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ANISOTROPIC SAMPLING GRIDS FOR LENGTH ESTIMATION

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ABSTRACT

In this paper, we introduce a new estimator for curve length based on the number of intersections of an anisotropic test system of parallel lines with the curve. The estimator is similar to a Horvitz-Thompson estimator since the outcome of the experiment is weighted by the probability of the realization of the random angle of the test system. It is shown how the directional distribution of the angle of the sampling grid can be chosen in order to minimize the variance of this estimator if a-priori information of the directional distribution of the curve is known. For special curves the variance is calculated and it is shown that the variance of our estimator is smaller than the variance of the usual estimator based on an isotropic grid.

Keywords: stereology, length estimation, surface area estimation, anisotropic sampling grid, directional distribution.

INTRODUCTION

Classical stereological estimators are based on the intersection of an uniform random test system and an object. For estimating the length and surface area of an object, the test system must additionally be isotropic, see Baddeley and Jensen (2005). These estimators are unbiased due to the sampling procedure. They do not, however, use any a-priori knowledge of the object of interest. These isotropic uniform random test systems are used since decades for the estimation of curve length in stereology. By counting the number of intersection points with the test system and multiplying this number with a constant one gets an unbiased estimator for curve length. This constant depends only on the mean length per unit area of the test system. However, if the curve from which the length is estimated is extremely anisotropic, then this estimator has very high variance. This is due to the isotropic rotation, as both very small and very high numbers of intersections are equally likely and hence the variation in the intersection counts is very high.

There are several spatial structures which are highly anisotropic. Examples are nerve fibres in skin tissue or surface area in rolled steel. The a-priori information concerning the anisotropy of the objects can be used to construct anisotropic sampling grids which lead to considerably reduced estimation variance compared to the classical isotropic sampling grids, as it is better to choose the random rotation of the test system in a way such that situations with a high number of intersections are more likely since then more information is used. Non-uniform systematic sampling was studied in Dorph-Petersen *et al.* (2000) and based on a-priori knowledge of the object under study, more efficient estimators than the classical ones were constructed. In this paper, we also propose to use a-priori information of the object in order to get more efficient estimators than the classical ones. Here, we consider objects where the directional distribution of the boundary is known. The directional distribution of a curve is the distribution of the tangent angle at a uniform random point on the curve.

In the following, we will construct an unbiased estimator for anisotropic uniform random test systems if the directional distribution has a probability density. Later we will show how to choose this density in an optimal way if the directional distribution of the curve is known.
Optimal here means that we have minimal variance for a given test system if only the directional distribution of the test system can be chosen but the test system is uniform random and the distance between the lines is fixed. Although we are only considering test systems of parallel lines in this paper, the ideas can also be extended to other test systems, see the discussion section.

PRELIMINARIES

The problem considered in this paper is the estimation of the length $\nu_1(Y)$ of a curve $Y \subset \mathbb{R}^2$. To estimate this length we intersect the curve with a random test system $\Lambda_{z,t}$. We will consider both isotropic and anisotropic uniform random test systems $\Lambda_{z,t}$ of parallel lines a distance h apart. The construction of such a test system goes as follows (Cruz-Orive (2002); Baddeley and Jensen (2005)). First we construct two independent random variables, z, uniformly distributed in the interval $[0,h)$, and t, distributed according to a probability density f on $[0,\pi)$. Here, it is assumed that

$$f:[0,\pi)\to(0,\infty) \quad \text{and} \quad f(\phi)>0 \quad \forall\phi\in[0,\pi) \tag{1}$$

Let $L_1(z,t)$ denote a line with direction t and translated by z. Further, let $L_1^{\perp}(0,t)$ denote the orthogonal complement of the line $L_1(0,t)$ with angle t going through the origin. We identify $L_1^{\perp}(0,t)$ with $\mathbb{R}$ and assume that $z\in L_1^{\perp}(0,t)$. Then

$$\Lambda_{z,t}=\bigcup_{k\in\mathbb{Z}} L_1(t,z+k\cdot h) \tag{2}$$

If $f\equiv 1/\pi$ we have an isotropic uniform random (IUR) test system, otherwise we call the test system anisotropic uniform random (AUR). For IUR test systems $\Lambda_{z,t}$ we get the classical unbiased estimator $\widehat{v}_1(Y)$ for $v_1(Y)$ with

$$\widehat{v}_1(Y):=\frac{\pi}{2}h\,v_0(Y\cap\Lambda_{z,t}) \tag{3}$$

Here, $v_0(Y\cap\Lambda_{z,t})$ denotes the number of intersection points. In the next section we will introduce a generalization of this estimator for AUR test systems.

THE ESTIMATOR

Here, we consider a uniform random test system with random orientation t distributed according to a probability density f on $[0,\pi)$ with $f(t)>0$ for all $t\in[0,\pi)$. Then we get the following unbiased estimator for the length of a curve.

Proposition 1 *Let $f(t)$ be the rotation density of the test system $\Lambda_{z,t}$, where z is a uniform random translation. Then*

$$\widehat{v}_1(Y,f):=\frac{h}{2f(t)}v_0(Y\cap\Lambda_{z,t}) \tag{4}$$

is an unbiased estimator of $v_1(Y)$ for a curve Y.

Proof: It holds that

$$\begin{aligned}\mathbb{E}\widehat{v}_1(Y,f) &= \frac{h}{2}\int_0^{\pi}\int_0^{h}\frac{v_0(Y\cap\Lambda_{z,t})}{f(t)}f(t)\frac{dz}{h}\,dt\\ &= \frac{1}{2}\int_0^{\pi}\int_0^{h}v_0(Y\cap\Lambda_{z,t})\,dz\,dt\end{aligned}$$

The unbiasedness of the estimator can now be proven in the same way as for the isotropic estimator. □

Note that the estimator based on an AUR test system can be interpreted as a continuous version of a Horvitz-Thompson estimator where we divide by the sampling probability which here is given by the density function $f(t)$.

The next proposition states a formula for the second moment of the estimator.

Proposition 2 *The second moment of the estimator in (4) is given by*

$$\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right) = \frac{1}{4}\int_0^{\pi}\frac{1}{f(t)}\mathbb{E}_z\left(h^2v_0^2(Y\cap\Lambda_{z,t})\right)dt,$$

where the inner expectation is with respect to z uniformly distributed in $[0,h)$.

Proof: This follows directly from

$$\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right)=\mathbb{E}_t\left[\mathbb{E}\left(\widehat{v}_1^2(Y,f)|t\right)\right]$$

and

$$\begin{aligned}\mathbb{E}\left(\widehat{v}_1^2(Y,f)|t\right) &= \mathbb{E}_z\left(\widehat{v}_1^2(Y,f)\right)\\ &= \frac{1}{4\cdot f(t)^2}\mathbb{E}_z\left(h^2v_0^2(Y\cap\Lambda_{z,t})\right)\end{aligned}$$

Here, $\mathbb{E}_t(\cdot),\mathbb{E}_z(\cdot)$ are expectations with respect to t,z, respectively. □

As an example, we will now consider two special cases of curves, convex curves and line segments.

Proposition 3 *If Y is a convex curve, or the boundary of a convex object, the second moment of the estimator in (4) is given by*

$$\begin{aligned}&\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right) \\ &= \int_0^{\pi}\frac{1}{f(t)}\mathrm{Var}_z\,\widehat{v}_1(Y_t')\,dt+\int_0^{\pi}\frac{1}{f(t)}v_1^2(Y_t')\,dt\end{aligned} \tag{5}$$

If Y is a line segment, we get

$$\begin{aligned}&\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right) \\ &= \frac{1}{4}\left(\int_0^{\pi}\frac{1}{f(t)}\mathrm{Var}_z\,\widehat{v}_1(Y_t')\,dt+\int_0^{\pi}\frac{1}{f(t)}v_1^2(Y_t')\,dt\right)\end{aligned} \tag{6}$$

Here, Y_t' is the orthogonal projection of Y onto the orthogonal complement of the test system and $\widehat{v}_1(Y_t')$ is the estimator of the length of Y in $\mathbb{R}^1$ based on a uniform random point grid.

Proof: Let us first consider the case where Y is the boundary of a convex object. Then, almost surely, the number of intersections of a line from the test system and Y is 0 or 2. It is 2 if the translation of the line lies in the projection of Y onto the orthogonal complement of the line. This gives

$$\begin{aligned}
&\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right)\\
&= \frac{1}{4}\int_0^{\pi} \frac{1}{f(t)} \mathbb{E}_z\left(h^2 v_0^2(Y\cap\Lambda_{z,t})\right)\,dt\\
&= \int_0^{\pi} \frac{1}{f(t)}\left(\mathrm{Var}_z\,\widehat{v}_1(Y_t') + \left(\mathbb{E}_z\widehat{v}_1(Y_t')\right)^2\right)dt\\
&= \int_0^{\pi} \frac{1}{f(t)} \mathrm{Var}_z\,\widehat{v}_1(Y_t')\,dt + \int_0^{\pi}\frac{1}{f(t)} v_1^2(Y_t')\,dt
\end{aligned}$$

If Y is a line segment, then the number of intersections is almost surely 0 or 1 and we get the result in the same way as above. □

The result of Proposition 3 can be easily generalized to arbitrary curves if the total projection is used instead of the orthogonal projection.

OPTIMAL CHOICE OF THE DIRECTIONAL DISTRIBUTION

The results for the second moments given in the previous section correspond to the decomposition of the variance into two parts, where one part is due to the projection and the other due to the estimation of the projection by point counting. For a convex curve Y, that is

$$\mathrm{Var}\widehat{v}_1(Y,f) = \mathrm{Var}\mathbb{E}\left(\widehat{v}_1(Y,f)|t\right) + \mathbb{E}\mathrm{Var}\left(\frac{1}{f(t)}\widehat{v}_1(Y_t')|t\right) \tag{7}$$

For non-convex curves Y instead of Y_t' the half total projection onto $L_1^{\perp}(0,t)$ has to be used here and in the rest of the section, but the results are still valid. In the following we will assume that the position of the curve Y is unknown. Furthermore the distance h between the lines is fixed. Under this assumptions it is reasonable to try to minimize the variance due to projection only since the variance due to point counting is unknown. If the density g of the directional distribution of Y is known, the density f should be chosen to minimize the variance due to projection. That is, we want to find a probability density f^* with the property

$$\mathrm{Var}\mathbb{E}\left(\widehat{v}_1(Y,f^*)|t\right) = \min_{f\in F}\mathrm{Var}\mathbb{E}\left(\widehat{v}_1(Y,f)|t\right)$$

where F is the set of all probability densities on $[0,\pi)$. The variance of a random variable X is 0 if and only if X is constant almost surely. In our case this corresponds to

$$\begin{aligned}
\mathbb{E}\left(\widehat{v}_1(Y,f)|t\right) &= \mathbb{E}_z\left(\frac{h}{2}\frac{v_0(Y\cap\Lambda_{z,t})}{f(t)}\right)\\
&= \frac{1}{f(t)} v_1(Y_t') \overset{!}{\equiv} const
\end{aligned}$$

That is, we should choose f^* such that

$$f^*(t) \propto v_1(Y_t') = \frac{v_1(Y)}{2}\int_0^{\pi} g(s)|\sin(s-t)|\,ds$$

By normalizing the right hand side, we get

$$f^*(t) = \frac{1}{2}\int_0^{\pi} g(s)|\sin(s-t)|\,ds \tag{8}$$

Figure 1 shows the minimizing density f^* if the density g is a von Mises distribution. The von Mises distribution plays the role of the normal distribution for circular distributions (Mardia (1972)). Its probability density is

$$g(t) = \frac{1}{C_\kappa}\exp(-\kappa\cdot\cos(2\cdot(t-\phi))) \tag{9}$$

where κ is the concentration parameter, C_κ is a normalization constant and ϕ the preferred direction.

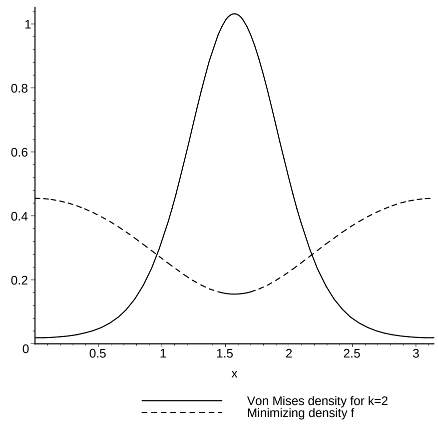

Fig. 1. *The density g for a von Mises distribution and the minimizing density f^*.*

A closer look at the definition of f^* reveals that it is given by a convolution of g with sin, so it is a smoothened version of g. This can be seen is Figure 1, where we see that f^* is more equally distributed over $[0,\pi)$ than g. This seems to be a good property since we have to divide by $f^*(t)$ in the estimator $\widehat{v}_1(Y,f^*)$.

CONVEX CURVES FOR A GIVEN DIRECTIONAL DISTRIBUTION

In order to calculate the second moments in Proposition 3 we need parametric equations of curves for a given density of the directional distribution. Suppose the directional distribution is given by a probability density g on $[0,\pi)$. We assume that $g(t+\pi)=g(t)$, i.e. that g is periodic on $\mathbb{R}$ with period π. Then we can construct a convex curve Y with the following parametric equation:

$$x(t) := \begin{pmatrix} \int_0^t g(s)\cos s\, ds \\ \int_0^t g(s)\sin s\, ds \end{pmatrix} \quad , \quad t\in[0,2\pi]$$

This curve is closed since $x(2\pi)=x(0)$ and the length is 2 since

$$\int_0^{2\pi} \| x'(t) \| \, dt = \int_0^{2\pi} f(t) dt = 2$$

Furthermore, at point t the tangent of $x(t)$ is equal to $t \bmod \pi$ since

$$x'(t) = g(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

For an interval $[a,b) \subset [0,\pi)$, the probability that the tangent of a uniform random point of Y is in $[a,b)$ can be calculated,

$$\begin{aligned}&\mathbb{P}(\mathrm{Tan}[Y,x] \in [a,b)) \\ &= \frac{1}{2}\int_0^{2\pi} \| x'(t) \| \, \mathbb{1}(t\in[a,b)\cup[a+\pi,b+\pi))\, dt \\ &= \frac{1}{2}\left(\int_a^b g(t)dt + \int_{a+\pi}^{b+\pi} g(t)dt\right) = \int_a^b g(t)dt\end{aligned}$$

Here, we use the periodicity of g. Since this holds for all intervals $[a,b)$, the two measures have to coincide on the Borel-σ-algebra on $[0,\pi)$. This means that if Y is given by the parametric equation $x(t)$, we have a convex curve with directional distribution according to the density g. By multiplying $x(t)$ by $1/2\cdot v_1(\widetilde{Y})$, we get a curve $\widetilde{Y}$ of length $v_1(\widetilde{Y})$ with the same directional distribution. This representation of a convex curve will be used in the next section to calculate the variance in some examples.

EXAMPLES

CONVEX CURVE INTERSECTED BY AN AUR TEST SYSTEM

Now we will calculate the second moment for our estimator for a convex curve Y with distribution density g. We assume that the curve Y is given in the parametric form explained in the preceding section. An example of such a curve with von Mises directional distribution is shown in Figure 2.

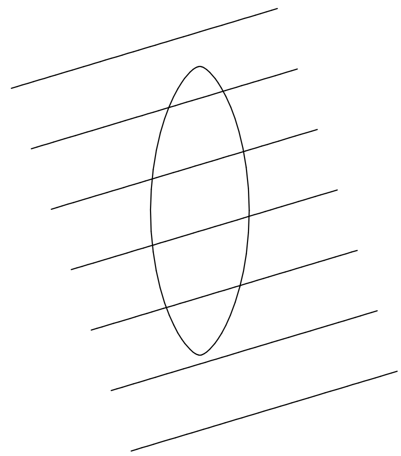

Fig. 2. *Anisotropic curve intersected by a test system.*

The parametrization of Y is given by

$$x(t) := \frac{v_1(Y)}{2} \begin{pmatrix} \int_0^t g(s)\cos s\, ds \\ \int_0^t g(s)\sin s\, ds \end{pmatrix} \quad , \quad t\in[0,2\pi]$$

The length $v_1(Y_t')$ of the orthogonal projection of Y for a given t is

$$v_1(Y_t') = \frac{v_1(Y)}{2}\int_0^{\pi} g(s)|\sin(s-t)|\, ds$$

We can use the representation of the second moment from Proposition 3, i.e.

$$\begin{aligned}&\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right) \qquad (10)\\ &= \int_0^{\pi} \frac{1}{f(t)} \mathrm{Var}_z\, \widehat{v}_1(Y_t')\, dt + \int_0^{\pi} \frac{1}{f(t)} v_1^2(Y_t')\, dt\end{aligned}$$

In the rest of this section we will consider densities f_1 and f^* defined by

$$\begin{aligned} f_1(t) &:= \frac{1}{\pi} \\ f^*(t) &:= \frac{1}{2}\int_0^{\pi} g(s)|\sin(s-t)|\, ds \end{aligned}$$

Let us first consider the second integral in (10). Then we get on the one hand

$$\int_0^\pi \frac{1}{f_1(t)} v_1^2(Y_t')\,dt$$
$$= \frac{\pi}{4} v_1^2(Y) \int_0^\pi \left(\int_0^\pi g(s)|\sin(s-t)|\,ds \right)^2 dt$$

and on the other hand

$$\int_0^\pi \frac{1}{f^*(t)} v_1^2(Y_t')\,dt = v_1^2(Y) \int_0^\pi f^*(t)\,dt = v_1^2(Y)$$

which is obvious due to the selection of f^*. If g is von-Mises distributed, we get for f_1 by numerical integration

$$\int_0^\pi \frac{1}{f_1(t)} v_1^2(Y_t')\,dt \approx \begin{cases} 1.412 \frac{\pi}{4} v_1^2(Y) & , \quad \text{if } \kappa = 2 \\ 1.504 \frac{\pi}{4} v_1^2(Y) & , \quad \text{if } \kappa = 5 \\ 1.536 \frac{\pi}{4} v_1^2(Y) & , \quad \text{if } \kappa = 10 \end{cases}$$

Now we consider the first integral in (10). There we have to calculate the variance due to estimation of the length of the projection by a point grid. This is (Baddeley and Jensen (2005))

$$\mathrm{Var}_z\, \widehat{v}_1(Y_t')$$
$$= h^2 \left(p(\lfloor \frac{v_1(Y_t')}{h} \rfloor + 1)^2 + (1-p)(\lfloor \frac{v_1(Y_t')}{h} \rfloor)^2 \right)$$
$$- v_1^2(Y_t')$$

where

$$p = \frac{v_1(Y_t')}{h} - \lfloor \frac{v_1(Y_t')}{h} \rfloor$$

With this the integral can be calculated numerically for f_1 and f^* to get the exact variance including the usual fluctuation of the variance of estimators based on systematic sampling (also called 'Zitterbewegung'). We can use the following approximation formula (Cruz-Orive (1989; 2002))

$$\mathrm{Var}_z\, \widehat{v}_1(Y_t') \approx 0.1667\, h^2$$

This is independent of $v_1(Y_t')$, so we only have to calculate

$$\int_0^\pi \frac{1}{f_1(t)}\,dt = \pi^2$$

and

$$\int_0^\pi \frac{1}{f^*(t)}\,dt \approx \begin{cases} 11.272 & , \quad \text{if } \kappa = 2 \\ 13.082 & , \quad \text{if } \kappa = 5 \\ 14.441 & , \quad \text{if } \kappa = 10 \end{cases}$$

Here we thus get

$$\int_0^\pi \frac{1}{f_1(t)}\,dt \le \int_0^\pi \frac{1}{f^*(t)}\,dt$$

but with the factor h^2 in the approximation this contribution to the variance tends to zero as h tends to zero whereas the contribution due to the projection is constant. The variances for both estimators are plotted with respect to h in Figures 3 and 4 for a von Mises distributed convex curve with parameter $\kappa = 2$ and $\kappa = 5$ respectively. One can see that the variance for f^* is always lower.

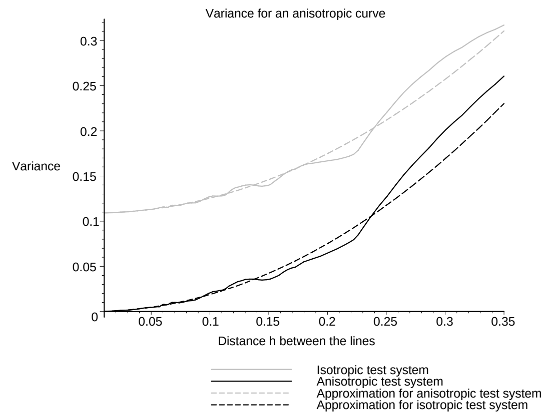

Fig. 3. *Variance for* $\kappa = 2$ *for a curve of length* 1.

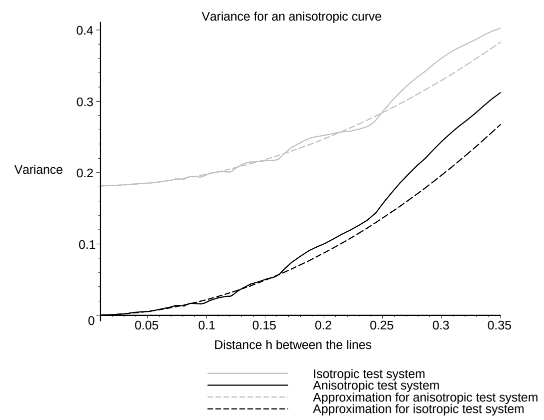

Fig. 4. *Variance for* $\kappa = 5$ *for a curve of length* 1.

Let us now we assume that we use the 'wrong' anisotropic test system. In Figure 5, we have plotted the variance for a von Mises distributed convex curve with Parameter $\kappa = 5$ for the estimator fitted to $\kappa = 2$ and the estimator fitted to $\kappa = 10$.

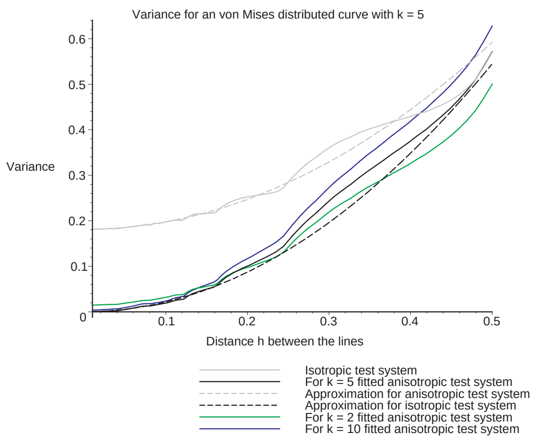

Fig. 5. *Variance for* $\kappa = 5$ *for a curve of length* 1 *including wrong fitted test systems.*

The estimators with the anisotropic test systems behave very similar for small *h*, even if the test system is not fitted to the right distribution. It seems very robust to the wrong directional distribution. In Figure 6 the densities f^* for $\kappa = 2, 5, 10$ are plotted with the densities of g for these parameters. There is no great difference for f^* for different parameters, although there is a big difference for g.

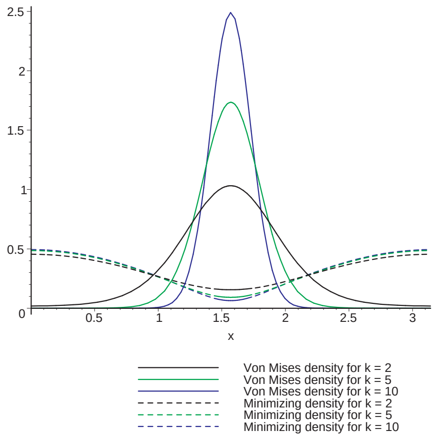

Fig. 6. *Densities for different parameters.*

We can go further and assume that we fit the density f^* to the wrong direction. This is plotted in figure 7. There we assumed that the direction is chosen with an error of 0.3.

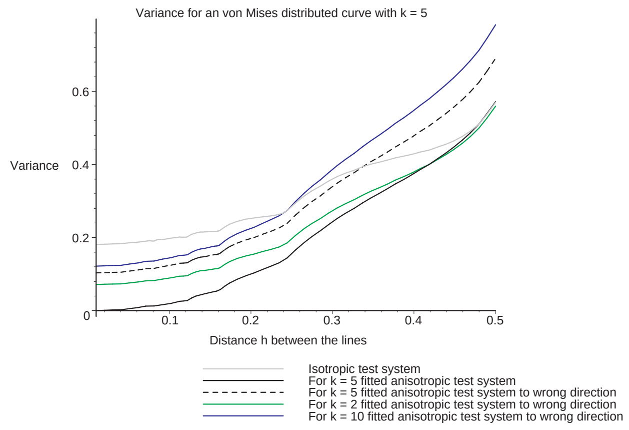

Fig. 7. *Variance for* $\kappa = 5$ *for a curve of length* 1 *including wrong fitted test systems with wrong directions.*

There we see that even for the wrong direction the anisotropic estimators are better for realistic values of h. Note that $v_1(Y) = 1$, i.e. the maximal diameter is less than $1/2$. The considered densities are plotted in figure 8 to get an idea of the error.

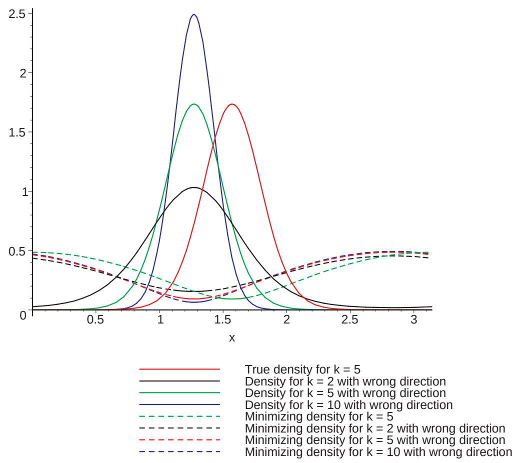

Fig. 8. *Densities for different parameters in figure 7.*

One could also consider a random processes of such a curve Y. Assume that we want to estimate the mean length per unit area of such a process. If we have a Boolean model (Stoyan *et al.* (1995)) of Y we get similar results for the variance since the objects are independent of each other.

RANDOMLY ORIENTATED LINE SEGMENTS

Now we consider the variance for a line segment Y_s with direction s which is random and distributed according to the density g. We use the same densities

f_1 and f^* as in the preceding example. Proposition 3 yields

$$\mathbb{E}\left(\widehat{v}_1^2(Y,f)\right) = \frac{1}{4}\int_0^\pi\int_0^\pi \frac{g(s)}{f(t)}\mathrm{Var}_z\,\widehat{v}_1(Y'_{s,t})\,dt\,ds + \frac{1}{4}\int_0^\pi\int_0^\pi \frac{g(s)}{f(t)} v_1^2(Y'_{s,t})\,dt\,ds$$

The length of the orthogonal projection is in this case $v_1(Y'_{s,t}) = v_1(Y)|\sin(t-s)|$. With this the second integral can be calculated as before. The first integral can also be calculated by numerical integration in the same way as for a convex curve and the same approximation formula can be considered. The results are plotted in Figure 9 for a von Mises distributed line segment with parameter $\kappa = 5$.

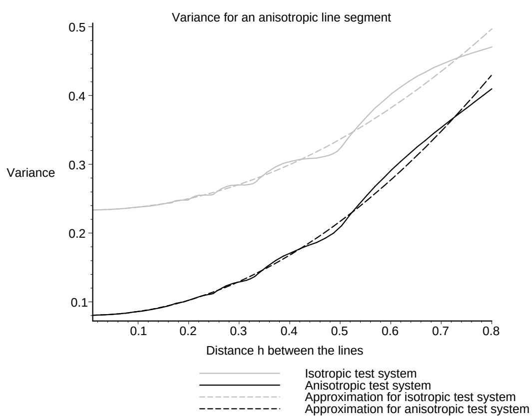

Fig. 9. *The variance for* $\kappa = 5$ *for a line segment of length* 1.

For other values of κ the results are similar. Again, Boolean models could also be considered.

DISCUSSION

We have introduced an unbiased estimator based on anisotropic uniform random sampling grids. This is a generalization of the existing estimators for curve length in $\mathbb{R}^2$ since these are included as special cases. It is also shown how to choose the directional distribution of the test system such that the variance is minimized if the directional distribution of the curve is known. For two special cases, namely a deterministic bounded closed curve with von Mises distributed tangents and a line segment with von Mises distributed orientation, the variance was calculated and it was shown that the variance of the estimator based on the anisotropic test system is less than the variance of the isotropic test system, even if the density of the directional distribution of the test system is fitted to a wrong directional distribution of the curve. It seems that our estimator is robust against such errors in the a-priori assumption, but it seems that it is more sensitive to errors in the choice of the preferred direction than in the shape of the distribution.

The ideas of this paper can also be generalized to estimators based on lines or planes in $\mathbb{R}^3$ for curve length and surface area estimation. It is also possible to generalize these ideas to estimators based on uniform random test systems not consisting of lines but of 1-dimensional probes with directional distribution according to our variance minimizing density f^* similar to the cycloid arcs, see Baddeley *et al.* (1986). Then each individual intersection point hast to be weighted with the value $1/f(\phi)$, where ϕ is the angle of the tangent at the intersection point of the test system. We have done simulation studies for this problem, but there no unique statements for the variance can be done since the variance also depends on the position of the probes with respect to each other. In many examples, however, the anisotropic estimator gives a better estimate than the isotropic estimator.

Additionally, these ideas could be extended to the estimators from local stereology, see Jensen (1998).

To conclude, we have shown that using a-priori information of a curve can reduce the variance of curve length estimation significantly and it is our belief that these estimators can easily be applied if a-priori information of the preferred direction and distribution of the object of interest is available.

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