_0}{2K}$. This is exactly the statement with $\delta_s \le \sqrt{\frac{K_0}{2K}}$. This finishes the proof of Thr. 4.2.

4.6 Perturbative result for the BE: proof of Thr. 4.3

Recall the definition of the seminorm $\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}}$:

$$\|\cdot\|^2_{\mathcal{H}^s_{\varepsilon,\perp}} = \sum_{\substack{|j|+|l|\le s\\|j|\ge 1}} b^{(s)}_{j,l}\|\partial^j_l(I-\pi_L)\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\le s}\alpha^{(s)}_l\|\partial^0_l\cdot\|^2_{L^2_{x,v}} + \sum_{i=1}^d\sum_{\substack{|l|\le s\\c_i(l)>0}} a^{(s)}_{i,l}\,\varepsilon(\partial^{\delta_i}_{l-\delta_i}\cdot,\,\partial^0_l\cdot)_{L^2_{x,v}}.$$

The main ingredient of the proof is a proposition giving an apriori estimate on a solution to the linearized Boltzmann equation

$$\partial_t h + \frac{1}{\varepsilon}v\cdot\nabla_x h = \frac{1}{\varepsilon^2}L(h) + \frac{1}{\varepsilon}\Gamma(h,h).$$

In the proposition we will show the equivalence of $\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}}$ and $\|\cdot\|_{H^s_{x,v}}$. The core idea is as follows. If we choose coefficients in the definition of $\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}}$ such that

$$\|\cdot\|_{\mathcal{H}^s_{1,\perp}} \sim \sum_{\substack{|j|+|l|\le s\\|j|\ge 1}}\|\partial^j_l(I-\pi_L)\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\le s}\|\partial^0_l\cdot\|^2_{L^2_{x,v}}$$

then it also holds

$$\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}} \sim \sum_{\substack{|j|+|l|\le s\\|j|\ge 1}}\|\partial^j_l(I-\pi_L)\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\le s}\|\partial^0_l\cdot\|^2_{L^2_{x,v}} \qquad 0<\varepsilon\le\varepsilon_0$$

for a suitable $\varepsilon_0 > 0$, with coefficients independent of ε. Moreover, the fact that $N(L)$ has finite dimension implies that

$$\forall s \in \mathbb{N} \;\; \exists C_{\pi s} > 0 : \;\forall j, l, \; |j| + |l| = s, \; \forall h \in H^s_{x,v}, \quad \|\partial^j_l \pi_L(h)\|^2_{L^2_{x,v}} \leq C_{\pi s} \|\partial^0_l \pi_L(h)\|^2_{L^2_{x,v}}.$$

From the above estimate and the decomposition $h = \pi_L(h) + h^\perp$ it follows

$$\|\partial^j_l h\|^2_{L^2_{x,v}} \leq C_{\pi s} \|\partial^0_l h\|^2_{L^2_{x,v}} + \|\partial^j_l h^\perp\|^2_{L^2_{x,v}} \leq \|h\|^2_{H^s_{x,v}},$$

which means that $\|\cdot\|_{\mathcal{H}^s_{\varepsilon,\perp}} \sim \|\cdot\|_{H^s_{x,v}}$ with coefficients independent of ε.

Proposition 4.3. *Assume L is a linear operator satisfying the conditions (H1'), (H2'), (H3), and that Γ is a bilinear operator satisfying (H5). Then there exists $\varepsilon_d \in (0, 1]$ such that for all $s \in \mathbb{Z} \cap [1, \infty)$ and for all $h_{in} \in N(G_\varepsilon)^\perp$, the solution h to*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(h, h), \qquad h(0) = h_{in},$$

there exist constants $K^{(s)}_0, K^{(s)}_1, b^{(s)}_{j,l}, \alpha^{(s)}_l, a^{(s)}_{i,l} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_d]$:

- $\|\cdot\|_{\mathcal{H}^s_{\varepsilon\perp}} \sim \|\cdot\|_{H^s_{x,v}}$,
- *For every $h_{in} \in H^s_{x,v} \cap N(G_\varepsilon)^\perp$,*

$$\frac{d}{dt} \|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leq -K^{(s)}_0 \left(\frac{1}{\varepsilon^2} \|h^\perp\|^2_{H^s_\Lambda} + \sum_{1 \leq |l| \leq s} \|\partial^0_l h\|^2_{L^2_{x,v}} \right) + K^{(s)}_1 \mathcal{G}^s_{x,v}(h, h)^2.$$

(Main ideas of the) Proof. We are going to prove the Proposition by induction on $s \geq 1$. We point out (again) that, since $N(G_\varepsilon)^\perp$ is invariant w.r.t. to the evolution of the system (i.e. $h_{in} \in N(G_\varepsilon) \Rightarrow h(t) \in N(G_\varepsilon)^\perp$ for all $t > 0$), we can use the inequalities and properties stated in Section 4.3.

Step 1: $s = 1$**.** It holds

$$\|h\|^2_{\mathcal{H}^1_{\varepsilon\perp}} = A \|h\|^2_{L^2_{x,v}} + \alpha \|\nabla_x h\|^2_{L^2_{x,v}} + b \|\nabla_v h^\perp\|^2_{L^2_{x,v}} + a\varepsilon (\nabla_x h, \nabla_v h)_{L^2_{x,v}},$$

and $A, \alpha, b > 0$. Therefore we can employ the estimates derived in Section 4.3. Taking a linear combination of (60), (61), (68) with coefficients A, α, b (respectively) yields

$$\begin{aligned}\frac{d}{dt} \|h\|^2_{\mathcal{H}_{\varepsilon\perp}} \leq\; & \frac{1}{\varepsilon^2} (K^\perp_1 b - \lambda A) \|h^\perp\|^2_\Lambda + \frac{1}{\varepsilon^2} (K^\perp e a - \lambda \alpha) \|\nabla_x h^\perp\|^2_\Lambda \\ & + \frac{1}{\varepsilon^2} \left(\frac{a}{4eC_{\pi 1} C_\pi C_p} \frac{a}{e} - b \frac{\nu^\Lambda_3}{2} \right) \|\nabla_v h^\perp\|^2_\Lambda + \left(K^\perp_{dx} b - \frac{a}{2} \right) \|\nabla_x h\|^2_{L^2_{x,v}} \\ & + K(A, \alpha, b, a) \mathcal{G}^1_{x,v}(h, h)^2. \end{aligned} \tag{78}$$

Again, by arguing in a similar way to the proof of Thr. 4.1 (case $s = 1$), we can choose the free parameters in a suitable way and obtain: there exists $s_0, K_1 > 0$ such that, for every $\varepsilon \in (0, 1]$,

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_{\varepsilon\perp}} \leq -s_0\left(\frac{1}{\varepsilon^2}\|h^\perp\|^2_\Lambda + \frac{1}{\varepsilon^2}\|\nabla_x h^\perp\|^2 + \frac{1}{\varepsilon^2}\|\nabla_v h^\perp\|^2_\Lambda + \|\nabla_x h\|^2_{L^2_{x,v}}\right) + K_1\mathcal{G}_{x,v}(h,h)^2,$$
$$\|h\|^2_{\mathcal{H}^1_{1\perp}} \sim \|h\|^2_{L^2_{x,v}} + \|\nabla_x h\|^2_{L^2_{x,v}} + \|\nabla_v h^\perp\|^2_{L^2_{x,v}}.$$

This is the statement for $s = 1$.

Step 2: $s > 1$. Again, we assume the statement is true up to $s - 1$, and we suppose L satisfies (H1'), (H2'), (H3). This part is similar in the philosophy to the proof of Thr. 4.1, Step 2. We omit it to avoid technicalities. □

Exponential decay. We already know from Thr. 4.2 that the linearized Boltzmann equation has a solution h for any given $h_{in} \in H^s_{x,v}$ as long as $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leq \delta_s$. Moreover we know from Prop. 4.3 that

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leq -K_0^{(s)}\left(\frac{1}{\varepsilon^2}\|h^\perp\|^2_{H^s_\Lambda} + \sum_{1\leq|l|\leq s}\|\partial^0_l h\|^2_{L^2_{x,v}}\right) + K_1^{(s)}\mathcal{G}_{x,v}(h,h)^2.$$

The equivalence of the $L^2_{x,v}$ and Λ norms on the fluid part (i.e. (58)) implies, for $|l| \geq 1$,

$$\|\partial^0_l h\|^2_\Lambda \leq C(\|\partial^0_l h^\perp\|^2_\Lambda + \|\partial^0_l h\|^2_{L^2_{x,v}}),$$

while for $l = 0$ the Poincaré inequality (59) and (58) yield

$$\|h\|^2_\Lambda \leq C(\|h^\perp\|^2_\Lambda + \|\nabla_x h\|^2_{L^2_{x,v}}).$$

It follows

$$\begin{aligned}\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} &\leq -K_0^{(s)}\left(\sum_{\substack{|j|+|l|\leq s\\|j|\geq 1}}\|\partial^j_l h^\perp\|^2_\Lambda + \sum_{|l|\leq s}\|\partial^0_l h\|^2_\Lambda\right) + K_1^{(s)}\mathcal{G}_{x,v}(h,h)^2\\ &\leq -\tilde{K}_0^{(s)}\|h\|^2_{H^s_\Lambda} + K_1^{(s)}\mathcal{G}_{x,v}(h,h)^2.\end{aligned}$$

Since Γ satisfies (H4), for $s \geq s_0$ (defined in (H4)) we get

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leq (K_1^{(s)}C_\Gamma^2 C\|h\|^2_{H^s_{x,v}} - \tilde{K}_0^{(s)})\|h\|^2_{H^s_\Lambda}.$$

Therefore if we choose the initial datum h_{in} such that

$$\|h_{in}\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leq \frac{\tilde{K}_0^{(s)}}{2K_1^{(s)}C_\Gamma^2 C}$$

it follows that $t \in (0,\infty) \mapsto \|h(t)\|_{\mathcal{H}^s_{\varepsilon\perp}}$ is decreasing, and so

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \le -\frac{\tilde{K}_0^{(s)}}{2K_1^{(s)}C_\Gamma^2 C}\|h\|^2_{H^s_\Lambda} \qquad t>0.$$

Furthermore, $\|h\|_{H^s_\Lambda}$ dominates $\|h\|_{H^s_{x,v}}$, which is equivalent to $\|h\|_{\mathcal{H}^s_{\varepsilon\perp}}$ by Prop. 4.3. Gronwall's lemma yields the exponential decay. This finishes the proof of Thr. 4.3.

4.7 Incompressible Navier-Stokes limit: proof of Thr. 4.4.

The first step in the proof of the theorem is to derive a finite time convergence rate. Then we will combine this fact with the exponential decay of the solution and we will get a global-in-time convergence result.

Throught this section we assume $s \ge s_0$ (defined in (H4)), $0 < \varepsilon \le \varepsilon_d$, $h_{in} \in H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \le \delta_s$. From Thr 4.2 we know that a solution h_ε to the linearized Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x h_\varepsilon = \frac{1}{\varepsilon^2}L(h_\varepsilon) + \frac{1}{\varepsilon}\Gamma(h_\varepsilon,h_\varepsilon), \qquad h_\varepsilon(0) = h_{in}, \tag{79}$$

exists. Moreover, since $\|h_\varepsilon(t)\|_{\mathcal{H}^s_\varepsilon} \le e^{-\tau_s t}\|h_{in}\|_{\mathcal{H}^s_\varepsilon}$ for $t>0$ and

$$\|\cdot\|_{\mathcal{H}^s_\varepsilon} \sim \left[\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\le s}\|\partial^0_l\cdot\|^2_{L^2_{x,v}} + \varepsilon^2\sum_{\substack{|j|+|l|\le s\\ |j|\ge 1}}\|\partial^j_l\cdot\|^2_{L^2_{x,v}}\right]^{\frac{1}{2}},$$

it follows that h_ε is bounded in $L^\infty_t H^s_x L^2_v$, therefore is weakly-* convergent to h in the same space (up to a subsequence). Furthermore from the bound for h_ε in $L^\infty_t H^s_x L^2_v$ and (79) we obtain a bound for $\partial_t h_\varepsilon$ in a suitable space. The two bounds for h_ε and $\partial_t h_\varepsilon$ yield, thanks to Aubin-Lions Lemma, strong convergence (up to subsequences) for h_ε in $C([0,T],L^\infty_x L^2_v)$ for any $T>0$. Since we know the weak limit of h_ε is h, it follows

$$\forall T>0, \quad V_T(\varepsilon) \equiv \sup_{t\in[0,T]}\|h_\varepsilon(t)-h(t)\|_{L^\infty_x L^2_v} \to 0 \qquad \text{as } \varepsilon\to 0.$$

This fact is the starting point of the argument that follows. We are going to prove the following

Theorem 4.6. *Let $s \ge s_0$, $h_{in} \in H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \le \delta_s$. Then $h_\varepsilon \rightharpoonup^* h$ weakly* in $L^\infty_t H^s_x L^2_v$, $h \in N(L)$, $\nabla_x\cdot u = 0$, $\rho+\theta = 0$.*
Furthermore, $\int_0^T h\,dt \in H^s_x L^2_v$ and $C>0$ exists such that

$$\left\|\int_0^T (h_\varepsilon - h)dt\right\|_{H^s_x L^2_v} \le C\max\{\sqrt{\varepsilon}, \sqrt{T\varepsilon}, TV_T(\varepsilon)\}, \qquad T>0.$$

The convergence of h_ε in $L^2_{[0,T]}H^s_xL^2_v$ is strong if and only if $L(h_{in})=0$, $\nabla_x\cdot u_{in}=0$, $\rho_{in}+\theta_{in}=0$ (initial layer conditions). In this case

$$\|h-h_\varepsilon\|_{L^2_{[0,T]}H^s_xL^2_v}\leq C\max\{\sqrt{\varepsilon},\sqrt{T}V_T(\varepsilon)\},\qquad T>0.$$

Finally, in the case the convergence of h_ε in $L^2_{[0,T]}H^s_xL^2_v$ is strong and $h_{in}\in H^{s+\delta}_xL^2_v$ for some $\delta>0$, it holds

$$\sup_{t\in[0,T]}\|h-h_\varepsilon\|_{H^s_xL^2_v}\leq C\max\{\varepsilon^{\min\{\delta,1/2\}},V_T(\varepsilon)\},\qquad T>0.$$

Proof. We already know from the results about the linear case that the linear operator $G_\varepsilon=\varepsilon^{-2}L-\varepsilon^{-1}v\cdot\nabla_x$ generates a semigroup e^{tG_ε} on $H^s_{x,v}$. It is therefore possible to apply Duhamel's principle and rewrite the equation for h_ε as follows:

$$h_\varepsilon=e^{tG_\varepsilon}h_{in}+\int_0^t\frac{1}{\varepsilon}e^{(t-s)G_\varepsilon}u_\varepsilon(s)ds=U^\varepsilon h_{in}+\Psi^\varepsilon[u_\varepsilon],\tag{80}$$

where

$$U^\varepsilon\equiv e^{tG_\varepsilon},\qquad u_\varepsilon\equiv\Gamma(h_\varepsilon,h_\varepsilon),\qquad \Psi^\varepsilon[u]\equiv\int_0^t\frac{1}{\varepsilon}e^{(t-s)G_\varepsilon}u(s)ds\quad\forall u=u(x,v,t).$$

In [4] we find a study of the Fourier transform with respect to x of the semigroup e^{tG_ε}, which will allow us to study the strong limit of $U^\varepsilon h_{in}$ and $\Psi^\varepsilon(u_\varepsilon)$ as $\varepsilon\to 0$. Let $\mathcal{F}_x$ the discrete Fourier transform, i.e. the Fourier transform acting on functions defined on the torus $\mathbb{T}^d$. We will denote with $n\in\mathbb{Z}^d$ the tranformed variable.

We employ the following version of [4, Thr. 3-1].

Theorem 4.7 (High-low frequencies decomposition of $\hat{U}^\varepsilon$)**.** *There exists $n_0\in\mathbb{Z}\cap[1,\infty)$ and there exist C^∞ functions $\lambda_{-1},\lambda_0,\lambda_1,\lambda_2:[-n_0,n_0]\to\mathbb{C}$, and C^0 functions $e_{-1},e_0,\dots,e_d:[-n_0,n_0]\times\mathbb{S}^{d-1}\to L^2_v$ such that*

1. *for $j=-1,0,1,2$ it holds $\lambda_j(\zeta)=i\alpha_j\zeta-\beta_j\zeta^2+\gamma_j(\zeta)$, with $\alpha_j\in\mathbb{R}$, $\alpha_0=\alpha_2=0$, $\beta_j<0$ and $|\gamma_j(\zeta)|\leq C_\gamma|\zeta|^3$.*
2. *For $j=-1,\dots,d$ it holds $e_j(\zeta,\omega)=e_{0j}(\omega)+\zeta e_{1j}(\omega)+\omega^2e_{2j}(\zeta,\omega)$, with $e_{0,-1}(\omega)(v)=e_{0,1}(-\omega)(v)=A(1-\omega\cdot v+(|v|^2-d)/2)\mu(v)^{1/2}$.*
3. *It holds $e^{tG_\varepsilon}=\mathcal{F}_x^{-1}\hat{U}(\varepsilon n,v,t/\varepsilon^2)\mathcal{F}_x$ and the term $\hat{U}(n,v,t)$ decomposes as*

$$\hat{U}(n,v,t)=\sum_{j=-1}^{2}\hat{U}_j(n,v,t)+\hat{U}_R(n,v,t),$$

with the terms $\hat{U}_{-1},\dots,\hat{U}_2$ satisfying, for $j=-1,0,1,2$,

$$\hat{U}_j(n,v,t)=\chi_{\{|n|\leq n_0\}}e^{t\lambda_j(|n|)}P_j\left(|n|,\frac{n}{|n|}\right)(v),$$

$$P_j\left(|n|, \frac{n}{|n|}\right) = \begin{cases} e_j\left(|n|, \frac{n}{|n|}\right) \otimes e_j\left(|n|, -\frac{n}{|n|}\right) & \text{for } j = -1, 0, 1, \\ \sum_{s=2}^{d} e_s\left(|n|, \frac{n}{|n|}\right) \otimes e_s\left(|n|, -\frac{n}{|n|}\right) & \text{for } j = 2, \end{cases}$$

$$P_j\left(|n|, \frac{n}{|n|}\right) = P_{0,j}\left(\frac{n}{|n|}\right) + |n| P_{1,j}\left(\frac{n}{|n|}\right) + |n|^2 P_{2,j}\left(|n|, \frac{n}{|n|}\right),$$

$$P_{0,j}\left(\frac{n}{|n|}\right)\Big|_{N(L)^\perp} \equiv 0 \quad \text{and} \quad \sum_{s=-1}^{2} P_{0,s}\left(\frac{n}{|n|}\right) = \Pi_L,$$

while $\hat{U}_R$ satisfies the following decay condition

$$\|\hat{U}_R(n, v, t)\|_{L^2_v} \le C e^{-\sigma t}, \qquad t > 0, \quad n \in \mathbb{Z}^d,$$

for a suitable choice of the constants $C_R, \sigma > 0$.

The above result states that we can divide the eigenvalues of the linear operator G_ε into "small" and "big" eigenvalues (low and high frequencies decomposition). The former are located close to the origin in the complex plane and are smooth perturbations of the eigenvalues of the homogeneous operator, while the latter are negative and therefore yield a strong semigroup property for the remainder $\hat{U}_R$. In other words, the spectrum of the whole linear operator can be seen as a perturbation of the spectrum of the homogeneous linear operator.

This result gives us all the tools we need to take the limit $\varepsilon \to 0$ in the Boltzmann equation. Moreover, since the semigroup commutes with the $x-$derivatives, it is enough to study the convergence in $L^2_x L^2_v$ to obtain the desired results in $H^s_x L^2_v$.

Study of the linear part. The terms $\hat{U}_j(\varepsilon n, v, t/\varepsilon^2)$, $j = -1, 0, 1, 2$ can be decomposed as sum of 4 contributions:

$$\hat{U}_j(t/\varepsilon^2, \varepsilon n, v) = \sum_{s=0}^{3} U^\varepsilon_{sj}(n, v, t), \qquad j = -1, 0, 1, 2.$$

To avoid technicalities, we do not write down the definition of the terms U^ε_{sj}. We just state (without proof) the relevant properties that we are going to need. We collect these properties in 3 lemmas.

Lemma 4.4 (About U^ε_{0j})**.** *For $j = \pm 1$ there exists $C_0 > 0$ such that, for all $T \in [0, \infty) \cup \{\infty\}$*

$$\left\| \int_0^T U^\varepsilon_{0j} h_{in} dt \right\|^2_{L^2_x L^2_v} \le C_0 \varepsilon^2 \|h_{in}\|^2_{L^2_x L^2_v}.$$

Moreover we have a strong convergence in the $L^2_{[0,\infty)} L^2_x L^2_v$ norm if and only if $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$. In this case $U^\varepsilon_{0j} h_{in} = 0$.

Lemma 4.5 (About U^ε_{lj}, $l = 1, 2, 3$)**.** *For $l = 1, 2, 3$, $j = -1, 0, 1, 2$, the following estimates hold for U^ε_{lj}.*

1. A constant $C_l > 0$ exists such that, for all $T > 0$,

$$\left\| \int_0^T U^\varepsilon_{lj} h_{in} dt \right\|^2_{L^2_x L^2_v} \le C_l \varepsilon^2 \|h_{in}\|^2_{L^2_x L^2_v}.$$

2. A constant $C'_l > 0$ exists such that

$$\left\| U^\varepsilon_{lj} h_{in} \right\|^2_{L^2_{(0,\infty)} L^2_x L^2_v} \le C'_l \varepsilon^2 \|h_{in}\|^2_{L^2_x L^2_v}.$$

3. For every $\delta > 0$ a constant $C_{l,\delta} > 0$ exists such that

$$\left\| U^\varepsilon_{lj} h_{in} \right\|^2_{L^\infty_{(0,\infty)} L^2_x L^2_v} \le C_{l,\delta} \varepsilon^{2\delta} \|h_{in}\|^2_{H^\delta_x L^2_v}.$$

Lemma 4.6 (About $\hat{U}_R$)**.** *The operator $U^\varepsilon_R(n, v, t) \equiv \hat{U}_R(\varepsilon n, v, t/\varepsilon^2)$ satisfies the following inequalities.*

1. A constant $C_4 > 0$ exists such that, for all $T > 0$,

$$\left\| \int_0^T U^\varepsilon_R h_{in} dt \right\|^2_{L^2_x L^2_v} \le C_4 T \varepsilon^2 \|h_{in}\|^2_{L^2_x L^2_v}.$$

2. A constant $C'_4 > 0$ exists such that

$$\|U^\varepsilon_R h_{in}\|^2_{L^2_{(0,\infty)} L^2_x L^2_v} \le C'_4 \varepsilon^2 \|h_{in}\|^2_{L^2_x L^2_v}.$$

3. For every $t_0 > 0$ a constant $C_r > 0$ exists such that

$$\|U^\varepsilon_R h_{in}\|^2_{L^\infty_{(t_0,\infty)} L^2_x L^2_v} \le C_r t_0^{-1/2} \varepsilon \|h_{in}\|^2_{L^2_x L^2_v}.$$

Furthermore, the strong convergence in the time interval $(0, \infty)$ holds true if and only if $L(h_{in}) = 0$. In this case, for every $\delta > 0$ there exists a constant $C_{\delta,R} > 0$ such that

$$\|U^\varepsilon_R h_{in}\|^2_{L^2_{(0,\infty)} L^2_x L^2_v} \le C_{\delta,R} \varepsilon^{2\delta} \|h_{in}\|^2_{H^\delta_x L^2_v}.$$

The final ingredient we need to deduce the convergence for the linear term is the following remark: Thr. 4.7 states that $\alpha_0 = \alpha_2 = 0$, therefore U^ε_{00}, U^ε_{02} do not depend on ε. By putting this remark and Lemmas 4.4–4.6 together we conclude that, as $\varepsilon \to 0$,

$$e^{tG_\varepsilon} h_{in} \to \mathcal{F}_x^{-1} \left[e^{-\beta_0 t |n|^2} P_{00} \left(\frac{n}{|n|} \right) + e^{-\beta_2 t |n|^2} P_{02} \left(\frac{n}{|n|} \right) \right] \mathcal{F}_x h_{in}. \tag{81}$$

The above convergence is strong in $L^2_t H^s_x L^2_v$ if $h_{in} \in H^{s+\delta}_x L^2_v$ and both conditions in Lemmas 4.4, 4.6 are fullfilled, i.e. $L(h_{in}) = 0$, $\nabla_x \cdot u_{in} = 0$, $\rho_{in} + \theta_{in} = 0$. Moreover, any time average of $e^{tG_\varepsilon} h_{in}$ is strongly convergent.

Finally, this result also allows us to deduce that the value of $\lim_{\varepsilon\to 0} e^{tG_\varepsilon} h_{in}$ is the orthogonal projection of h_{in} onto the space $\{g \in N(L) \;:\; \nabla_x \cdot u_g = 0, \quad \rho_g + \theta_g = 0\}$.

Study of the bilinear part. Recall the definition of $u_\varepsilon = \Gamma(h_\varepsilon, h_\varepsilon)$. As a consequence of hypothesis H5, $u_\varepsilon \in N(L)^\perp$. However, we know that $P_{0,j}(n/|n|)|_{N(L)^\perp} \equiv 0$. It follows that

$$P_j\left(|\varepsilon n|, \frac{n}{|n|}\right)\hat{u}_\varepsilon = |\varepsilon n| P_{1j}\left(\frac{n}{|n|}\right)\hat{u}_\varepsilon + |\varepsilon n|^2 P_{2j}\left(|\varepsilon n|, \frac{n}{|n|}\right)\hat{u}_\varepsilon.$$

We can decompose $\psi^\varepsilon[u_\varepsilon]$ as

$$\Psi^\varepsilon[u_\varepsilon] = \int_0^t \frac{1}{\varepsilon} e^{(t-s)G_\varepsilon} u_\varepsilon(s) ds = \sum_{j=-1}^{2}\sum_{l=0}^{4} \psi^\varepsilon_{lj}[u_\varepsilon] + \psi^\varepsilon_R[u_\varepsilon],$$

as in the linear case. For the sake of simplicity, we do not write down the explicit expression for ψ^ε_{lj} (see [2] for details), while

$$\psi^\varepsilon_R[u_\varepsilon] = \int_0^t \frac{1}{\varepsilon} U^\varepsilon_R(t-s) u_\varepsilon(s) ds.$$

Again, as in the linear case, ψ^ε_{00}, ψ^ε_{02} do not depend on ε.

We state suitable bounds for the terms $(\psi^\varepsilon_{lj})_{l,j}$, ψ^ε_R in three Lemmas.

Lemma 4.7 (About ψ^ε_{0j})**.** *For $j = \pm 1$ a constant $\tilde{C}_0 > 0$ exists such that, for every $T > 0$,*

$$\left\| \int_0^T \psi^\varepsilon_{0j}(u_\varepsilon) dt \right\|^2_{L^2_x L^2_v} \le \tilde{C}_0 T^2 \varepsilon^2 E(h_\varepsilon)^2.$$

Lemma 4.8 (About ψ^ε_{lj})**.** *For $j = -1, 0, 1, 2$, $l = 1, 2, 3$ the following bounds hold:*

1. *A constant $\tilde{C}_l > 0$ exists such that, for every $T > 0$,*

$$\left\| \int_0^T \psi^\varepsilon_{lj}[u_\varepsilon] dt \right\|^2_{L^2_x L^2_v} \le \tilde{C}_l T \varepsilon^2 E(h_\varepsilon)^2.$$

2. *A constant $\tilde{C}'_l > 0$ exists such that, for every $T > 0$,*

$$\left\| \psi^\varepsilon_{lj}[u_\varepsilon] \right\|^2_{L^2_{[0,T]} L^2_x L^2_v} \le \tilde{C}'_l \varepsilon^2 E(h_\varepsilon)^2.$$

3. *For every (fractional) multi-index $\delta \in [0,1]^d$, $0 \le |\delta| \le 1$, there exists a constant $C_{l,\delta} > 0$ such that*[2]

$$\left\| \psi^\varepsilon_{lj}[u_\varepsilon] \right\|^2_{L^\infty_t L^2_x L^2_v} \le \tilde{C}_{l,\delta} \varepsilon^{2\delta} E(\partial^0_\delta h_\varepsilon)^2.$$

[2] The fractional derivative ∂^0_δ is defined by means of the Fourier transform $\mathcal{F}_x$.

Lemma 4.9 (About ψ_R^ε)**.** *The following bounds hold.*

1. *A constant* $\tilde{C}_4 > 0$ *exists such that, for every* $T > 0$,

$$\left\| \int_0^T \psi_R^\varepsilon[u_\varepsilon] dt \right\|_{L_x^2 L_v^2}^2 \le \tilde{C}_4 T \varepsilon E(h_\varepsilon)^2.$$

2. *A constant* $\tilde{C}_4' > 0$ *exists such that, for every* $T > 0$,

$$\|\psi_R^\varepsilon[u_\varepsilon]\|_{L^2_{[0,T]} L_x^2 L_v^2}^2 \le \tilde{C}_4' \varepsilon E(h_\varepsilon)^2.$$

3. *A constant* $C_4'' > 0$ *exists such that*

$$\left\|\psi_{lj}^\varepsilon[u_\varepsilon]\right\|_{L_t^\infty L_x^2 L_v^2}^2 \le \tilde{C}_4'' \varepsilon E(h_\varepsilon)^2.$$

The last ingredient we need is the following remark: From Theorems 4.2, 4.3 we know that $(h_\varepsilon)_{\varepsilon>0}$ is bounded in $L_t^\infty H_x^s L_v^2$, which implies that $E(h_\varepsilon)$ is bounded, too. This fact and the previous three lemmas yield the strong convergence of $\psi_{0,\pm1}^\varepsilon[u_\varepsilon]$, $\psi_{lj}^\varepsilon[u_\varepsilon]$ ($j = -1, 0, 1, 2,\ l = 1, 2, 3$) and $\psi_R^\varepsilon[u_\varepsilon]$ towards zero. Since ψ_{00}^ε, ψ_{02}^ε do not depend on ε, this implies the strong convergence of $\psi^\varepsilon[u_\varepsilon] - \psi[u_\varepsilon]$ towards 0, where ψ is defined as

$$\psi[u] = \mathcal{F}_x^{-1}(\psi_{00}^\varepsilon(u) + \psi_{02}^\varepsilon[u])\mathcal{F}_x.$$

Now we have to show that $\psi[u_\varepsilon] \to \psi[u]$ strongly as $\varepsilon \to 0$, with $u = \Gamma(h, h)$.

Remember what we already know:

$$\forall T > 0, \quad V_T(\varepsilon) \equiv \sup_{t\in[0,T]} \|h_\varepsilon(t) - h(t)\|_{L_x^\infty L_v^2} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The following estimates for the difference $\psi[u_\varepsilon] - \psi[u]$ can be shown.

Lemma 4.10 (Rates of convergence)**.** *The following relations hold:*

1. *A constant* $\tilde{C}_5 > 0$ *exists such that, for every* $T > 0$,

$$\left\| \int_0^T \psi[u_\varepsilon] dt - \int_0^T \psi[u] dt \right\|_{L_x^2 L_v^2}^2 \le \tilde{C}_5 T^2 V_T(\varepsilon)^2.$$

2. *A constant* $\tilde{C}_5' > 0$ *exists such that, for every* $T > 0$,

$$\|\psi[u_\varepsilon] - \psi[u]\|_{L^2_{[0,T]} L_x^2 L_v^2}^2 \le \tilde{C}_5' T V_T(\varepsilon)^2.$$

3. *A constant* $C_5'' > 0$ *exists such that, for every* $T > 0$,

$$\|\psi[u_\varepsilon](T) - \psi[u](T)\|_{L_x^2 L_v^2}^2 \le \tilde{C}_5'' V_T(h_\varepsilon)^2.$$

Putting the previous lemmas and the study of the linear case (Lemmas 4.4–4.6) yield the proof of Thr. 4.6. □

Thanks to Thr. 4.6 we can control the convergence $h_\varepsilon \to h$ in any finite time interval $[0, T]$. It is possible to show that, for a hard potential collision kernel,

$$\forall T > 0, \quad V_T(\varepsilon) \leq C_V \varepsilon.$$

Moreover, Thr. 4.2 implies exponential decay for both h and h_ε, so

$$\|h_\varepsilon(T) - h(T)\|_{H^s_x L^2_v} \leq 2\|h_{in}\|_{\mathcal{H}^s_\varepsilon} e^{-\tau_s T}.$$

Define

$$T_M \equiv -\frac{1}{\tau_s} \log \left(\frac{\varepsilon}{2\|h_{in}\|_{\mathcal{H}^s_\varepsilon}} \right).$$

It follows

$$\forall t \geq T_M, \quad \|h_\varepsilon(t) - h(t)\|_{H^s_x L^2_v} \leq \varepsilon.$$

Now it is enough to apply Thr. 4.6 with T replaced by T_M to conclude the proof of Thr. 4.4.

Appendix: validation of the assumptions.

We present here some commonly used kinetic models for which the hypocoercivity assumptions hold.

1. Linear relaxation:
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \left(\mu \int_{\mathbb{R}^d} f_* dv_* - f \right).$$

2. Linear Fokker-Planck equation:
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_v \cdot (\nabla_v f + v f).$$

3. Semiclassical relaxation:
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (\mu(1 - \delta f) f_* - \mu_*(1 - \delta f_*) f) dv_*.$$

4. Boltzmann equation with angular cutoff:
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} b(\cos\theta) |v - v_*|^\gamma (f' f'_* - f f_*) dv_* d\sigma.$$

5. Landau equation:
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_v \cdot \int_{\mathbb{R}^d} \Phi(v - v_*) |v - v_*|^{\gamma+2} (f_* (\nabla f) - f (\nabla f)_*) dv_*.$$

Hypothesis (H4), (H5) are trivially satisfied for the linear models 1, 2. Moreover, for model 3 it is straightforward to see that $\|\cdot\|_{\Lambda_v}$ is just the L^2_v norm and (H4) holds (just apply Cauchy-Schwartz inequality). We are going to prove (H5) for the Semiclassical relaxation model and (H4), (H5) for the Boltzmann equation. Showing (H4), (H5) for the Landau equation goes beyond the scope of these Lecture Notes; the curious Reader can find the proof in [2]. Also, the proof of (H1)–(H3), (H1'), (H2') can be found e.g. in [8].

Semiclassical relaxation. The equilibrium of the semiclassical relaxation model is not a Maxwellian like in the Boltzmann case. One can prove (by considering the relation $\int_{\mathbb{R}^d} fQ(f,f)dv = 0$) that the equilibrium for model 3 reads as

$$f^{FD} = \frac{k_\infty \mu}{1+\delta k_\infty \mu},$$

where k_∞ depends on f_0. The function f^{FD} is the so-called *Fermi-Dirac* distribution and plays an important role in the modeling of quantum phenomena, e.g. charge transport in semiconductors.

A possible (actually, good) linearization of the semiclassical relaxation equation is

$$f = f^{FD} + \varepsilon \frac{\sqrt{K_\infty \mu}}{1+\delta k_\infty \mu} h.$$

By empoying this linearization we obtain an equation identical to (47) with the operators L and Γ replaced by $L^{(SC)}$ and $\Gamma^{(SC)}$, respectively. We point out that $N(L^{(SC)}) = \text{Span}(f^{FD}/\sqrt{\mu})$, which means that (H3) is not fulfilled. However, $f^{FD}/\sqrt{\mu} \le Ce^{-|v|^2/4}$ and so we can still use the estimates in Section 4.3 and, as a consequence, all the Theorems we proved hold also for the semi-classical relaxation model.

Let us now show that the bilinear operator $\Gamma^{(SC)}$ satisfies (H5). It is easy to see that $\Gamma^{(SC)}$ is defined as

$$\Gamma^{(SC)}(g,h) = \frac{\delta\sqrt{k_\infty}}{2} \int_{\mathbb{R}^d} \frac{\mu_* - \mu}{1+\varepsilon k_\infty \mu_*}(hg_* + h_* g)\sqrt{\mu_*}dv_*.$$

By bultiplying the above equality times a function f, integrating in $\mathbb{R}^d$ and exchanging $v \leftrightarrow v^*$ we get

$$(\Gamma^{(SC)}(g,h), f)_{L^2_v} = \frac{\delta\sqrt{k_\infty}}{4} \int_{\mathbb{R}^d\times\mathbb{R}^d} (\mu_* - \mu)(hg_* + h_* g)\left(f\frac{f^{FD}_*}{\sqrt{\mu_*}} - f_* \frac{f^{FD}}{\sqrt{\mu}}\right) dvdv_*.$$

Choosing $f \in N(L^{(SC)})$ yields $(\Gamma^{(SC)}(g,h), f)_{L^2_v} = 0$, i.e. (H5) holds.

Boltzmann operator with angular cutoff and hard potential. We remind the Reader that this case is characterized by $\gamma > 0$. Given the property $\mu_*\mu'_* = \mu\mu'$, Γ is given by

$$\Gamma(g,h) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\sqrt{\mu_*}(g'_* h' + g' h'_* - g_* h - g h_*)dv_* d\sigma.$$

For every $\psi \in L^2_v$ it holds

$$\int_{\mathbb{R}^d} \Gamma(g,h)(v)\psi(v)dv$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B(g'_*h' + g'h'_* - g_*h - gh_*)(\sqrt{\mu_*}\psi + \sqrt{\mu}\psi_* - \sqrt{\mu'_*}\psi' - \sqrt{\mu'}\psi'_*)dvdv_*d\sigma.$$

It is known [8] that $N(L) = \mathrm{Span}(\sqrt{\mu}, \sqrt{\mu}v_1, \dots, \sqrt{\mu}v_d, \sqrt{\mu}|v|^2)$. Therefore choosing $\psi \in N(L)$ yields $(\Gamma(g,h),\psi)_{L^2_v} = 0$, i.e. (H5) holds.

Now we show (H4). Split Γ as sum $\Gamma(g,h) = \Gamma^+(g,h) + \Gamma^-(g,h)$, with

$$\Gamma^+(g,h) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\sqrt{\mu_*}g'_*h'dv_*d\sigma,$$
$$\Gamma^-(g,h) = -\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\sqrt{\mu_*}g_*hdv_*d\sigma.$$

Let us define the new variable $u = v - v_*$ (relative velocity). Therefore $v' = v + f_1(u,\sigma)$, $v'_* = v + f_2(u,\sigma)$ for some suitable functions f_1, f_2, and B is a function of u, σ: $B = b(\cos\theta)|u|^\gamma$. Let j, l multi-indexes such that $|j| + |l| \le s$. Differentiating $\Gamma^-(g,h)$ yields

$$\partial^j_l\Gamma^-(g,h) = -\frac{1}{2}\sum_{\substack{j_0+j_1+j_2=j\\ l_1+l_2=l}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)|u|^\gamma\partial^{j_0}_0\sqrt{\mu(v-u)}\,\partial^{j_1}_{l_1}g_*\,\partial^{j_2}_{l_2}h\,dud\sigma.$$

It is quite clear that, for a suitable constant $C > 0$,

$$|\partial^{j_0}_0\sqrt{\mu(v-u)}| \le C\mu(v-u)^{1/4}.$$

Furthermore, since $\gamma > 0$,

$$|u|^\gamma\mu(v-u)^{1/4} \le C(1+|v|)^\gamma\mu(v-u)^{1/8}.$$

From the above relations and the fact that $b(\cos\theta) \le C$ (by Grad's angular cutoff (51)) we deduce

$$|(\partial^j_l\Gamma^-(g,h),f)_{L^2_{x,v}}| \le C\sum_{\substack{j_0+j_1+j_2=j\\ l_1+l_2=l}} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|)^\gamma|\partial^{j_2}_{l_2}h||f|\left(\int_{\mathbb{R}^d}\mu_*^{1/8}|\partial^{j_1}_{l_1}g_*|dv_*\right)dvdx$$
$$\le \mathcal{G}^s(g,h)\|f\|_\Lambda,$$

with

$$\mathcal{G}^s(g,h) = C\sum_{|j_1|+|j_2|+|l_1|+|l_2|\le s}\left(\int_{\mathbb{T}^d}\|\partial^{j_2}_{l_2}h\|^2_{\Lambda_v}\|\partial^{j_1}_{l_1}g\|^2_{\Lambda_v}dx\right)^{1/2}$$

and

$$\|f\|_{\Lambda_v} = \int_{\mathbb{R}^d} f^2(1+|v|^2)^\gamma dv.$$

Recall the Sobolev embedding $W_x^{1,s/2} \hookrightarrow L_x^\infty$ for s large enough. Clearly either $|j_1| + |l_1| \leq s/2$ or $|j_2| + |l_2| \leq s/2$ is fulfilled. If for example $|j_1| + |l_1| \leq s/2$ it follows

$$\begin{aligned}
\|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 \leq \sup_{x\in\mathbb{T}^d} \|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 \leq C_s \left\| \|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 \right\|_{W_x^{1,s/2}} = \sum_{|p|\leq s/2} \int_{\mathbb{T}^d} \left| \partial_p \int_{\mathbb{R}^d} |\partial_{l_1}^{j_1} g|^2 dv \right| dx \\
\leq C_s \sum_{|p|\leq s/2} \sum_{p_1+p_2=p} \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{l_1+p_1}^{j_1} g||\partial_{l_1+p_2}^{j_1} g| dxdv \\
\leq C_s \sum_{|j_3|+|l_3|\leq s} \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{l_3}^{j_3} g|^2 dxdv = C_s \|g\|_{H_{x,v}^s}^2.
\end{aligned}$$

In the other case, the same computations yield

$$\|\partial_{l_2}^{j_2} g\|_{L_v^2}^2 \leq C_s \|h\|_{H_{x,v}^s}^2.$$

In the case $j = 0$ we can repeat the above argument and find that, since we consider terms with no $v-$derivatives and the computations we make do not produce $v-$derivatives, we can control the terms by means of the $x-$derivatives only. Putting all these estimates together yields (H4) for Γ^-.

We deal with the second term Γ^+ in a similar way. We have thus shown that (H4) holds for the Boltzmann equation with Grad's cutoff and hard potential.

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