Sectorial forms and degenerate differential operators

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Abstract

If a is a densely defined sectorial form in a Hilbert space which is possibly not closable, then we associate in a natural way a holomorphic semigroup generator with a. This allows us to remove in several theorems of semigroup theory the assumption that the form is closed or symmetric. Many examples are provided, ranging from complex sectorial differential operators, to Dirichlet-to-Neumann operators and operators with Robin or Wentzell boundary conditions.

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1 Introduction

Form methods are most efficient to solve evolution equations in a Hilbert spaces H. The theory establishes a correspondence between closable sectorial forms and holomorphic semigroups on H which are contractive on a sector (see Kato [Kat], Tanabe [Tan] and Ma–Röckner [MaR], for example). The aim of this article is to extend the theory in two directions and apply the new criteria to differential operators. Our first result shows that the condition of closability can be omitted completely. To be more precise, consider a sesquilinear form

$$a\colon D(a) \times D(a) \to \mathbf{C}$$

where $D(a)$ is a dense subspace of a Hilbert space H. The form a is called **sectorial** if there exist a (closed) sector

$$\Sigma_\theta = \{r\, e^{-\alpha} : r \geq 0,\ |\alpha| \leq \theta\}$$

with $\theta \in [0, \frac{\pi}{2})$, and $\gamma \in \mathbf{R}$, such that $a(u) - \gamma \|u\|_H^2 \in \Sigma_\theta$ for all $u \in D(a)$, where $a(u) = a(u,u)$. We shall show that one can define an operator A in H associated with a as follows. Let $x, f \in H$. Then $x \in D(A)$ and $Ax = f$ by definition if and only if there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that $(\operatorname{Re} a(u_n))_n$ is bounded, $\lim_{n\to\infty} u_n = x$ in H and $\lim_{n\to\infty} a(u_n, v) = (f, v)_H$ for all $v \in D(a)$. It is part of the following theorem that f is independent of the sequence $u_1, u_2, \ldots$.

Theorem 1.1 (Incomplete case) *The operator A is well defined and $-A$ generates a holomorphic C_0-semigroup e^{-tA} on the interior of $\Sigma_{\frac{\pi}{2}-\theta}$.*

This is a special case of Theorem 3.2 below, but we give a short proof already in Section 2. Recall that the form a is called **closable** if for every Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that $\lim_{n\to\infty} u_n = 0$ in H one has $\lim_{n\to\infty} a(u_n) = 0$. Here $D(a)$ carries the natural norm $\|u\|_a = (\operatorname{Re} a(u) + (1-\gamma)\, \|u\|_H^2)^{1/2}$. In Theorem 1.1 we do not assume that a is closable. Nonetheless, if $u_1, u_2, \ldots$ is a bounded sequence in $D(a)$ such that $\lim_{n\to\infty} u_n = 0$ in H and in addition there exists an $f \in H$ such that for all $v \in D(a)$ one has the limit $\lim_{n\to\infty} a(u_n, v) = (f, v)_H$ then necessarily $f = 0$. This is precisely the fact that A is well defined.

For our second extension of the theorem we consider the **complete case**, where the form a is defined on a Hilbert space V. However, we do not assume that V is embedded in H, but merely that there exists a not necessarily injective operator j from V into H. This case is actually the first we consider in Section 2. It is used for the proof of Theorem 1.1 given in Section 2. In Section 3 we give a common extension of both Theorems 1.1 and the main theorem of Section 2. It turns out that many examples can be treated by our extended form method and Section 4 is devoted to several applications. Our most substantial results concern degenerate elliptic differential operators of second order with complex measurable coefficients on an open set Ω in $\mathbf{R}^d$. If the coefficients satisfy merely a sectoriality condition (which can be very degenerate including the case where the coefficients are zero on some part of Ω), then Theorem 1.1 shows right away that the corresponding operator generates a holomorphic C_0-semigroup on $L_2(\Omega)$. We are able to give quite precise properties of the associated operator and semigroup. In particular we prove a Davies–Gaffney type estimate which gives us locality properties and in case of Neumann boundary conditions and real coefficients, the invariance of the constant functions. This extends results for positive symmetric forms on $\mathbf{R}^d$ in [ERSZ2] and [ERSZ1]. We also extend the criteria for closed convex sets due to Ouhabaz [Ouh] to our more general situation and show that the semigroup is submarkovian if the coefficients are real (but possibly non-symmetric). As a second application, we present an easy and direct treatment of the Dirichlet-to-Neumann operator on a Lipschitz domain Ω. Here it is essential to allow non injective $j: D(a) \to H$. As a result, we obtain submarkovian semigroups on $L_p(\partial\Omega)$. Most interesting are Robin boundary conditions which we consider in Subsection 4.3 on an open bounded set Ω of $\mathbf{R}^d$ with the $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$. Using Theorem 1.1 we obtain directly a holomorphic semigroup on $L_2(\Omega)$. Moreover, for every element in the domain of the generator there is a unique trace in $L_2(\partial\Omega, \sigma)$ realising Robin boundary conditions. Such boundary conditions on rough domains had been considered before by Daners [Dan] and [ArW]. We also give a new simple proof for the existence of a trace for such general domains. We use these results on the trace to consider Wentzell boundary conditions in Subsection 4.5. These boundary conditions obtained much attention recently [FGGR] [VoV]. By our approach we may allow degenerate coefficients for the elliptic operator and the boundary condition. Our final application in Subsection 4.2 concerns multiplicative perturbation of the Laplacian.

Throughout this paper we use the notation and conventions as in [Kat]. Moreover, the field is $\mathbf{C}$, except if indicated explicitly.

2 Generating theorems for complete forms

The first step in the proof of Theorem 1.1 is the following extension of the 'French' approach to closed sectorial forms (see Dautray–Lions [DaL] Chapter XVIIA Example 3, Tanabe [Tan] Sections 2.2 and 3.6, and Lions [Lio]). It is a generation theorem for forms with a complete form domain. It differs from the usual well-known result for closed forms in the following point. We do not assume that the form domain is a subspace of the given Hilbert space, but that there exists a linear mapping j from the form domain into the Hilbert space. Moreover, we do not assume that the mapping is injective. In the injective case, and also in the general case by restricting j to the orthogonal complement of its kernel, we could reduce our result to the usual case. It seems to us simpler to give a direct proof, though, which is adapted from [Tan], Section 3.6, Application 2, treating the usual case.

Let V be a normed space and $a: V \times V \to \mathbf{C}$ a sesquilinear form. Then a is called **continuous** if there exists a $c > 0$ such that

$$|a(u,v)| \le c\,\|u\|_V\,\|v\|_V \qquad (1)$$

for all $u, v \in V$. Let H be a Hilbert space and $j: V \to H$ be a bounded linear operator. The sesquilinear form $a: V \times V \to \mathbf{C}$ is called j-**elliptic** if there exist $\omega \in \mathbf{R}$ and $\mu > 0$ such that

$$\operatorname{Re} a(u) + \omega\,\|j(u)\|_H^2 \ge \mu\,\|u\|_V^2 \qquad (2)$$

for all $u \in V$. The form a is called **coercive** if (2) is valid with $\omega = 0$.

An operator $A: D(A) \to H$ with $D(A) \subset H$ is called **sectorial** if there are $\gamma \in \mathbf{R}$, called a **vertex**, and $\theta \in [0, \frac{\pi}{2})$, called a **semi-angle**, such that

$$(Au,u) - \gamma\,\|u\|_H^2 \in \Sigma_\theta$$

for all $u \in D(A)$. Moreover, A is called m-**sectorial** if it is sectorial and $(\lambda I - A)$ is surjective for some $\lambda \in \mathbf{R}$ with $\lambda < \gamma$. Then an operator A on H is m-sectorial if and only if $-A$ generates a holomorphic C_0-semigroup S with $S_t = e^{-tA}$, which is quasi-contractive on some sector, i.e. there exist $\theta \in (0, \frac{\pi}{2})$ and $\omega \in \mathbf{R}$ such that $\|e^{-\omega z} S_z\|_{\mathcal{L}(H)} \leq 1$ for all $z \in \Sigma_\theta$.

A practical theorem is as follows.

Theorem 2.1 *Let H, V be Hilbert spaces and $j: V \to H$ be a bounded linear operator such that $j(V)$ is dense in H. Let $a: V \times V \to \mathbf{C}$ be a continuous sesquilinear form which is j-elliptic.*

(a) *There exists a unique operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if*

> *there exists a $u \in V$ such that $j(u) = x$ and $a(u, v) = (f, j(v))_H$ for all $v \in V$.*

(b) *The operator A of Statement* (a) *is m-sectorial.*

We call the operator A in Statement (a) of Theorem 2.1 the **operator associated with** (a, j).

This theorem will be a corollary of the next theorem. In the definition of j-elliptic the assumption is that (2) is valid for all $u \in V$. For a variation of the Dirichlet-to-Neumann operator in Subsection 4.4 this condition is too strong. One only needs (2) to be valid for all u in a suitable subspace $V(a)$ of V which we next introduce. Set

$$D_H(a) = \{u \in V : \text{there exists an } f \in H \text{ such that } a(u, v) = (f, j(v))_H \text{ for all } v \in V\}$$

and

$$V(a) = \{u \in V : a(u, v) = 0 \text{ for all } v \in \ker j\}.$$

Clearly $D_H(a) \subset V(a)$ and $V(a)$ is closed in V.

Theorem 2.2 *Let H, V be Hilbert spaces and $j: V \to H$ be a bounded linear operator such that $j(V)$ is dense in H. Let $a: V \times V \to \mathbf{C}$ be a continuous sesquilinear form. Suppose that there exist $\omega \in \mathbf{R}$ and $\mu > 0$ such that*

$$\operatorname{Re} a(u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2 \tag{3}$$

for all $u \in V(a)$. Then one has the following.

(a) *There exists a unique operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if*

> *there exists a $u \in V$ such that $j(u) = x$ and $a(u, v) = (f, j(v))_H$ for all $v \in V$.*

(b) *The operator A of Statement* (a) *is m-sectorial.*

(c) *The restriction map $j|_{D_H(a)}: D_H(a) \to H$ is injective.*

Again we call the operator A in Statement (a) of Theorem 2.2 the **operator associated with** (a,j).

Proof The proof consists of several steps.

Step 1 First, we prove that the restriction map $j|_{D_H(a)} \colon D_H(a) \to H$ is injective. Let $u \in D_H(a)$ and suppose that $j(u) = 0$. Let $f \in H$ be such that $a(u,v) = (f, j(v))_H$ for all $v \in V$. Then by (3) one deduces that

$$\mu \, \|u\|_V^2 \leq \operatorname{Re} a(u) + \omega \, \|j(u)\|_H^2 = \operatorname{Re}(f, j(u))_H + \omega \, \|j(u)\|_H^2 = 0.$$

So $\|u\|_V = 0$ and $u = 0$.

Step 2 Next we prove Statement (a). If $u \in V$ then it follows from the density of $j(V)$ in H that there exists at most one $f \in H$ such that $a(u,v) = (f, j(v))_H$ for all $v \in V$. But $j|D_H(a)$ is injective. Therefore we can define the operator A by $D(A) = j(D_H(a))$ and

$$a(u,v) = (Aj(u), j(v))_H \qquad \text{for all } u \in D_H(a) \text{ and } v \in V. \tag{4}$$

(We emphasize that (4) is restricted to $u \in D_H(a)$ and need not to be valid for all $u \in V$ with $j(u) \in D(A)$. An example will be given in Example 3.14.)

Step 3 We shall prove that $-A$ generates a holomorphic C_0-semigroup. Let $\lambda \in \mathbf{C}$, $u \in D_H(a)$ and set $f = (\lambda I + A) j(u)$. Then by (3)

$$\begin{aligned} \mu \, \|u\|_V^2 &\leq \operatorname{Re} a(u) + \omega \, \|j(u)\|_H^2 \\ &= \operatorname{Re}(Aj(u), j(u))_H + \omega \|j(u)\|_H^2 \\ &= \operatorname{Re}(f, j(u))_H + (\omega - \operatorname{Re} \lambda) \|j(u)\|_H^2 \\ &\leq \|f\|_H \, \|j(u)\|_H \end{aligned}$$

if $\operatorname{Re} \lambda \geq \omega$. Moreover,

$$\lambda \, \|j(u)\|_H^2 + a(u,u) = (f, j(u))_H.$$

So

$$\begin{aligned} |\lambda| \, \|j(u)\|_H^2 &\leq |(f, j(u))_H| + |a(u,u)| \\ &\leq \|f\|_H \, \|j(u)\|_H + c \, \|u\|_V^2 \\ &\leq \Big(1 + \frac{c}{\mu}\Big) \|f\|_H \, \|j(u)\|_H, \end{aligned}$$

where $c > 0$ is the constant as in (1). Therefore $|\lambda| \, \|j(u)\|_H \leq (1 + \frac{c}{\mu}) \|f\|_H$ and $\lambda I + A$ is injective if in addition $\lambda \neq 0$. We claim that the range of $\lambda I + A$ equals H if $\operatorname{Re} \lambda > \omega$. Let $f \in H$. Then the form b on V with $b(u,v) = a(u,v) + \lambda \, (j(u), j(v))_H$ is continuous and coercive. Hence by the Lax–Milgram theorem there exists a unique $u \in V$ such that $b(u,v) = (f, j(v))_H$ for all $v \in V$. Therefore $j(u) \in D(A)$ and $(\lambda I + A) j(u) = f$. So $\lambda I + A$ is invertible and $|\lambda| \, \|(\lambda I + A)^{-1} f\|_H \leq (1 + \frac{c}{\mu}) \|f\|_H$ for all $f \in H$. Then $D(A)$ is dense in H by [ABHN] Proposition 3.3.8. Thus $-A$ generates a holomorphic semigroup on H. This proves Theorem 2.2 and then also Theorem 2.1. □

Although Theorem 1.1 is a special case of Theorem 3.2, a short direct proof can be given at this stage.

Proof of Theorem 1.1 Denote by V the completion of $(D(a), \|\cdot\|_a)$. The injection of $(D(a), \|\cdot\|_a)$ into H is continuous. Hence there exists a $j \in \mathcal{L}(V, H)$ such that $j(u) = u$ for all $u \in D(a)$. Since a is sectorial, there exists a unique continuous extension $\tilde{a} \colon V \times V \to \mathbf{C}$. This extension is continuous and j-elliptic. Let A be the operator associated with $(\tilde{a}, j)$. If $u_1, u_2, \ldots \in D(a)$ with $\lim_{n\to\infty} u_n$ convergent in H and $\operatorname{Re} a(u_1), \operatorname{Re} a(u_2), \ldots$ bounded, then $u_1, u_2, \ldots$ is bounded in $D(a)$. Therefore it has a weakly convergent subsequence in V. It follows from the density of $D(a)$ in V that A equals the operator from Theorem 1.1. In particular, the operator is well defined. Now the result follows from Theorem 2.1. □

We emphasize that in the Theorem 1.1 we do not assume that the form a is closable.

We return to the situation of Theorem 2.2. One might wonder whether the estimate (3) valid for all $u \in V(a)$ in Theorem 2.2 can be weakened by a condition

$$\operatorname{Re} a(u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2$$

valid for all $u \in W$, where W is a subspace of V such that $V = W \oplus \ker j$. The next example shows that this is not possible.

Example 2.3 Let H be an infinite dimensional Hilbert space and let T be an unbounded self-adjoint operator in H with $T \geq I$. Let $V = D(T) \times D(T)$ with the graph norm $(u_1, u_2) \mapsto (\|u_1\|_2^2 + \|Tu_1\|_2^2 + \|u_2\|_2^2 + \|Tu_2\|_2^2)^{1/2}$. Define $j \colon V \to H$ by $j(u_1, u_2) = u_1$. Then $j(V) = D(T)$ is dense in H. Fix $\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{0\}$. Define the sesquilinear form $a \colon V \times V \to \mathbf{C}$ by

$$a(u, v) = \lambda_1 (Tu_1, Tv_1)_H + \lambda_2 (Tu_2, Tv_2)_H.$$

Then $V(a) = D(T) \times \{0\}$ and the restriction of j to $V(a)$ is injective. Arguing as in Step 2 of the proof of Theorem 2.2 it follows that one can define in a unique manner an operator A associated with (a, j).

Let $u = (u_1, u_2) \in D_H(a)$, set $x = j(u)$ and $f = Ax$. Then $a(u, v) = (f, j(v))_H = (f, v_1)_H$ for all $v \in V$. Let $e \in D(T)$. If $v = (e, 0)$ one deduces that $\lambda_1 (Tu_1, Te)_H = (f, e)_H$. So $u_1 \in D(T^2)$ and $f = \lambda_1 T^2 u_1$. Moreover, if $v = (0, e)$ then $\lambda_2 (Tu_2, Te)_H = 0$, so $u_2 = 0$. Therefore $D_H(a) = D(T^2) \times \{0\}$ and $Au_1 = \lambda_1 T^2 u_1$. We have proved that $A = \lambda_1 T^2$. It follows that A is m-sectorial if and only if $\lambda_1 > 0$.

Next let $W = \{(u_1, u_2) \in V : u_1 = u_2\}$ and choose $\lambda_1 = -1$ and $\lambda_2 = 3$. Then $V = W \oplus \ker j$ and $\operatorname{Re} a(u) + 2\|j(u)\|_H^2 = \|u\|_V^2$ for all $u \in W$. But the operator $-A$ does not generate a holomorphic semigroup. □

If the form a is j-elliptic and if $\tau \in \mathbf{C}$ then obviously the operator $A + \tau I$ is associated with (b, j), where b is the j-elliptic form $b(u, v) = a(u, v) + \tau \, (j(u), j(v))_H$ on V.

Although it is very convenient that we do not assume that the operator j is injective, the first statement in the next proposition shows that in general one might assume that j is injective, by considering a different form. The proposition is a kind of uniqueness result. It determines the dependence of the operator on the choice of V

Proposition 2.4 *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Then one has the following.*

(a) *$V(a) = \overline{D_H(a)}$, where the closure is taken in V. Moreover, $j|_{V(a)}$ is injective and A equals the operator associated with $(a|_{V(a)\times V(a)}, j|_{V(a)})$.*

(b) *If U is a closed subspace of V such that $D_H(a) \subset U$, then A equals the operator associated with $(a|_{U\times U}, j|_U)$. If, in addition, the restriction $j|_U$ is injective, then $U = \overline{D_H(a)}$.*

(c) *If U is a closed subspace of $V(a)$ such that $j(U)$ is dense in H and A is the operator associated with $(a|_{U\times U}, j|_U)$, then $U = V(a)$.*

Proof Clearly $V(a)$ is closed. If $u \in V(a)$ and $j(u) = 0$ then $a(u) = 0$. The j-ellipticity of a then implies that $u = 0$. So $j|_{V(a)}$ is injective. Moreover, $D_H(a) \subset V(a)$. Then the rest of Statement (a) follows from Statement (b).

Proof of Statement (b). Note that $j(U)$ and $j(V(a))$ both contain $j(D_H(a)) = D(A)$. Therefore $j(U)$ and $j(V(a))$ are dense in H. Let $b_1 = a|_{U\times U}$ and $b_2 = a|_{V(a)\times V(a)}$. Moreover, let B_1 and B_2 be the operators associated with $(b_1, j|_U)$ and $(b_2, j|_{V(a)})$. Then for all $u \in D_H(a)$ one deduces that $(Aj(u), j(v))_H = a(u, v) = b_1(u, v)$ for all $v \in U$. Therefore $u \in D_H(b_1)$ and $B_1 j(u) = Aj(u)$. So $A \subset B_1$. But both $-A$ and $-B_1$ are semigroup generators. Therefore $B_1 = A$. Similarly, $A = B_2$. Finally, if j is injective on U then it follows from the inclusion $V(a) \subset U$ and the uniqueness theorem for closed sectorial forms, [Kat] Theorem VI.2.7 that $U = V(a)$. This proves Statement (b).

Statement (c) follows from Statement (b) with a replaced by $a_{U\times U}$. □

It is easy to construct examples with $V(a) \neq V$. Therefore the injectivity condition in Proposition 2.4(b) is necessary.

The next theorem bridges the current operators associated with (a, j) and the closed sectorial forms in Kato [Kat] Section VI.2.

Theorem 2.5 *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Then the following holds.*

(a) $\ker j \oplus V(a) = V$.

(b) *Let a_c be the form on H defined by*

$$D(a_c) = j(V) \text{ and } a_c(j(u), j(v)) = a(u, v) \quad (u, v \in V(a)).$$

Then a_c is the unique closed, sectorial form such that A is associated with a_c.

Proof '(a)'. Let $\omega \in \mathbf{R}$ and $\mu > 0$ be as in (2). Define the sesquilinear form b with $D(b) = V$ by $b(u, v) = a(u, v) + (\omega + 1)\,(j(u), j(v))_H$. Then $V(a) = V(b)$. So we can assume that $\omega = -1$, and the form a is coercive. Denoting the real part of a by h, then $\langle u, v\rangle := h(u, v)$ defines an equivalent scalar product on V. So we may assume that $\|u\|_V = \|u\|_h$ for all $u \in V$. Let $V_1 = \ker j$ and $V_2 = (\ker j)^\perp$. Moreover, let π_1 and π_2 be the projection from V onto V_1 and V_2, respectively. Then $h(u_1, v_2) = 0$ for all $u_1 \in V_1$ and $v_2 \in V_2$. There exists an invertible operator $B \in \mathcal{L}(V)$ such that $a(u, v) = h(Bu, v)$ for all $u, v \in V$. Let $B_{11} = \pi_1 \circ B|_{V_1} \in \mathcal{L}(V_1)$ and $B_{12} = \pi_1 \circ B|_{V_2} \in \mathcal{L}(V_2, V_1)$. If $(u_1, u_2) \in V_1 \times V_2$ then $u_1 + u_2 \in V(a)$ if and only if $0 = h(Bu, v_1) = h((B_{11}u_1 + B_{12}u_2), v_1)$ for all $v_1 \in V_1$. So

$$V(a) = \{u_1 + u_2 : (u_1, u_2) \in V_1 \times V_2 \text{ and } B_{11}u_1 + B_{12}u_2 = 0\}.$$

Since

$$\operatorname{Re} h(B_{11}u_1, u_1) = \operatorname{Re} h(\pi_1(Bu_1), u_1) = \operatorname{Re} h(Bu_1, u_1)$$
$$= \operatorname{Re} a(u_1, u_1) \geq \mu \|u\|_V^2 = \mu (\operatorname{Re} h(u_1) + \|u_1\|_H^2)$$

for all $u_1 \in V_1$, it follows from the Lax–Milgram theorem that also B_{11} is invertible. Thus for all $u_2 \in V_2$ there exists a $u_1 \in V_1$ such that $u_1 + u_2 \in V(a)$. Consequently, $j(V(a)) = j(V_2) = j(V)$. This implies that $\ker j + V(a) = V$. This sum is direct by Proposition 2.4(a).

'(b)'. Define on $j(V(a))$ the scalar product carried over from $V(a)$ by j. Then the form a_c is clearly continuous and elliptic, which is the same as sectorial and closed (cf. Lemma 3.1). The operator A is clearly the operator associated with a_c. □

In the sequel we call the form a_c in Theorem 2.5 the **classical form** associated with (a, j). It equals the classical form associated with the m-sectorial form A. One can decompose the form $a = h + ik$ in its real and imaginary parts, where $h, k \colon D(a) \times D(a) \to \mathbf{C}$ are symmetric sesquilinear forms. We write $\Re a = h$ and $\Im a = k$. The proof of Theorem 2.5 also allows to estimate the real part of the classical form of a by the classical form of the real part of a.

Proposition 2.6 *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Suppose $\omega \leq -1$ in (2). Let h be the real part of a and h_c the classical form associated with (h, j). Then $D(a_c) = D(h_c)$. Moreover, there exists a constant $C > 0$ such that $\operatorname{Re} a_c(x) \leq C\, h_c(x)$ for all $x \in j(V)$.*

Proof The first statement is obvious since $D(a_c) = j(V) = D(h_c)$. We use the notation introduced in the proof of Theorem 2.5. Moreover, we may assume that the inner product on V is given by $(u, v) \mapsto h(u, v)$. Let $u \in V(a)$. Then $B_{11}u_1 + B_{12}u_2 = 0$, where $u_1 = \pi_1(u)$ and $u_2 = \pi_2(u)$. So $u_1 = -B_{11}^{-1}B_{12}u_2$. Moreover, $j(u) = u_2 = j(u_2)$ and $u_2 \in V(h)$. So $a_c(j(u)) = a(u)$ and $h_c(j(u)) = h_c(j(u_2)) = h(u_2) = \|u_2\|_V^2$. Since the operators B, B_{11}^{-1} and B_{12} are bounded one estimates

$$\operatorname{Re} a_c(j(u)) = \operatorname{Re} a(u) = \operatorname{Re} h(Bu, u) = \operatorname{Re}(Bu, u)_V \leq \|Bu\|_V \|u\|_V \leq \|B\| \|u\|_V^2$$
$$= \|B\| (\|u_1\|_V^2 + \|u_2\|_V^2) \leq \|B\| (\|B_{11}^{-1}\|_2^2 \|B_{12}\|^2 + 1) \|u_2\|_V^2 = C\, h_c(j(u))$$

where $C = \|B\| (\|B_{11}^{-1}\|_2^2 \|B_{12}\|^2 + 1)$. □

The next lemma gives a sufficient condition for the resolvents to be compact.

Lemma 2.7 *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). If j is compact then $(\lambda I + A)^{-1}$ is compact for all $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > \omega$, where ω is as in (2).*

Proof By the Lax–Milgram theorem there exists a $B \in \mathcal{L}(H, V)$ such that

$$(f, j(v))_H = a(Bf, v) + \lambda\, (j(Bf), j(v))_H$$

for all $f \in H$ and $v \in V$. Then $B(H) \subset D_H(a)$ and $(A + \lambda I) j(Bf) = f$ for all $f \in H$. Therefore $(\lambda I + A)^{-1} = j \circ B$ is compact. □

Remark 2.8 If B is the operator associated with (a^*, j) where a^* is the j-elliptic form on V given by $a^*(u, v) = \overline{a(v, u)}$, then A^* is an extension of B. But both $-A^*$ and $-B$ are generators of semigroups. Therefore A^* is the operator associated with (a^*, j).

In [Ouh] Theorem 2.2 there is a characterization of closed convex subsets which are invariant under the semigroup S. Using the two statements of Theorem 2.5, the theorem of Ouhabaz can be reformulated in the current context. Recall that a sesquilinear form b is called **accretive** if $\mathrm{Re}\, b(u) \geq 0$ for all $u \in D(b)$.

Proposition 2.9 *Suppose the form a is j-elliptic, let A be the operator associated with (a, j) and S the semigroup generated by $-A$. Moreover, suppose that a is accretive. Let $C \subset H$ be a closed convex set and let $P: H \to C$ be the orthogonal projection. Then the following conditions are equivalent.*

(i) *$S_t C \subset C$ for all $t > 0$.*

(ii) *For all $u \in V$ there exists a $w \in V$ such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \mathrm{Re}\, a(w, u - w) \geq 0.$$

(iii) *For all $u \in V$ there exists a $w \in V$ such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \mathrm{Re}\, a(u, u - w) \geq 0.$$

(iv) *There exists a dense subset D of V such that for all $u \in D$ there exists a $w \in V$ such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \mathrm{Re}\, a(w, u - w) \geq 0.$$

Proof '(i)⇒(ii)'. Let $u \in V$. By Theorem 2.5 there exists a $u' \in V(a)$ such that $j(u') = j(u)$. Then $Pj(u') \in D(a_c)$ by [Ouh] Theorem 2.2 1)⇒2). So there exists a $w \in V(a)$ such that $Pj(u') = j(w)$. Then $\mathrm{Re}\, a(w, u' - w) = \mathrm{Re}\, a_c(j(w), j(u') - j(w)) = \mathrm{Re}\, a_c(Pj(u'), j(u') - Pj(u')) \geq 0$ again by [Ouh] Theorem 2.2 1)⇒2). But $a(w, u - u') = 0$ since $w \in V(a)$ and $u - u' \in \ker j$. So $\mathrm{Re}\, a(w, u - w) \geq 0$.

'(ii)⇒(iii)'. Trivial, since $\mathrm{Re}\, a(u - w, u - w) \geq 0$.

'(iii)⇒(i)'. Let $u \in V(a)$. By assumption there exists a $w \in V$ such that $Pj(u) = j(w)$ and $\mathrm{Re}\, a(u, u - w) \geq 0$. Let $w' \in V(a)$ be such that $j(w) = j(w')$. Then $a(u, w - w') = 0$ since $u \in V(a)$ and $w - w' \in \ker j$. So $\mathrm{Re}\, a(u, u - w') \geq 0$ and $\mathrm{Re}\, a_c(j(u), j(u) - Pj(u)) \geq 0$. Then the implication follows from [Ouh] Theorem 2.2 3)⇒1).

'(ii)⇒(iv)'. Trivial.

'(iv)⇒(ii)'. Since a is continuous there exists a $c > 0$ such that $|a(u, v)| \leq c\, \|u\|_V\, \|v\|_V$ for all $u, v \in V$. Let $u \in V$. There exists $u_1, u_2, \ldots \in D$ such that $\lim u_n = u$ in V. For all $n \in \mathbf{N}$ there exists by assumption a $w_n \in V$ such that $Pj(u_n) = j(w_n)$ and $\mathrm{Re}\, a(w_n, u_n - w_n) \geq 0$. Let μ and ω be as in (2). Then

$$\begin{aligned}
\mu\, \|w_n\|_V^2 &\leq \mathrm{Re}\, a(w_n) + \omega\, \|j(w_n)\|_H^2 \\
&= \mathrm{Re}\, a(w_n, u_n) - \mathrm{Re}\, a(w_n, u_n - w_n) + \omega\, \|j(w_n)\|_H^2 \\
&\leq \mathrm{Re}\, a(w_n, u_n) + \omega\, \|j(w_n)\|_H^2 \\
&\leq c\, \|w_n\|_V\, \|u_n\|_V + \omega\, \|Pj(u_n)\|_H^2
\end{aligned}$$

for all $n \in \mathbf{N}$. Since $\{u_n : n \in \mathbf{N}\}$ is bounded in V and $\{Pj(u_n) : n \in \mathbf{N}\}$ is bounded in H by continuity of j and P, it follows that the set $\{w_n : n \in \mathbf{N}\}$ is bounded in V. So there exist $w \in V$ and a subsequence $w_{n_1}, w_{n_2}, \ldots$ of $w_1, w_2, \ldots$ such that $\lim_{k\to\infty} w_{n_k} = w$ weakly in V. Then $\lim_{k\to\infty} Pj(u_{n_k}) = \lim j(w_{n_k}) = j(w)$ weakly in H. Since C is closed and convex it follows that $j(w) \in C$. Alternatively, the continuity of j and P gives $\lim_{n\to\infty} Pj(u_n) = Pj(u)$ strongly in H. So $Pj(u) = j(w)$. Since $\operatorname{Re} a(w_n, u_n - w_n) \geq 0$ one has $\operatorname{Re} a(w_n) \leq \operatorname{Re} a(w_n, u_n)$ for all $n \in \mathbf{N}$. Moreover, $\lim_{k\to\infty} \operatorname{Re} a(w_{n_k}, u_{n_k}) = \operatorname{Re} a(w, u)$. In addition, since a is accretive and j-elliptic it follows that $v \mapsto (\operatorname{Re} a(v) + \varepsilon \, \|j(v)\|_H^2)^{1/2}$ is an equivalent norm associated with an inner product on V for all $\varepsilon > 0$. Therefore $\operatorname{Re} a(w) \leq \liminf_{k\to\infty} \operatorname{Re} a(w_{n_k})$. So $\operatorname{Re} a(w) \leq \operatorname{Re} a(w, u)$ and $\operatorname{Re} a(w, u - w) \geq 0$ as required. □

3 Generating theorems in the incomplete case

First we reformulate the complete case.

Let $a \colon D(a) \times D(a) \to \mathbf{C}$ be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. We say that a is a **j-sectorial form** if there are $\gamma \in \mathbf{R}$, called a **vertex**, and $\theta \in [0, \frac{\pi}{2})$, called a **semi-angle**, such that

$$a(u) - \gamma \, \|j(u)\|_H^2 \in \Sigma_\theta$$

for all $u \in D(a)$. If a is j-sectorial with vertex γ then we define a seminorm $\| \cdot \|_a$ on $D(a)$ by

$$\|u\|_a^2 = \operatorname{Re} a(u) + (1 - \gamma) \, \|j(u)\|_H^2. \tag{5}$$

Again we do not include the γ in the notation. Then $\| \cdot \|_a$ is a norm if and only if $\operatorname{Re} a(u) = j(u) = 0$ implies $u = 0$ for all $u \in D(a)$. A j-sectorial form a is called **closed** if $\| \cdot \|_a$ is a norm and $(D(a), \| \cdot \|_a)$ is a Hilbert space.

The alluded reformulation is as follows.

Lemma 3.1 *Let V be a vector space, $a \colon V \times V \to \mathbf{C}$ a sesquilinear form, H a Hilbert space and $j \colon V \to H$ a linear map. Then the following are equivalent.*

(i) *The form a is j-sectorial and closed.*

(ii) *There exists a norm $\| \cdot \|_V$ on V such that V is a Banach space, the map j is bounded from $(V, \| \cdot \|_V)$ into H, the form a is j-elliptic and a is continuous.*

Moreover, if Condition (ii) *is valid, then the norms $\| \cdot \|_a$ and $\| \cdot \|_V$ are equivalent.*

Proof The easy proof is left to the reader. □

In this section we drop the assumption that $(D(a), \| \cdot \|_a)$ is closed. So H is a Hilbert space, $a \colon D(a) \times D(a) \to \mathbf{C}$ is a sesquilinear form, $j \colon D(a) \to H$ is a linear map and we assume that a is merely j-sectorial and $j(D(a))$ is dense in H. We will again associate a sectorially bounded holomorphic semigroup generator on H. The next theorem is an extension of Theorem 1.1. The construction in the proof might seem to be long, but each step is totally natural. Note that if j is injective, then the quotient map in the construction is superfluous.

Theorem 3.2 *Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Assume that a is j-sectorial and $j(D(a))$ is dense in H. Then one has the following.*

(a) *There exists a unique operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that*
  - (I) $\lim_{n\to\infty} j(u_n) = x$ *in H,*
  - (II) $\sup_{n\in\mathbf{N}} \operatorname{Re} a(u_n) < \infty$, *and,*
  - (III) $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ *for all $v \in D(a)$.*

(b) *The operator A of Statement* (a) *is m-sectorial.*

In a natural way one can define the notion of Cauchy sequence in a semi-normed vector space. We will see in the proof of the theorem that for all $x \in D(A)$ one can actually find a Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that (I) and (III) are valid with $f = Ax$.

Proof of Theorem 3.2 Define

$$W = \{u \in D(a) : (u, u)_a = 0\},$$

where $(\,\cdot, \cdot\,)_a$ is the semi-inner product defined by

$$(u, v)_a = (\Re a)(u, v) + (1 - \gamma)\,(j(u), j(v))_H,$$

and γ is as in (5). Note that $\|u\|_a^2 = (u, u)_a$. Then W is a closed subspace of $D(a)$. Set $V_0 = D(a)/W$. If $q \colon D(a) \to V_0$ is the quotient map, then $(\,\cdot, \cdot\,)_{V_0} \colon V_0 \times V_0 \to \mathbf{C}$ given by

$$(q(u), q(v))_{V_0} := (u, v)_a$$

defines an inner product on V_0. We denote by V the completion of V_0 and consider V_0 as a subspace of V. Note that $\|u\|_a = \|q(u)\|_V$ for all $u \in D(a)$. Since $j(u) = 0$ for all $u \in W$ there exists a unique map $j_0 \colon V_0 \to H$ such that

$$j_0(q(u)) = j(u)$$

for all $u \in D(a)$. Then $\|j_0(q(u))\|_H = \|j(u)\|_H \leq \|u\|_a = \|q(u)\|_{V_0}$ for all $u \in D(a)$. Hence there exists a unique contraction $\tilde{j} \in \mathcal{L}(V, H)$ which extends j_0. Next, since

$$|a(u, v) - \gamma\,(j(u), j(v))_H| \leq (1 + \tan\theta)\|u\|_a\,\|v\|_a$$

for all $u, v \in D(a)$, where θ is the semi-angle of a and we used the estimate (1.15) of Subsection VI.1.2 in [Kat], there exists a unique sesquilinear form a_0 on V_0 such that

$$a_0(q(u), q(v)) = a(u, v)$$

for all $u, v \in D(a)$. Then

$$|a_0(q(u), q(v)) - \gamma\,(j_0(q(u)), j_0(q(v)))_H| \leq (1 + \tan\theta)\|q(u)\|_{V_0}\,\|q(v)\|_{V_0}$$

for all $u, v \in D(a)$. Hence a_0 is continuous with respect to $\|\cdot\|_{V_0}$. Therefore a_0 has a unique continuous extension $\tilde{a} \colon V \times V \to \mathbf{C}$ which is $\tilde{j}$-sectorial. Moreover, if $u \in D(a)$ then

$$\operatorname{Re} a_0(q(u)) + (1 - \gamma)\|\tilde{j}(q(u))\|_H^2 = \operatorname{Re} a(u) + (1 - \gamma)\|j(u)\|_H^2 = \|u\|_a^2 = \|q(u)\|_V^2.$$

By density, this implies that $\tilde{a}$ is $\tilde{j}$-elliptic. Now let A be the operator associated with $(\tilde{a}, \tilde{j})$.

Let $x, f \in H$. We next show that the statements

(i) $x \in D(A)$ and $Ax = f$,

(ii) there exists a Cauchy sequence $u_1, u_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim j(u_n) = x$ and $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ for all $v \in D(a)$, and

(iii) there exists a bounded sequence $u_1, u_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim j(u_n) = x$ and $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ for all $v \in D(a)$

are equivalent.

'(i)⇒(ii)'. It follows from the definition (4) that there exists a $\tilde{u} \in V$ such that $\tilde{j}(\tilde{u}) = x$ and $\tilde{a}(\tilde{u}, \tilde{v}) = (f, \tilde{j}(\tilde{v}))_H$ for all $\tilde{v} \in V$. Then there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that $\lim q(u_n) = \tilde{u}$ in V. Hence $u_1, u_2, \ldots$ is a Cauchy sequence in $(D(a), \|\cdot\|_a)$. Moreover,

$$(f, j(v))_H = (f, \tilde{j}(q(v)))_H = \tilde{a}(\tilde{u}, q(v)) = \lim \tilde{a}(q(u_n), q(v)) = \lim a(u_n, v)$$

for all $v \in D(a)$ and $\lim j(u_n) = \lim \tilde{j}(q(u_n)) = \tilde{j}(\tilde{u}) = x$ in H.

'(ii)⇒(iii)'. Trivial.

'(iii)⇒(i)'. Since $q(u_1), q(u_2), \ldots$ is a bounded sequence in V_0 the weak limit $\tilde{u} = \lim q(u_n)$ exists in V after passing to a subsequence, if necessary. Then $\tilde{j}(\tilde{u}) = \lim \tilde{j}(q(u_n)) = \lim j(u_n) = x$ weakly in H. Moreover,

$$\tilde{a}(\tilde{u}, q(v)) = \lim \tilde{a}(q(u_n), q(v)) = \lim a(u_n, v) = (f, j(v))_H = (f, \tilde{j}(q(v)))_H$$

for all $v \in D(a)$. Since $q(D(a))$ is dense in V one deduces that $\tilde{a}(\tilde{u}, \tilde{v}) = (f, \tilde{j}(\tilde{v}))_H$ for all $\tilde{v} \in V$. So $x \in D(A)$ and $Ax = f$ as required.

We have proved the existence of the operator A in Statement (a) of the theorem. The uniqueness is easy, since $j(D(a))$ is dense in H. Now Statement (b) follows from Theorem 2.2. □

We call the operator A in Statement (a) of Theorem 3.2 the **operator associated with** (a, j). Note that there is no confusion if $D(a)$ was provided with a Hilbert space structure such that j is continuous, a is continuous and a is j-elliptic.

Remark 3.3 Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Suppose that a is j-sectorial. Let D be **core of** $D(a)$, i.e. a dense subspace of $D(a)$. Then $j(D)$ is dense in H and the operator associated with (a, j) equals the operator associated with $(a|_{D\times D}, j|_D)$. This follows immediately from the Cauchy-type characterization in Theorem 3.2(a).

Remark 3.4 Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Assume that a is j-sectorial and $j(D(a))$ is dense in H. Then a^* is j-sectorial. Moreover, if B is the operator associated with (a^*, j) and A is the operator associated with (a, j), then $B = A^*$. In fact, using the notation as in the proof of Theorem 3.2 it follows that A is the operator associated with $(\tilde{a}, \tilde{j})$. Starting with a^* one has $(u, v)_{a^*} = (u, v)_a$ for all $u, v \in D(a) = D(a^*)$. Therefore one obtains the same space W, map j_0, inner product space V_0 and completion V. But $\widetilde{a^*} = (\tilde{a})^*$. So by construction the operator B is associated with $(\widetilde{a^*}, \tilde{j}) = ((\tilde{a})^*, \tilde{j})$. Hence $B = A^*$ by Remark 2.8. In particular, if a is symmetric then A is self-adjoint.

Remark 3.5 It follows from the construction that the operator $\lambda I + A$ is invertible for all $\lambda > (-\gamma) \vee 0$ if A is the operator associated with a j-sectorial form a with vertex γ.

Example 3.6 Let a be a sesquilinear form with domain $D(a)$. Suppose there exists a $\theta \in [0, \frac{\pi}{2})$ such that $a(u) \in \Sigma_\theta$ for all $u \in D(a)$. Let H and H' be Hilbert spaces and let $M \in \mathcal{L}(H, H')$ be invertible. Let $j: D(a) \to H$ be linear and set $j' = M \circ j$. Then a is both j-sectorial and j'-sectorial. Moreover, the seminorms $u \mapsto \sqrt{\operatorname{Re} a(u) + \|j(u)\|_H^2}$ and $u \mapsto \sqrt{\operatorname{Re} a(u) + \|j'(u)\|_{H'}^2}$ on $D(a)$ are equivalent, so they determine the same Cauchy sequences. Suppose $j(D(a))$ is dense in H. Let A and A' be the operators associated with (a, j) and (a, j'), respectively. Then $A = M^* A' M$.

Indeed, let $x \in D(A)$. Set $f = Ax$. Then there exists a Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that $\lim j(u_n) = x$ in H and $\lim a(u_n, v) = (f, j(v))_H$ for all $v \in D(a)$. Then $\lim j'(u_n) = Mx$ in H' and $\lim a(u_n, v) = (f, j(v))_H = (f, M^{-1} j'(v))_H = ((M^{-1})^* f, j'(v))_{H'}$ for all $v \in D(a)$. So $Mx \in D(A')$ and $A'Mx = (M^{-1})^* f$. Therefore $M^* A' Mx = Ax$ and $A \subset M^* A' M$. Similarly $A' \subset (M^{-1})^* A M^{-1}$. Hence $A = M^* A' M$. □

The next theorem is of the nature of [Kat] Theorem VIII.3.6. If $F_1, F_2, \ldots$ are subsets of a set F then define $\liminf_{n\to\infty} F_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty F_k$.

Theorem 3.7 *Let a be a sesquilinear form, H a Hilbert space and $j: D(a) \to H$ a linear map. Assume that a is j-sectorial with vertex γ. For all $n \in \mathbf{N}$ let a_n be a sesquilinear form with $D(a_n) \subset D(a)$. Suppose that there exist $\theta \in [0, \frac{\pi}{2})$ and for all $n \in \mathbf{N}$ a $\gamma_n \in \mathbf{R}$ such that*

$$a_n(u) - a(u) - \gamma_n \|j(u)\|_H^2 \in \Sigma_\theta \tag{6}$$

for all $u \in D(a_n)$. Assume that $\lim_{n\to\infty} \gamma_n = 0$. Moreover, suppose that there exists a core D of $D(a)$ such that $D \subset \liminf_{n\to\infty} D(a_n)$ and $\lim_{n\to\infty} \operatorname{Re} a_n(u) = \operatorname{Re} a(u)$ for all $u \in D$. Finally, suppose that $j(D(a_n))$ is dense in H for all $n \in \mathbf{N}$. Let A be the operator associated with (a, j) and for all $n \in \mathbf{N}$ let A_n be the operator associated with $(a_n, j|_{D(a_n)})$. Fix $\lambda > (-\gamma) \vee 0$. Then

$$\lim_{n\to\infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f$$

for all $f \in H$.

Proof Without loss of generality we may assume that $\gamma = 0$. Then a_n is j-sectorial with vertex γ_n and $D(a_n)$ has the norm $\|u\|_{a_n}^2 = \operatorname{Re} a_n(u) + (1 - \gamma_n) \|j(u)\|_H^2$. We use the construction as in the proof of Theorem 3.2. For the form a we construct W, q, V_0, V, $\tilde{j}$, $\tilde{a}$ and for the form a_n we construct W_n, q_n, V_{n0}, V_n, $\tilde{j}_n$, $\tilde{a}_n$.

Let $n \in \mathbf{N}$. It follows from (6) that $\|u\|_a^2 \leq \|u\|_{a_n}^2$ for all $u \in D(a_n)$. Therefore $W_n \subset W$ and there exists a unique $\Phi_n \in \mathcal{L}(V_{n0}, V)$ such that $\Phi_n(q_n(u)) = q(u)$ for all $u \in D(a_n)$. Then there exists a unique $\widetilde{\Phi}_n \in \mathcal{L}(V_n, V)$ such that $\widetilde{\Phi}_n(q_n(u)) = q(u)$ for all $u \in D(a_n)$. Therefore $\tilde{j}_n(q_n(u)) = j(u) = \tilde{j}(q(u)) = \tilde{j}(\widetilde{\Phi}_n(q_n(u)))$ for all $u \in D(a_n)$ and by density $\tilde{j}_n = \tilde{j} \circ \widetilde{\Phi}_n$. Define the sectorial form $b_n: D(a_n) \times D(a_n) \to \mathbf{C}$ by

$$b_n(u, v) = a_n(u, v) - a(u, v) - \gamma_n (j(u), j(v))_H.$$

Then $|b_n(u)| \leq \|u\|_{a_n}^2$, so there exists a unique continuous accretive sectorial form $\tilde{b}_n: V_n \times V_n \to \mathbf{C}$ such that $\tilde{b}_n(q_n(u), q_n(v)) = b_n(u, v)$ for all $u, v \in D(a_n)$. Then

$$\tilde{a}_n(u, v) = \tilde{a}(\widetilde{\Phi}_n(u), \widetilde{\Phi}_n(v)) + \tilde{b}_n(u, v) + \gamma_n (\tilde{j}(u), \tilde{j}(v))_H \tag{7}$$

first for all $u, v \in q(D(a_n))$ and then by density for all $u, v \in V_n$.

In order not to duplicate too much of the proof for the current theorem for the proof of Theorem 3.8 we first prove a little bit more. Let $f, f_1, f_2, \ldots \in H$ and suppose that $\lim f_n = f$ weakly in H. For all $n \in \mathbf{N}$ there exists a unique $\tilde{u}_n \in D_H(\tilde{a}_n)$ such that $\tilde{j}_n(\tilde{u}_n) = (\lambda I + A_n)^{-1} f_n$. Set $u_n = \widetilde{\Phi}_n(\tilde{u}_n) \in V$. Then $\tilde{j}(u_n) = \tilde{j}_n(\tilde{u}_n)$. We show that there exists a subsequence (u_{n_k}) of (u_n) and a $u \in V$ such that $\lim u_{n_k} = u$ weakly in V and $\tilde{j}(u) = (\lambda I + A)^{-1} f$.

Since $\tilde{u}_n \in D_H(\tilde{a}_n)$ and $\lambda \tilde{j}_n(\tilde{u}_n) + A_n \tilde{j}_n(\tilde{u}_n) = f_n$ it follows from (4) that

$$\lambda \, (\tilde{j}_n(\tilde{u}_n), \tilde{j}_n(v))_H + \tilde{a}_n(\tilde{u}_n, v) = (f_n, \tilde{j}_n(v))_H \tag{8}$$

for all $v \in V_n$. Taking $v = \tilde{u}_n$ in (8) and using (7) we obtain

$$\begin{aligned} \tfrac{2\lambda}{3} \, \|\tilde{j}_n(\tilde{u}_n)\|_H^2 + \operatorname{Re} \tilde{a}(u_n) + \operatorname{Re} \tilde{b}_n(\tilde{u}_n) + (\tfrac{\lambda}{3} + \gamma_n) \, \|\tilde{j}_n(\tilde{u}_n)\|_H^2 & \\ = \operatorname{Re}(f_n, \tilde{j}_n(\tilde{u}_n))_H & \\ \leq \|f_n\|_H \, \|\tilde{j}_n(\tilde{u}_n)\|_H \leq \tfrac{\lambda}{3} \, \|\tilde{j}_n(\tilde{u}_n)\|_H^2 + \tfrac{3}{\lambda} \|f_n\|_H^2. & \end{aligned} \tag{9}$$

Since $\frac{\lambda}{3} + \gamma_n \geq 0$ for large n this implies that the set $\{\tilde{j}(u_n) : n \in \mathbf{N}\} = \{\tilde{j}_n(\tilde{u}_n) : n \in\}$ is bounded in H. Consequently, the two sets $\{\operatorname{Re} \tilde{a}(u_n) : n \in \mathbf{N}\}$ and $\{\operatorname{Re} \tilde{b}_n(\tilde{u}_n) : n \in \mathbf{N}\}$ are bounded. In particular the sequence $u_1, u_2, \ldots$ is bounded in V. Passing to a subsequence, if necessary, it follows that there exists a $u \in V$ such that $\lim u_n = u$ weakly in V. Then $\lim \tilde{j}(u_n) = \tilde{j}(u)$ weakly in H.

Let $n \in \mathbf{N}$. Then $\tilde{b}_n$ is $\tilde{j}_n$-sectorial with vertex 0 and semi-angle θ. Therefore

$$|\tilde{b}_n(\tilde{u}_n, v)| \leq (1 + \tan\theta) \Big(\operatorname{Re} \tilde{b}_n(\tilde{u}_n) \Big)^{1/2} \Big(\operatorname{Re} \tilde{b}_n(\tilde{v}) \Big)^{1/2}$$

for all $v \in V_n$. Now let $v \in D$. Then $\lim_{n\to\infty} \operatorname{Re} b_n(v) = 0$ by assumption. Hence $\lim_{n\to\infty} \tilde{b}_n(\tilde{u}_n, q_n(v)) = 0$. It follows from (7) and (8) that

$$\lambda \, (\tilde{j}(u_n), j(v))_H + \tilde{a}(u_n, q(v)) + \tilde{b}_n(\tilde{u}_n, q_n(v)) + \gamma_n \, (\tilde{j}(u_n), j(v))_H = (f_n, j(v))_H.$$

Taking the limit $n \to \infty$ gives

$$\lambda \, (\tilde{j}(u), j(v))_H + \tilde{a}(u, q(v)) = (f, j(v))_H \tag{10}$$

for all $v \in D$. Since D is a core for $D(a)$ one deduces that (10) is valid for all $v \in D(a)$ and then again by density one establishes that

$$\lambda \, (\tilde{j}(u), \tilde{j}(v))_H + \tilde{a}(u, v) = (f, \tilde{j}(v))_H \tag{11}$$

for all $v \in V$. Thus by definition of A, it follows that $\tilde{j}(u) = (\lambda I + A)^{-1} f$.

Now we prove the theorem. Let $f \in H$ and apply the above with $f_n = f$ for all $n \in \mathbf{N}$. In order to deduce that $\lim \tilde{j}(u_n) = \tilde{j}(u)$ strongly in H, by Proposition 3.6 in [HiL] it suffices to show that $\limsup \|\tilde{j}(u_n)\|_H \leq \|\tilde{j}(u)\|_H$.

Substituting $v = u_n$ in (11) gives

$$\lambda \, (\tilde{j}(u), \tilde{j}(u_n))_H + \tilde{a}(u, u_n) = (f, \tilde{j}(u_n))_H$$

for all $n \in \mathbf{N}$. Hence by (9) one deduces that

$$\begin{aligned}\lambda \|\tilde{j}(u_n)\|_H^2 &\leq \lambda \, \|\tilde{j}_n(\tilde{u}_n)\|_H^2 + \operatorname{Re} \tilde{b}_n(\tilde{u}_n) \\ &= \operatorname{Re} \Big((f, \tilde{j}(\tilde{u}_n))_H - \tilde{a}(u_n) \Big) - \gamma_n \, \|\tilde{j}(u_n)\|_H^2 \\ &= \operatorname{Re} \Big(\lambda \, (\tilde{j}(u), \tilde{j}(u_n))_H + \tilde{a}(u, u_n) - \tilde{a}(u_n) \Big) - \gamma_n \, \|\tilde{j}(u_n)\|_H^2\end{aligned}$$

for all $n \in \mathbf{N}$. But $\operatorname{Re} \tilde{a}(u) \leq \liminf \operatorname{Re} \tilde{a}(u_n)$ by [Kat], Lemma VIII.3.14a. Therefore $\limsup \lambda \, \|\tilde{j}(u_n)\|_H^2 \leq \operatorname{Re} \lambda \, \|\tilde{j}(u)\|_H^2 = \lambda \, \|\tilde{j}(u)\|_H^2$ and the strong convergence follows.

We have shown that there exists a subsequence $n_1, n_2, \ldots$ of the sequence $1, 2, \ldots$ such that $\lim_{k \to \infty} (\lambda I + A_{n_k})^{-1} f = (\lambda I + A)^{-1} f$. But this implies that

$$\lim_{n \to \infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f$$

and the proof of the theorem is complete. □

For compact maps one obtains a stronger convergence in Theorem 3.7.

Theorem 3.8 *Assume the notation and conditions of Theorem* 3.7. *Suppose in addition that the map $j \colon D(a) \to H$ is compact, i.e. it maps bounded subsets of $D(a)$ into totally bounded subsets of H. If $\lambda > (-\gamma) \vee 0$ then*

$$\lim_{n \to \infty} \|(\lambda I + A_n)^{-1} - (\lambda I + A)^{-1}\| = 0.$$

Proof Suppose not. Then there exist $\varepsilon > 0$, $n_1, n_2, \ldots \in \mathbf{N}$ and $f_1, f_2, \ldots \in H$ such that $n_k < n_{k+1}$, $\|f_k\|_H \leq 1$ and $\|(\lambda I + A_{n_k})^{-1} f_k - (\lambda I + A)^{-1} f_k\| \geq \varepsilon$ for all $k \in \mathbf{N}$. Passing to a subsequence, if necessarily, there exists an $f \in H$ such that $\lim_{k \to \infty} f_{n_k} = f$ weakly in H. For all $k \in \mathbf{N}$ there exists a $\tilde{u}_k \in D_H(\tilde{a}_{n_k})$ such that $\tilde{j}_{n_k}(u_k) = (\lambda I + A_{n_k})^{-1} f_k$. Let $u_k = \widetilde{\Phi}_{n_k}(\tilde{u}_k)$, where we use the notation as in the proof of Theorem 3.7. Then it follows from the first part of the proof of Theorem 3.7 that there exists a $u \in V$ such that, after passing to a subsequence if necessarily, $\lim_{k \to \infty} u_k = u$ weakly in V and $\tilde{j}(u) = (\lambda I + A)^{-1} f$. Since j is compact, the map $\tilde{j}$ is compact. Therefore

$$\lim_{k \to \infty} (\lambda I + A_{n_k})^{-1} f_k = \lim \tilde{j}(u_k) = \tilde{j}(u) = (\lambda I + A)^{-1} f$$

strongly in H. Moreover, $\lim_{k \to \infty} (\lambda I + A)^{-1} f_k = (\lambda I + A)^{-1} f$ by Lemma 2.7. So $\lim_{k \to \infty} \|(\lambda I + A_{n_k})^{-1} f_k - (\lambda I + A)^{-1} f_k\| = 0$. This is a contradiction. □

Theorem 3.7 has as corollary that under a mild additional condition the operator A can be viewed as a kind of viscosity operator. If a is symmetric and j is the identity map then this theorem is a generalization of Corollary 3.9 in [ERS], which followed from [Kat] Theorem VIII.3.11. Note that [Kat] Theorem VIII.3.11 is a special case of Theorem 3.7. The point in the following corollary is that the form a is merely j-sectorial, but not necessarily j-elliptic.

Corollary 3.9 *Let V, H be Hilbert spaces and $j \in \mathcal{L}(V, H)$ with $j(V)$ dense in H. Let $a \colon V \times V \to \mathbf{C}$ be a continuous j-sectorial form with vertex γ. Let $b \colon V \times V \to \mathbf{C}$ be a j-elliptic continuous form. Suppose that there exists a $\theta \in [0, \frac{\pi}{2})$ such that $b(u) \in \Sigma_\theta$ for* *all $u \in V$. For all $n \in \mathbf{N}$ define $a_n = a + \frac{1}{n} b$. Then a_n is j-elliptic. Let A_n be the operator associated with (a_n, j) and let A be the operator associated with (a, j). Then*

$$\lim_{n\to\infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f$$

for all $\lambda > (-\gamma) \vee 0$ and $f \in H$.

Remark 3.10 The condition $\lim_{n\to\infty} \gamma_n = 0$ is necessary in general in Theorem 3.7. It is not sufficient to assume that the a_n are uniformly j-sectorial in the sense that there exist one $\gamma' \in \mathbf{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $a_n(u) - \gamma' \, \|j(u)\|_H^2 \in \Sigma_\theta$ for all $n \in \mathbf{N}$ and $u \in D(a_n)$, together with a core condition on D and the condition $\lim_{n\to\infty} \operatorname{Re} a_n(u) = \operatorname{Re} a(u)$ for all $u \in D$. A counter example is if $a_n(u,v) = a(u,v) + i \, (j(u), j(v))_H$ for all $n \in \mathbf{N}$.

We next consider the classical form associated with the m-sectorial form A.

Proposition 3.11 *Let a be an accretive sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Suppose the form a is j-sectorial and $j(D(a))$ is dense in H. Let A be the operator associated with (a, j). Then one has the following.*

(a) *There exists a unique closable sectorial form a_r with form domain $j(D(a))$ such that A is associated with $\overline{a_r}$.*

(b) $D(\overline{a_r}) = \{x \in H :$ *there exists a bounded sequence $u_1, u_2, \ldots \in D(a)$ such that $x = \lim_{n\to\infty} j(u_n)$ in $H\}$.*

(c) *There exists a $c > 0$ such that $\|j(u)\|_{a_r} \leq c \, \|u\|_a$ for all $u \in D(a)$. In particular, if D is a core for $D(a)$ then $j(D)$ is a core for $\overline{a_r}$.*

(d) *Let h be the real part of a and let h_r be defined similarly as in Statement* (a). *Then $D(\overline{a_r}) = D(\overline{h_r})$.*

Proof '(a)'. We use the notation as in the proof of Theorem 3.2. Let b be the closed sectorial form associated with A, i.e. the classical form associated with $(\tilde{a}, \tilde{j})$ given in Theorem 2.5(b) by $D(b) = \tilde{j}(V) = \tilde{j}(V(\tilde{a}))$ and $b(\tilde{j}(u), \tilde{j}(v)) = \tilde{a}(u,v)$ for all $u, v \in V(\tilde{a})$. Then $j(D(a)) = \tilde{j}(q(D(a))) \subset \tilde{j}(V) = D(b)$. We show that $j(D(a))$ is a core of b. Let $x \in D(b)$. There exists a unique $u \in V(\tilde{a})$ such that $\tilde{j}(u) = x$. There exist $u_1, u_2, \ldots \in D(a)$ such that $\lim q(u_n) = u$ in V. Let π_2 be the projection of V onto $V(\tilde{a})$ along the decomposition $V = \ker \tilde{j} \oplus V(\tilde{a})$. Clearly $\pi_2(u) = u$. In addition, π_2 is continuous and $j(u_n) = \tilde{j}(q(u_n)) = \tilde{j}(\pi_2(q(u_n)))$ for all $n \in \mathbf{N}$. Therefore $\|x - j(u_n)\|_{D(b)} = \|\pi_2(u) - \pi_2(q(u_n))\|_{V(\tilde{a})} \leq \|\pi_2\| \, \|u - q(u_n)\|_V$ for all $n \in \mathbf{N}$, from which one deduces that $\lim j(u_n) = x$ in $D(b)$. We have shown that $D(a)$ is a core of $D(b)$. Let $a_r = b|_{D(a)}$. Then $b = \overline{a_r}$. This proves existence of a_r. The uniqueness is obvious from [Kat] Theorem VI.2.7.

'(b)'. '$\subset$'. Let $x \in D(\overline{a_r}) = D(b)$. Let $u_1, u_2, \ldots \in D(a)$ and $u \in V(\tilde{a})$ be as in the proof of Statement (a). Then $\lim j(u_n) = x$ in $D(b)$, therefore also in H. Moreover, $\lim q(u_n) = u$ in V. So the sequence $q(u_1), q(u_2), \ldots$ is bounded in V. But $\|u_n\|_a = \|q(u_n)\|_V$ for all $n \in \mathbf{N}$. Thus the sequence $u_1, u_2, \ldots$ satisfies the requirements.

'$\supset$'. Let $u_1, u_2, \ldots$ be a bounded sequence in $D(a)$, $x \in H$ and suppose that $\lim j(u_n) = x$ in H. Then $q(u_1), q(u_2), \ldots$ is a bounded sequence in V. So passing to a subsequence if necessary, there exists a $v \in V$ such that $\lim q(u_n) = v$ weakly in V. Then $\tilde{j}(v) = \lim j(u_n)$ weakly in H. Hence $x = \tilde{j}(v) \in \tilde{j}(V) = D(\overline{a_r})$.

'(d)'. The construction in the proof of Theorem 3.2 with h instead of a leads to the same closed space W, then the same normed space V_0 and the same Banach space V. Let $\tilde{h}: V \times V \to \mathbf{C}$ be the unique continuous form on V such that $\tilde{h}(q(u), q(v)) = h(u, v)$ for all $u, v \in V$. Then $\tilde{h} = \Re\tilde{a}$, the real part of $\tilde{a}$. Let h_c be the classical form associated with $(\tilde{h}, \tilde{j})$. Then $h_c = \overline{h_r}$ and $b = \overline{a_r}$ by part (a). Then Statement (d) follows from Proposition 2.6.

'(c)'. Again by Proposition 2.6 there exists a $c \geq 1$ such that $\operatorname{Re} b(x) \leq c\, h_c(x)$ for all $x \in \tilde{j}(V)$. But $h_c(\tilde{j}(u)) \leq \tilde{h}(u) = \operatorname{Re}\tilde{a}(u)$ for all $u \in V$. So $\|\tilde{j}(u)\|_b \leq c\,\|u\|_{\tilde{a}}$ for all $u \in V$. Then $\|j(u)\|_{a_r} \leq c\,\|q(u)\|_{\tilde{a}} = c\,\|u\|_a$ for all $u \in D(a)$. The last assertion in Statement (c) is an immediate consequence. □

We call a_r the **regular** and $\overline{a_r}$ the **relaxed** form of the j-sectorial form a. This terminology coincides with the one employed by Simon [Sim2] in the symmetric case if $D(a) \subset H$ and j is the identity map. If a is **positive**, i.e. if the numerical range $\{a(u) : u \in D(a)\}$ is contained in $[0, \infty)$, then Simon characterizes the regular part of a as the largest closable form lying below a for the order relation $b_1 \leq b_2$ if and only if $D(b_2) \subset D(b_1)$ and $b_1(u) \leq b_2(u)$ for all $u \in D(b_2)$. Of course, such an order relation is not possible to define for sectorial forms. It seems to us, though, that the direct formula in Theorem 1.1 expressing the generator directly in terms of the form a, is frequently more useful than the computation of a_r. For positive a Simon proved Proposition 3.11(b) in [Sim1], Theorem 3. Note that for general a (but still j the inclusion), the form a is closable if and only if a_r coincides with a on $D(a)$.

Let a be a densely defined sectorial form and A its associated operator, as above. If the form a is symmetric, then the associated operator A is self-adjoint. But the converse is not true if the form a not closable. In order to see this, it suffices to consider the form $(1 + i)a$ where a is the form as in Example 3.14 below.

For general j-sectorial forms we also consider invariance of closed convex subsets.

Proposition 3.12 *Let a be an accretive sesquilinear form, H a Hilbert space and $j: D(a) \to H$ a linear map. Suppose the form a is accretive, j-sectorial and $j(D(a))$ is dense in H. Let A be the operator associated with (a, j) and S the semigroup generated by $-A$. Let $C \subset H$ be a closed convex set and let $P: H \to C$ be the orthogonal projection. Then the following are equivalent.*

(i) *$S_t C \subset C$ for all $t > 0$.*

(ii) *for all $u \in D(a)$ there exists a Cauchy sequence $w_1, w_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(w_n) = Pj(u)$ in H and $\lim_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geq 0$.*

(iii) *for all $u \in D(a)$ there exists a bounded sequence $w_1, w_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(w_n) = Pj(u)$ in H and $\limsup_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geq 0$.*

Proof We use the notation as in the proof of Theorem 3.2. Clearly the form $\tilde{a}$ is accretive by continuity and density of V_0. We shall prove the equivalence with Condition (iv) in Proposition 2.9 for $D = V_0 = q(D(a))$, $\tilde{a}$ and $\tilde{j}$.

'(i)⇒(ii)'. Let $u \in D(a)$. By Proposition 2.9(i)⇒(iv) there exists a $w \in V$ such that $\tilde{j}(w) = Pj(u)$ and $\operatorname{Re}\tilde{a}(w, q(u) - w) \geq 0$. There are $w_1, w_2, \ldots \in D(a)$ such that $\lim q(w_n) = w$ in V. Then the sequence $w_1, w_2, \ldots$ satisfies the requirements.

'(ii)⇒(iii)'. Trivial.

'(iii)⇒(i)'. Let $u \in D(a)$. By assumption there exists a bounded sequence $w_1, w_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim j(w_n) = Pj(u)$ in H and $\limsup_{n\to\infty} a(w_n, u - w_n) \geq 0$. Then $q(w_1), q(w_2), \ldots$ is a bounded sequence in V, so passing to a subsequence if necessary, it follows that it is weakly convergent. Let $w = \lim_{n\to\infty} q(w_n)$ weakly in V. Then $\tilde{j}(w) = \lim j(w_n)$ weakly in H, so $\tilde{j}(w) = Pj(u) = P\tilde{j}(q(u))$. Moreover, $\tilde{a}(w, q(u)) = \lim \tilde{a}(q(w_n), q(u))$ and $\operatorname{Re} \tilde{a}(w,w) = \Re\tilde{a}(w) \leq \liminf \Re\tilde{a}(q(w_n))$ by [Kat] Lemma VIII.3.14a. So $\operatorname{Re} \tilde{a}(w, q(u) - w) \geq \limsup_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geq 0$. Then Condition (i) follows from Proposition 2.9(iv)⇔(i). □

Remark 3.13 Clearly Condition (ii) in Proposition 3.12 is valid if for all $u \in D(a)$ there exists a $w \in D(a)$ such that $j(w) = Pj(u)$ and $\operatorname{Re} a(w, u - w) \geq 0$.

We end this section with several remarks in the framework of Theorem 1.1. So from now on we assume that $D(a) \subset H$, the form a is densely defined, sectorial and that j is the inclusion map. We emphasize that we do not assume that the form a is closable. Let us give some further comments on the definition of the operator A associated with a. We might at first associate a *minimal operator* $A_{\min}$ with a in the following way. For all $u, f \in H$ we say by definition, that $u \in D(A_{\min})$ and $Au = f$ if and only if

$$a(u, v) = (f, v)_H \text{ for all } v \in D(a). \tag{12}$$

Thus the operator A is an extension of $A_{\min}$ consisting all *approximate solutions* of the problem (12). If the form is closed, then $A_{\min} = A$. The following example shows that the minimal operator may be trivial in the sense that $A_{\min} = \{0\}$ even if the form is *definite* in the sense that $a(u) = 0$ implies that $u = 0$.

Example 3.14 Let $H = L_2(0,1)$, $D(a) = C[0,1]$ and

$$a(u, v) = \sum_{n=1}^{\infty} 2^{-n}\, u(q_n)\, \overline{v(q_n)}$$

where $\{q_n : n \in \mathbf{N}\} = [0,1] \cap \mathbf{Q}$ with $q_n \neq q_m$ for all $n, m \in \mathbf{N}$ with $n \neq m$.

Now let $u \in D(A_{\min})$ and set $f = Au$. Then $a(u,v) = (f,v)_H$ for all $v \in D(a)$, so

$$\sum_{n=1}^{\infty} 2^{-n}\, u(q_n)\, \overline{v(q_n)} = \int_0^1 f(t)\, \overline{v(t)}\, dt$$

for all $v \in C[0,1]$. Since on the left hand side we apply a discrete measure and on the right hand side a continuous measure to v, it follows that $f = 0$ and $u(q_n) = 0$ for all $n \in \mathbf{N}$. Hence $u = 0$. We have shown that $D(A_{\min}) = \{0\}$.

In this example one can calculate the operator A associated with a. We use the notation as introduced in the proof of Theorem 1.1. First we characterize the completion of $D(a)$. Note that

$$\|u\|_a^2 = \int_0^1 |u|^2\, dx + \sum_{n=1}^{\infty} 2^{-n}\, |u(q_n)|^2.$$

We claim that the completion V of $D(a)$ is the space $L_2(0,1) \oplus K$, where K is the Hilbert space

$$K = \{\xi \in \mathbf{C}^{\mathbf{N}} : \sum_{n=1}^{\infty} 2^{-n}\, |\xi_n|^2 < \infty\}.$$

Clearly the mapping $\Phi\colon D(a) \to L_2(0,1) \oplus K$ given by $\Phi(u) = (u, (u(q_n))_{n\in\mathbf{N}})$ is an isometry. Let $F = \overline{\Phi(D(a))}$. We shall show that $F = L_2(0,1) \oplus K$. Let $m \in \mathbf{N}$ and $e_m = (0,\ldots,0,1,0\ldots,0,\ldots)$. We aim to show that $(0,e_m) \in F$. For all $k \in \mathbf{N}$ let $u_k \in C[0,1]$ be such that $0 \le u_k \le 1$, $u_k(q_m) = 1$ and $\operatorname{supp} u_k \subset (q_m - \frac{1}{k}, q_m + \frac{1}{k})$. Then $\lim_{k\to\infty} u_k = 0$ in $L_2(0,1)$. We show that $(u_k(q_n))_{n\in\mathbf{N}}$ converges to e_m in K as $k \to \infty$. Let $\varepsilon > 0$. There exists an $N > m$ such that $\sum_{n=N}^{\infty} 2^{-n} \le \varepsilon$. Next, there exists a $k_0 \in \mathbf{N}$ such that $\frac{1}{k_0} < |q_n - q_m|$ for all $n \in \{1,\ldots,N\}\setminus\{m\}$. Then $\|(u_k(q_n)_{n\in\mathbf{N}}) - e_m\|_K^2 \le \sum_{n\ge N} 2^{-n} \le \varepsilon$ for all $k \in \mathbf{N}$ with $k \ge k_0$. This proves that $(0,e_m) \in F$. Hence $(0,\xi) \in F$ for all $\xi \in K$. Let $u \in C[0,1]$. Then $(u,0) = (u,(u(q_n))_{n\in\mathbf{N}}) - (0,(u(q_n))_{n\in\mathbf{N}}) \in F$. Since $C[0,1]$ is dense in $L^2(0,1)$, it follows that $L^2(0,1) \oplus \{0\} \subset F$. This shows that $F = L^2(0,1) \oplus K$.

So $V = L^2(0,1) \oplus K$. Then the map $j \in \mathcal{L}(V,H)$ is given by $j(u,\xi)) = u$. The extension $\tilde{a}$ of a to $V \times V$ is given by

$$\tilde{a}((u,\xi),(v,\eta)) = \sum_{n=1}^{\infty} 2^{-n}\, \xi_n\, \overline{\eta_n}.$$

Let A be the operator associated with $(\tilde{a}, j)$. Let $(u,\xi) \in D_H(\tilde{a})$ and set $f = Aj(u,\xi)$. Then

$$(f,v)_{L^2} = (f, j(v,\eta))_{L^2} = \tilde{a}((u,\xi),(v,\eta)) = \sum_{n=1}^{\infty} 2^{-n}\, \xi_n\, \overline{\eta_n}$$

for all $(v,\eta) \in V$. Choose $v = 0$ and $\eta = \xi$. Then it follows that $\xi = 0$. Hence $(f,v)_{L^2} = 0$ for all $v \in L^2(0,1)$. Thus $f = 0$. It follows that $Au = A(j(u,\xi)) = 0$ and $D_H(\tilde{a}) \subset L_2(0,1) \times \{0\}$. Conversely, let $u \in L^2(0,1)$. Then $(u,0) \in V$ and $(0, j(v,\eta))_{L^2} = 0 = \tilde{a}((u,0),(v,\eta))$ for all $(v,\eta) \in V$. Thus $(u,0) \in D_H(\tilde{a})$ and $D_H(\tilde{a}) = L_2(0,1) \times \{0\}$. The operator A associated with $(\tilde{a}, j)$ is 0.

Finally, let $(u,\xi) \in V$. If $\xi \ne 0$ then $j(u,\xi) \in D(A)$ but

$$\tilde{a}((u,\xi),(v,\eta)) \ne 0 = (A(j(u,\xi), j(v,\eta)))$$

if $(v,\eta) = (u,\xi)$. This is an example which shows that in general (4) is restricted to $u \in D_H(\tilde{a})$. □

The form in the example is not closable. But even if the form a is closable, the domain $D(A_{\min})$ of $A_{\min}$ is not a core of A, in general. We give an example

Example 3.15 Let $H = L^2(\Omega)$ where Ω is a bounded open set in $\mathbf{R}^d$ and let $D(a) = \mathcal{D}(\Omega)$ be the space of all test functions. Define $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$ for all $u,v \in \mathcal{D}(\Omega)$. Then $D(A_{\min}) = \mathcal{D}(\Omega)$ and $A_{\min} u = -\Delta u$. The associated operator A is the Dirichlet Laplacian which has domain $D(A) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$. Both the Dirichlet as the Neumann Laplacian are extensions of $-A_{\min}$. Therefore $\mathcal{D}(\Omega) = D(A_{\min})$ is not a core of the associated operator A. □

Proposition 3.12 has several consequences which will be useful for differential operators in the next section. If $(X,\mathcal{B},m)$ is a measure space and a is a sesquilinear form in $L_2(X)$, then we call a **real** if $\operatorname{Re} u \in D(a)$ and $a(\operatorname{Re} u) \in \mathbf{R}$ for all $u \in D(a)$.

Corollary 3.16 *Let $(X,\mathcal{B},m)$ be a measure space and let a be a densely defined sectorial form in $L_2(X)$. Let A be the operator associated with a as in Theorem* 1.1 *and let S be the semigroup generated by the operator $-A$.*

(a) *If a is real then $S_t L_2(X,\mathbf{R}) \subset L_2(X,\mathbf{R})$ for all $t > 0$.*

(b) *If a is real, $u^+ \in D(a)$ and $a(u^+, u^-) \le 0$ for all $u \in D(a) \cap L_2(X,\mathbf{R})$, then S is positive. In particular, $|S_t u| \le S_t|u|$ for all $t > 0$ and $u \in L_2(X)$.*

(c) *If a is accretive, real, $u \wedge \mathbb{1} \in D(a)$ and $a(u \wedge \mathbb{1}, (u - \mathbb{1})^+) \ge 0$ for all $u \in D(a) \cap L_2(X,\mathbf{R})$, then S is* **submarkovian**, *i.e., $\|S_t u\|_\infty \le \|u\|_\infty$ for all $u \in L_2(X) \cap L_\infty(X)$.*

(d) *If a is accretive, real, $u \wedge \mathbb{1} \in D(a)$ and $a((u - \mathbb{1})^+, u \wedge \mathbb{1}) \ge 0$ for all $u \in D(a) \cap L_2(X,\mathbf{R})$, then $\|S_t u\|_1 \le \|u\|_1$ for all $u \in L_1(X) \cap L_2(X)$.*

Proof '(a)'. Replacing a by $(u,v) \mapsto a(u,v) + \gamma\,(u,v)_H$ we may assume that a is accretive. Let $u \in D(a)$. Set $w = \operatorname{Re} u$. Then $w \in D(a)$ and $\operatorname{Re} a(w, u - w) = \operatorname{Re} a(\operatorname{Re} u, i \operatorname{Im} u) = \operatorname{Im} a(\operatorname{Re} u, \operatorname{Im} u) = 0$. So by Proposition 3.12 the set $L_2(X,\mathbf{R})$ is invariant under S. (See also Remark 3.13.)

'(b)'. Again we may assume that a is accretive. Let $C = \{(\operatorname{Re} u)^+ : u \in L_2(X)\}$. Then C is closed and convex. Let P be the projection of $L_2(X)$ onto C. Let $u \in D(a)$. Then $Pu = (\operatorname{Re} u)^+ \in D(a)$. Moreover, $\operatorname{Re} a(Pu, u - Pu) = \operatorname{Re} a((\operatorname{Re} u)^+, -(\operatorname{Re} u)^- + i \operatorname{Im} u) = -a((\operatorname{Re} u)^+, (\operatorname{Re} u)^-) \ge 0$. So by Proposition 3.12 the set C is invariant under S.

'(c)'. Let $C = \{u \in L_2(X,\mathbf{R}) : u \le \mathbb{1}\}$. Then C is closed and convex in $L_2(X)$. The projection $P\colon L_2(X) \to C$ is given by $Pu = (\operatorname{Re} u) \wedge \mathbb{1}$. It follows by assumption and Proposition 3.12 that the set C is invariant under S. By linearity one deduces that $|S_t u| \le \mathbb{1}$ for all $t > 0$ and $u \in L_2(X,\mathbf{R})$ with $|u| \le \mathbb{1}$. Next, let $t > 0$, $u \in L_2(X,\mathbf{R})$ and assume $u \le 0$. If $n \in \mathbf{N}$ then $n\,u \in C$, so $n\,S_t u \in C$ and $S_t u \le \frac{1}{n}\mathbb{1}$. Therefore $S_t u \le 0$ and S is a positive semigroup. Finally, let $u \in L_2(X)$ and assume that $|u| \le \mathbb{1}$. Then $|S_t u| \le S_t|u| \le \mathbb{1}$ by Statement (b) and the above applied to $|u|$.

'(d)'. This follows by duality from Statement (c) and Remark 3.4. □

4 Examples

We illustrate the theorems of the previous sections by several examples.

4.1 Sectorial differential operators

First we consider differential operators on open sets in $\mathbf{R}^d$. We emphasize that the operators do not have to be symmetric and may have complex coefficients.

Lemma 4.1 *Let $\Omega \subset \mathbf{R}^d$ be open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_{1,\mathrm{loc}}(\Omega)$. Let $D(a)$ be a subspace of $L_2(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$. Assume that $\partial_i u \in L_{1,\mathrm{loc}}(\Omega)$ as distribution and*

$$\int_\Omega |(\partial_i u)\, a_{ij}\, \partial_j v| < \infty$$

for all $u,v \in D(a)$ and $i,j \in \{1,\ldots,d\}$. Define the form $a\colon D(a) \times D(a) \to \mathbf{C}$ by

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u)\, a_{ij}\, \overline{\partial_j v}.$$

Let $\theta \in [0, \frac{\pi}{2})$. Then the following are equivalent.

(i) *The form a is sectorial with semi-angle θ.*

(ii) *The form a is sectorial with vertex 0 and semi-angle θ.*

(iii) $\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\,\overline{\xi_j} \in \Sigma_\theta$ *for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$.*

Proof The implications (iii)⇒(ii)⇒(i) are trivial. Conversely, suppose there exists a $\gamma \in \mathbf{R}$ such that $a(u) - \gamma\,\|u\|_2^2 \in \Sigma_\theta$ for all $u \in C_c^\infty(\Omega)$. Let $\xi \in \mathbf{C}^d$ and $\tau \in C_c^\infty(\Omega)$. For all $\lambda \in \mathbf{R}$ define $u_\lambda \in C_c^\infty(\Omega)$ by $u_\lambda(x) = e^{i\lambda x\cdot\xi}\,\tau(x)$. Then

$$\sum_{i,j=1}^d \int_\Omega a_{ij}\,|\tau|^2\,\xi_i\,\overline{\xi_j} = \lim_{\lambda\to\infty} \lambda^{-2}\,a(u_\lambda) \in \Sigma_\theta.$$

Therefore (iii) follows. □

If one of the equivalent conditions of Lemma 4.1 is valid then we can apply Theorem 1.1 and we call the operator A associated with a a **sectorial differential operator**. Then $-A$ generates a holomorphic semigroup.

The assumptions on the domain $D(a)$ and the coefficients a_{ij} are very general. For example one can choose $D(a) = C_c^\infty(\Omega)$ together with the condition $a_{ij} \in L_{1,\mathrm{loc}}(\Omega)$, or alternatively one can choose $D(a) = H^1_{\mathrm{loc}}(\Omega)$ together with $a_{ij} \in L_{\infty,\mathrm{loc}}(\Omega)$. Or if $a_{ij} \in L_\infty(\Omega)$ one can choose for $D(a)$ any subspace of $H^1(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$.

In order to avoid too many cases we will not consider unbounded coefficients in this paper. We shall frequently use the approximation by strongly elliptic forms and operators. Let $\Omega \subset \mathbf{R}^d$ be open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0,\frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\,\overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define the form $a\colon D(a)\times D(a) \to \mathbf{C}$ by

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u)\,a_{ij}\,\overline{\partial_j v},$$

where $D(a)$ is a subspace of $H^1(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$. Let $l\colon D(a)\times D(a) \to \mathbf{C}$ be defined by $l(u,v) = \sum_{i=1}^d \int_\Omega \partial_i u\,\overline{\partial_i v}$. For all $n \in \mathbf{N}$ let $a^{(n)} = a + \frac{1}{n}\,l$. Then $a^{(n)}$ is strongly elliptic, i.e. there exists a $\mu > 0$ such that $\mathrm{Re}\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\,\overline{\xi_j} \geq \mu\,|\xi|^d$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. If A, A_n, S and $S^{(n)}$ are the associated operators and semigroups then the conditions of Theorem 3.7 are satisfied. In particular the A_n converge to A strongly in the resolvent sense and therefore $S_t^{(n)}$ converges strongly to S_t for all $t > 0$. Note that A_n is a kind of viscosity operator for A.

We next show that under a mild condition on the form domain $D(a)$ the semigroup associated with a sectorial differential operator satisfies Davies–Gaffney bounds. If F and G are two non-empty subsets of $\mathbf{R}^d$ then $d(F,G)$ denotes the Euclidean distance. The value of M can be improved significantly if the coefficients are real. (See [ERSZ2] Proposition 3.1.) In this paper the following version for complex coefficients suffices.

Theorem 4.2 *Let $\Omega \subset \mathbf{R}^d$ be open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0,\frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\,\overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define the form $a\colon D(a)\times D(a) \to \mathbf{C}$ by*

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u)\,a_{ij}\,\overline{\partial_j v},$$

where $D(a)$ is a subspace of $H^1(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$. Suppose $e^{\rho\psi} u \in D(a)$ for all $u \in D(a)$, $\rho \in \mathbf{R}$ and $\psi \in W^\infty_\infty(\mathbf{R}^d, \mathbf{R})$. Let S be the semigroup associated with a. Then

$$|(S_t u, v)| \leq e^{-\frac{d(\Omega_1, \Omega_2)^2}{4Mt}} \, \|u\|_2 \, \|v\|_2 \tag{13}$$

for all non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$, where $M = 3(1 + \tan\theta)^2 (1 + \sum_{i,j=1}^d \|a_{ij}\|_\infty)$.

Proof First suppose that the (a_{ij}) are strongly elliptic. Let $\rho > 0$ and $\psi \in W^\infty_\infty(\mathbf{R}^d, \mathbf{R})$ with $\|\nabla\psi\|_\infty \leq 1$. Define the form $a_\rho \colon D(a) \times D(a) \to \mathbf{C}$ by

$$a_\rho(u, v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u + \rho \, \psi_i \, u) \, a_{ij} \, \overline{\partial_j v - \rho \, \psi_j \, v},$$

where $\psi_i = \partial_i \psi$ for all $i \in \{1, \ldots, d\}$. Then

$$\begin{aligned} \operatorname{Re} a_\rho(u) = \operatorname{Re} a(u) + \rho \, \operatorname{Re} \int_\Omega \sum_{i,j=1}^d \psi_i \, u \, a_{ij} \, \overline{\partial_j u} - \rho \, \operatorname{Re} \int_\Omega \sum_{i,j=1}^d (\partial_i u) \, a_{ij} \, \psi_j \, \overline{u} \\ - \rho^2 \, \operatorname{Re} \int_\Omega \sum_{i,j=1}^d \psi_i \, a_{ij} \, \psi_j \, |u|^2 \end{aligned} \tag{14}$$

for all $u \in D(a)$. It follows from the estimate (1.15) of Subsection VI.1.2 in [Kat] that

$$\begin{aligned} \Big| \sum_{i,j=1}^d a_{ij}(x) \, \xi_i \, \overline{\eta_j} \Big| &\leq (1 + \tan\theta) \Big(\operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \xi_i \, \overline{\xi_j} \Big)^{1/2} \Big(\operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \eta_i \, \overline{\eta_j} \Big)^{1/2} \\ &\leq \varepsilon \operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \xi_i \, \overline{\xi_j} + \frac{(1 + \tan\theta)^2}{4\varepsilon} \, \operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \eta_i \, \overline{\eta_j} \end{aligned}$$

for all $\xi, \eta \in \mathbf{C}^d$, $\varepsilon > 0$ and a.e. $x \in \Omega$. Choosing $\xi_i = (\partial_i u)(x)$, $\eta_i = (\psi_i \, u)(x)$ and $\varepsilon = \frac{1}{4\rho}$ it follows that

$$\rho \Big| \int_\Omega \sum_{i,j=1}^d (\partial_i u) \, a_{ij} \, \psi_j \, \overline{u} \Big| \leq \tfrac{1}{4} \operatorname{Re} a(u) + (1 + \tan\theta)^2 \, \rho^2 \, \operatorname{Re} \int_\Omega \sum_{i,j=1}^d \psi_i \, a_{ij} \, \psi_j \, |u|^2.$$

Similarly the second term in (14) can be estimated. Hence

$$\begin{aligned} \operatorname{Re} a_\rho(u) &\geq \tfrac{1}{2} \operatorname{Re} a(u) - \Big(1 + 2(1 + \tan\theta)^2 \Big) \rho^2 \, \operatorname{Re} \int_\Omega \sum_{i,j=1}^d \psi_i \, a_{ij} \, \psi_j \, |u|^2 \\ &\geq \tfrac{1}{2} \operatorname{Re} a(u) - M \, \rho^2 \, \|u\|_2^2. \end{aligned} \tag{15}$$

Define $U_{\pm\rho} \colon L_2(\Omega) \to L_2(\Omega)$ by $U_{\pm\rho} v = e^{\pm\rho\psi} v$. Then $U_{\pm\rho} D(a) \subset D(a)$. Moreover, $a_\rho(u, v) = a(U_\rho u, U_{-\rho} v)$ for all $u, v \in D(a)$. Since the (a_{ij}) are strongly elliptic, the forms a and a_ρ are sectorial. Let A and A_ρ be the associated operators and let $S^{(\rho)}$ be the semigroup generated by $-A_\rho$. Then $A_\rho = U_{-\rho}\, A\, U_\rho$ and $S_t^{(\rho)} = U_{-\rho}\, S_t\, U_\rho$ for all $t > 0$. It follows from (15) that

$$\|S_t^{(\rho)}\|_{2\to 2} \le e^{M\,\rho^2\, t} \tag{16}$$

for all $t > 0$. Then

$$|(S_t u, v)| = |(S_t^{(\rho)}\, U_{-\rho} u, U_\rho v)| \le \|S_t^{(\rho)}\|_{2\to 2}\, \|U_{-\rho} u\|_2\, \|U_\rho v\|_2 \le e^{M\,\rho^2\, t}\, e^{-\rho\, d_\psi(\Omega_1,\Omega_2)}\, \|u\|_2\, \|v\|_2$$

for all $u \in L_2(\Omega_1)$ and $v \in L_2(\Omega_2)$, where $d_\psi(\Omega_1,\Omega_2) = \inf_{x\in\Omega_1} \psi(x) - \sup_{x\in\Omega_2} \psi(x)$. Minimizing over all $\psi \in W^\infty_\infty(\mathbf{R}^d)$ with $\|\nabla\psi\|_\infty \le 1$ gives

$$|(S_t u, v)| \le e^{M\,\rho^2\, t}\, e^{-\rho\, d(\Omega_1,\Omega_2)}\, \|u\|_2\, \|v\|_2$$

and choosing $\rho = \frac{d(\Omega_1,\Omega_2)}{2Mt}$ gives

$$|(S_t u, v)| \le e^{-\frac{d(\Omega_1,\Omega_2)^2}{4Mt}}\, \|u\|_2\, \|v\|_2$$

uniformly for all $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$.

Finally we drop the assumption that the (a_{ij}) are strongly elliptic. For all $n \in \mathbf{N}$ define $a_{ij}^{(n)} = a_{ij} + \frac{1}{n}\,\delta_{ij}$. Then $(a_{ij}^{(n)})$ is strongly elliptic. If $S^{(n)}$ is the associated semigroup then $\lim_{n\to\infty} S_t^{(n)} = S_t$ strongly for all $t > 0$ by Theorem 3.7. Hence the theorem follows. □

We next consider locality properties of the relaxed form $\overline{a_r}$ of the sectorial form a.

Corollary 4.3 *Assume the notation and assumptions of Theorem 4.2. Then $\overline{a_r}(u,v) = 0$ for all $u, v \in D(\overline{a_r})$ with compact disjoint support.*

Proof There exist open non-empty $\Omega_1, \Omega_2 \subset \mathbf{R}^d$ such that $\operatorname{supp} u \subset \Omega_1$, $\operatorname{supp} v \subset \Omega_2$ and $d(\Omega_1,\Omega_2) > 0$. Then it follows from Theorem 4.2 that there exists a $b > 0$ such that

$$|((I - S_t)u, v)| = |(S_t u, v)| \le e^{-bt^{-1}} \|u\|_2\, \|v\|_2$$

uniformly for all $t > 0$. Hence by [Ouh] Lemma 1.56 one deduces that

$$|\overline{a_r}(u,v)| = \lim_{t\downarrow 0} t^{-1} |((I - S_t)u, v)| \le \lim_{t\downarrow 0} t^{-1} e^{-bt^{-1}} \|u\|_2\, \|v\|_2 = 0$$

as required. □

If $\Omega \subset \mathbf{R}^d$ define

$$L_{2,c}(\Omega) = \{u \in L_2(\Omega) : \operatorname{supp} u \text{ is compact}\}.$$

Another corollary of Theorem 4.2 is that S_t maps $L_{2,c}(\Omega)$ into $L_1(\Omega)$. This is a special case of the following lemma.

For all $R > 0$ let B_R denote the open ball in $\mathbf{R}^d$ with centre 0 and radius R. Set $\chi_R = \mathbb{1}_{B_R}$.

Lemma 4.4 *Let $d \in \mathbf{N}$. There exists a constant $c_d > 0$ such that the following holds. Let $\Omega \subset \mathbf{R}^d$ be open and $T \in \mathcal{L}(L_2(\Omega))$. Let $c, N > 0$ and suppose that*

$$|(Tu, v)| \le c\, e^{-\frac{d(\Omega_1,\Omega_2)^2}{N}} \|u\|_2 \, \|v\|_2$$

for all non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$ and $v \in L_2(\Omega_2)$. Then $T L_{2,c}(\Omega) \subset L_1(\Omega)$ and

$$\|(\mathbb{1} - \chi_{2R}) T u\|_1 \le c\, c_d \, R^{-1} N^{\frac{d+2}{4}} \, e^{-\frac{R^2}{2N}} \, \|u\|_2$$

uniformly for all $R > 0$ and $u \in L_2(\Omega)$ with $\operatorname{supp} u \subset B_R$.

Proof Since $\chi_{2R} T u \in L_2(\Omega \cap B_{2R}) \subset L_1(\Omega)$ it suffices to show the estimate. Let $\varphi \in C_c(\Omega)$. Then

$$\begin{aligned}
|((\mathbb{1} - \chi_{2R}) T u, \varphi)| &= |(Tu, (\mathbb{1} - \chi_{2R})\varphi)| \\
&\le \sum_{n=1}^\infty |(Tu, (\chi_{(n+2)R} - \chi_{(n+1)R})\varphi)| \\
&\le \sum_{n=1}^\infty c\, e^{-\frac{n^2 R^2}{N}} \, \|u\|_2 \, \|(\chi_{(n+2)R} - \chi_{(n+1)R})\varphi\|_2 \\
&\le \sum_{n=1}^\infty c\, e^{-\frac{n^2 R^2}{N}} \, ((n+2)R)^{d/2} \, |B_1|^{1/2} \, \|u\|_2 \, \|\varphi\|_\infty \\
&\le 3^{d/2} |B_1|^{1/2} \, c\, e^{-\frac{R^2}{2N}} \|u\|_2 \, \|\varphi\|_\infty \sum_{n=1}^\infty e^{-\frac{n^2 R^2}{2N}} \, (nR)^{d/2}.
\end{aligned}$$

Let $c' > 0$ be such that $x^{d/4} \le c' e^x$ uniformly for all $x > 0$. Then c' can be chosen to depend only on d. Note that $\sum_{n=1}^\infty e^{-an^2} \le \int_0^\infty e^{-ax^2} \, dx = \sqrt{\frac{\pi}{4a}}$ for all $a > 0$. Therefore

$$\begin{aligned}
\sum_{n=1}^\infty e^{-\frac{n^2 R^2}{2N}} \, (nR)^{d/2} &= (4N)^{d/4} \sum_{n=1}^\infty e^{-\frac{n^2 R^2}{2N}} \Big(\frac{n^2 \, R^2}{4N} \Big)^{d/4} \\
&\le c' \, (4N)^{d/4} \sum_{n=1}^\infty e^{-\frac{n^2 R^2}{4N}} \le c' \, (4N)^{d/4} \Big(\frac{\pi \, N}{R^2} \Big)^{1/2}.
\end{aligned}$$

Then the lemma follows by taking the supremum over all φ with $\|\varphi\|_\infty \le 1$. □

As a consequence one deduces L_1-convergence of the viscosity semigroups on $L_{2,c}(\Omega)$. Recall that the coefficients in Theorem 4.2 are complex.

Lemma 4.5 *Assume the notation and assumptions of Theorem 4.2. For all $n \in \mathbf{N}$ let $a^{(n)} = a + \frac{1}{n} \, l$, where l is the form with $D(l) = D(a)$ and $l(u, v) = \sum_{i=1}^d \int_\Omega \partial_i u \, \overline{\partial_i v}$. Let $S^{(n)}$ be the semigroup associated with $a^{(n)}$. Then $\lim_{n\to\infty} S^{(n)}_t u = S_t u$ in $L_1(\Omega)$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$.*

Proof It follows from Theorem 4.2 that there exists an $M > 0$ such that

$$|(S_t u, v)| \vee |(S_t^{(n)} u, v)| \leq e^{-\frac{d(\Omega_1,\Omega_2)^2}{4Mt}} \, \|u\|_2 \, \|v\|_2$$

uniformly for all $n \in \mathbf{N}$, non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$, Let $c_d > 0$ be as in Lemma 4.4. Let $u \in L_{2,c}(\Omega)$ and $t > 0$. Then

$$\|(\mathbb{1} - \chi_{2R}) S_t^{(n)} u\|_1 \leq c_d \, R^{-1} (4Mt)^{\frac{d+2}{4}} \, e^{-\frac{R^2}{8Mt}} \, \|u\|_2$$

uniformly for all $n \in \mathbf{N}$ and $R > 0$ with $\operatorname{supp} u \subset B_R$. So $\lim_{R\to\infty} (\mathbb{1} - \chi_{2R}) S_t^{(n)} u = 0$ in $L_1(\Omega)$ uniformly in $n \in \mathbf{N}$. Similarly, $\lim_{R\to\infty} (\mathbb{1} - \chi_{2R}) S_t u = 0$ in $L_1(\Omega)$. So it suffices to prove that $\lim_{n\to\infty} \chi_{2R}(S_t^{(n)} u - S_t u) = 0$ for large $R > 0$. Since $\|\chi_{2R}(S_t^{(n)} u - S_t u\|_1 \leq |B_{2R}|^{1/2} \, \|S_t^{(n)} u - S_t u\|_2$ for all $n \in \mathbf{N}$ and $R > 0$, it follows from Theorem 3.7 that $\lim_{n\to\infty} \chi_{2R}(S_t^{(n)} u - S_t u) = 0$ in $L_1(\Omega)$ for all $R > 0$. □

For strongly elliptic operators one can strengthen the conclusions of Theorem 4.2.

Lemma 4.6 *Assume the notation and assumptions of Theorem* 4.2*. In addition suppose that the operator is strongly elliptic, i.e. there exists a* $\mu > 0$ *such that*

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(x) \, \xi_i \, \overline{\xi_j} \geq \mu \, |\xi|^2$$

for all $\xi \in \mathbf{C}^d$ *and a.e.* $x \in \Omega$*. Then one has the following.*

(a) $S_t L_2(\Omega) \subset H^1(\Omega)$ *for all* $t > 0$*.*

(b) *There exist* $c, M' > 0$ *such that*

$$|(\partial_i S_t u, v)| \leq c \, e^{-\frac{d(\Omega_1,\Omega_2)^2}{M't}} \, \|u\|_2 \, \|v\|_2$$

for all non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ *and* $t > 0$*.*

(c) *If* $u \in L_{2,c}(\Omega)$ *then* $S_t u, \partial_i S_t u \in L_1(\Omega)$ *for all* $t > 0$ *and* $i \in \{1, \ldots, d\}$*. Moreover,* $t \mapsto \|\partial_i S_t u\|_1$ *is locally bounded.*

Proof Statement (a) follows from strong ellipticity and Statement (c) is a consequence of Lemma 4.4 and the estimates of Theorem 4.2 and Statement (b). Therefore it remains to prove Statement (b).

We use the notation as in the proof of Theorem 4.2. Fix $\theta' \in (\theta, \frac{\pi}{2})$. For all $\varphi \in \mathbf{R}$ with $|\varphi| < \theta' - \theta$ define $a_{ij}^{[\varphi]} = e^{i\varphi} \, a_{ij}$ for all $i, j \in \{1, \ldots, d\}$. Then $\sum_{i,j=1}^d a_{ij}^{[\varphi]}(x) \, \xi_i \, \overline{\xi_j} \in \Sigma_{\theta'}$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Let $a^{[\varphi]}$ be the corresponding form with form domain $D(a)$. For all $\rho > 0$ let $a_\rho^{[\varphi]}$, $A^{[\varphi]}$, $A_\rho^{[\varphi]}$, $S^{[\varphi]}$ and $S^{[\varphi]\,\rho}$ be the form, operators and semigroups defined naturally as in the proof of Theorem 4.2. Then it follows from (16) that

$$\|S^{[\varphi]\,\rho}\|_{2\to 2} \leq e^{M_1 \, \rho^2 \, t}$$

for all $\rho, t > 0$ and $|\varphi| < \theta' - \theta$, where $M_1 = 3(1 + \tan\theta')^2 (1 + \sum_{i,j=1}^d \|a_{ij}\|_\infty)$. But $S_t^{[\varphi]\,\rho} = e^{-te^{i\varphi} A_\rho} = S_{te^{i\varphi}}^\rho$. So $\|S_{te^{i\varphi}}^\rho\|_{2\to 2} \leq e^{M_1 \, \rho^2 \, t}$ for all $t, \rho > 0$ and $|\varphi| < \theta' - \theta$. Since S^ρ is a holomorphic semigroup on the interior of $\Sigma_{\frac{\pi}{2} - \theta'}$ it follows that

$$S_t^\rho = \frac{1}{2\pi i} \int_{\Gamma_r(t)} \frac{1}{z - t} \, S_z^\rho \, dz$$

for all $t > 0$, where $\Gamma_r(t)$ is the circle centred at t and radius $r = c\,t$ and $c = \sin \frac{1}{2}(\frac{\pi}{2} - \theta')$. Therefore

$$\|A_\rho S^\rho_t\|_{2\to 2} \leq \frac{1}{2\pi} \int_{\Gamma_r(t)} \frac{1}{|z-t|^2} \, \|S^\rho_z\|_{2\to 2} \, d|z| \leq \frac{1}{c\,t} \, e^{M_2 \rho^2 t}$$

for all $\rho, t > 0$, where $M_2 = M_1(1+c)$. It then follows from (15) that

$$\begin{aligned}
\tfrac{1}{2}\,\mu \sum_{i=1}^d \|\partial_i \, S^\rho_t u\|_2^2 &\leq \operatorname{Re} a_\rho(S^\rho_t u, S^\rho_t u) + M\rho^2 \, \|S^\rho_t u\|_2^2 \\
&\leq \|A_\rho \, S^\rho_t u\|_2 \, \|S^\rho_t u\|_2 + M\rho^2 \, \|S^\rho_t u\|_2^2 \\
&\leq \frac{1}{c\,t} \, e^{M_2\,\rho^2\,t} \, e^{M\,\rho^2\,t} \|u\|_2^2 + M\rho^2 \, e^{2M\,\rho^2\,t} \|u\|_2^2 .
\end{aligned}$$

Hence there exist $c_3, M_3 > 0$ such that

$$\|\partial_i \, S^\rho_t\|_{2\to 2} \leq c_3 \, t^{-1/2} \, e^{M_3\,\rho^2\,t}$$

uniformly for all $i \in \{1, \ldots, d\}$ and $\rho, t > 0$. Since

$$\|U_{-\rho} \, \partial_i \, S_t \, U_\rho\|_{2\to 2} = \|(\partial_i + \rho\,\psi_i) S^\rho_t\|_{2\to 2} \leq \|\partial_i \, S^\rho_t\|_{2\to 2} + |\rho\,\psi_i| \, \|S^\rho_t\|_{2\to 2} \leq c_4 \, t^{-1/2} \, e^{M_4\,\rho^2\,t}$$

for suitable $c_4, M_4 > 0$, Statement (b) follows as at the end of the proof of Theorem 4.2.□

The conditions on the form domain in Theorem 4.2 are satisfied in case of Neumann boundary conditions, i.e. if $D(a) = H^1(\Omega)$. We next show that if $D(a) = H^1(\Omega)$ then a strong locality property is valid. We start with a lemma for (complex) strongly elliptic operators.

Lemma 4.7 *Let $\Omega \subset \mathbf{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega)$. Suppose there exists a $\mu > 0$ such that* $\operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \xi_i \, \overline{\xi_j} \geq \mu \, |\xi|^2$ *for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define $a \colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{C}$ by*

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u) \, a_{ij} \, \overline{\partial_j v}.$$

Let S be the semigroup associated with a. Then $(S_t u, \mathbb{1}) = (u, \mathbb{1})$ for all $u \in L_{2,c}(\Omega)$ and $t > 0$.

Proof Fix $\tau \in C_c^\infty(\mathbf{R}^d)$ such that $\tau|_{B_1} = \mathbb{1}$. For all $n \in \mathbf{N}$ define $\tau_n \in C_c^\infty(\mathbf{R}^d)$ by $\tau_n(x) = \tau(n^{-1}x)$. For all $n \in \mathbf{N}$ define $f_n \colon (0,\infty) \to \mathbf{C}$ by $f_n(t) = (S_t u, \tau_n \, \mathbb{1}_\Omega)$. Note that $\tau_n \, \mathbb{1}_\Omega \in H^1(\Omega) = D(a)$ for all $n \in \mathbf{N}$. Therefore

$$f_n'(t) = -a(S_t u, \tau_n \, \mathbb{1}_\Omega) = -\sum_{i,j=1}^d (\partial_i \, S_t u, a_{ij} \, \partial_j(\tau_n \, \mathbb{1}_\Omega)) = -\sum_{i,j=1}^d (\partial_i \, S_t u, a_{ij} \, (\partial_j \tau_n) \, \mathbb{1}_\Omega)$$

and

$$|f_n'(t)| \leq \sum_{i,j=1}^d \|\partial_i \, S_t u\|_1 \, \|a_{ij}\|_\infty \, n^{-1} \, \|\partial_j \tau\|_\infty$$

for all $n \in \mathbf{N}$ and $t > 0$, where we used that $\partial_i S_t u \in L_1(\Omega)$ by Lemma 4.6(c). So $\lim_{n\to\infty} f_n'(t) = 0$ locally uniform on $(0,\infty)$. In addition, $\lim_{n\to\infty} f_n(t) = (S_t u, \mathbb{1})$ for all $t \in (0,\infty)$. Therefore $t \mapsto (S_t u, \mathbb{1})$ is constant. Since $\lim_{t\downarrow 0}(S_t u, \mathbb{1}) = (u, \mathbb{1})$ the lemma follows. □

We are now able to prove strong locality for Neumann sectorial differential operators. Note that our conditions allow that the coefficients are 0 on part or even the entire domain.

Proposition 4.8 *Let $\Omega \subset \mathbf{R}^d$ open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0,\frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\,\overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define the form a with form domain $D(a) = H^1(\Omega)$ by*

$$a(u,v) = \sum_{i,j=1}^d \int_{\mathbf{R}^d} (\partial_i u)\, a_{ij}\, \overline{\partial_j v}.$$

Then one has the following.

(a) *$\overline{a_r}(u,v) = 0$ for all $u,v \in D(\overline{a_r})$ with compact support such that v is constant on a neighbourhood of the support of u.*

(b) *If S is the semigroup associated with a then $(S_t u, \mathbb{1}) = (u, \mathbb{1})$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$.*

Proof We first prove Statement (b). For all $n \in \mathbf{N}$ let $a^{(n)} = a + \frac{1}{n}\, l$, where l is the form with $D(l) = H^1(\Omega)$ and $l(u,v) = \sum_{i=1}^d \int_\Omega \partial_i u\, \overline{\partial_i v}$. Let $S^{(n)}$ be the semigroup associated with $a^{(n)}$. Then $(S_t u, \mathbb{1}) = \lim_{n\to\infty}(S_t^{(n)} u, \mathbb{1}) = (u, \mathbb{1})$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$ by Lemmas 4.5 and 4.7.

Next let $u,v \in D(\overline{a_r})$ with compact support such that v is constant on a neighbourhood of the support of u. Then there exists an open set U and a $\lambda \in \mathbf{C}$ such that $\operatorname{supp} u \subset U$ and $v(x) = \lambda$ for all $x \in U$. Therefore $(u,v) = \lambda\,(u,\mathbb{1}) = \lambda\,(S_t u, \mathbb{1})$ for all $t > 0$.

Let $c_d > 0$ be the constant in Lemma 4.4, which depends only on d. Moreover, set

$$M = 3(1+\tan\theta)^2 (1 + \sum_{i,j=1}^d \|a_{ij}\|_\infty).$$

Fix $R > 0$ such that $\operatorname{supp} u \subset B_R$.

Now let $t > 0$. Then

$$\begin{aligned} ((I - S_t)u, v) &= \lambda\,(S_t u, \mathbb{1}) - (S_t u, v) \\ &= \lambda\,(S_t u, \mathbb{1} - \chi_{2R}) + (S_t u, \lambda\,\chi_{2R} - v). \end{aligned}$$

We estimate the terms separate. First, S satisfies the Davies–Gaffney bounds (13) of Theorem 4.2. So one estimates

$$|(S_t u, \mathbb{1} - \chi_{2R})| \le \|(\mathbb{1} - \chi_{2R}) S_t u\|_1 \le c_d\, R^{-1} (4M\,t)^{\frac{d+2}{4}}\, e^{-\frac{R^2}{8Mt}}\, \|u\|_2$$

by Lemma 4.4. Next, let $D > 0$ be the distance between $\operatorname{supp} u$ and U^{c}. Then it follows from Theorem 4.2 that

$$\begin{aligned} |(S_t u, \lambda\,\chi_{2R} - v)| &\le e^{-\frac{D^2}{4Mt}}\, \|u\|_2\, \|\lambda\,\chi_{2R} - v\|_2 \\ &\le (|\lambda|\,(2R)^{d/2}\,|B_1|^{1/2} + \|v\|_2)\, e^{-\frac{D^2}{4Mt}}\, \|u\|_2. \end{aligned}$$

Therefore

$$t^{-1}|((I - S_t)u, v)| \le |\lambda|\, c_d\, R^{-1} t^{-1} (4M\,t)^{\frac{d+2}{4}}\, e^{-\frac{R^2}{8Mt}}\, \|u\|_2$$
$$+ (|\lambda|\,(2R)^{d/2}\,|B_1|^{1/2} + \|v\|_2)\, t^{-1} e^{-\frac{D^2}{4Mt}}\, \|u\|_2$$

for all $t > 0$. Since $\overline{a_r}(u, v) = \lim_{t\downarrow 0} t^{-1}((I - S_t)u, v)$ the proposition follows. □

Up to now the coefficients were allowed to be complex in this section. If the coefficients are real, but possibly not symmetric, then one has the following application of Corollary 3.16 and Proposition 4.8.

Corollary 4.9 *Let $\Omega \subset \mathbf{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega, \mathbf{R})$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x)\, \xi_i\, \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define the form $a: D(a) \times D(a) \to \mathbf{C}$ by*

$$a(u, v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u)\, a_{ij}\, \overline{\partial_j v},$$

where $D(a) = H^1(\Omega)$ or $D(a) = H^1_0(\Omega)$. Let S be the semigroup associated with a. Then S is real, positive and S extends consistently to a continuous contraction semigroup on $L_p(\Omega)$ for all $p \in [1, \infty]$. Moreover, if $D(a) = H^1(\Omega)$ then $S_t \mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $t > 0$.

Proof Only the last statement needs comments. Since $L_{2,c}(\Omega)$ is dense in $L_1(\Omega)$ one deduces from Proposition 4.8(b) that $(S_t u, \mathbb{1}) = (u, \mathbb{1})$ for all $u \in L_1(\Omega)$. Then the claim follows by duality and Remark 3.4. □

Thus for real coefficients and Neumann boundary conditions one has conservation of probability.

4.2 Multiplicative perturbation

We perturb the Dirichlet Laplacian by choosing a special function j. Let $\Omega \subset \mathbf{R}^d$ be open and bounded. Then we obtain a possibly degenerate operator as follows.

Proposition 4.10 *Let $m: \Omega \to (0, \infty)$ be such that $\frac{1}{m} \in L_{2,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(m\Delta m)$ on $L_2(\Omega)$ by the following. Let $w, f \in L_2(\Omega)$. Then we define $w \in D((m\Delta m))$ and $(m\Delta m)\, w = f$ if and only if $mw \in H^1_0(\Omega)$ and $\Delta(m\, w) = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(m\Delta m)$ is self-adjoint and $(m\Delta m)$ generates a positive semigroup S. Moreover, the set

$$C = \{f \in L_2(\Omega, \mathbf{R}) : f \le \tfrac{1}{m}\}$$

is invariant under S.

Proof Let $V = H^1_0(\Omega) \cap L_2(\Omega, \frac{1}{m^2}\, dx)$ and define $j \in \mathcal{L}(V, L_2(\Omega))$ by $j(u) = \frac{u}{m}$. Define $a: V \times V \to \mathbf{C}$ by $a(u, v) = \int_\Omega \nabla u\, \overline{\nabla v}$. Then a is continuous and symmetric. Since Ω is bounded it follows from the (Dirichlet type) Poincaré inequality that the norm

$$u \mapsto \int_\Omega |\nabla u|^2 + \int_\Omega \frac{|u|^2}{m^2}$$

is an equivalent norm on V. Therefore the form a is j-elliptic. Let A be the operator associated with (a, j). We shall show that $A = -(m\Delta m)$.

Let $w \in D(A)$ and write $f = Aw$. Then there exists a $u \in V$ such that $w = j(u) = \frac{u}{m}$ and $\int_\Omega \nabla u\, \overline{\nabla v} = \int_\Omega f\, \frac{\overline{v}}{m}$ for all $v \in V$. Observe that $\frac{f}{m} \in L_{1,\mathrm{loc}}(\Omega)$. Taking $v \in \mathcal{D}(\Omega)$ one deduces that $-\Delta u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Thus $w \in D((m\Delta m))$ and $-(m\Delta m)\, w = f$.

Conversely, let $w \in D((m\Delta m))$ and write $f = -(m\Delta m)\, w$. Set $u = m\, w \in H^1_0(\Omega)$. Then

$$a(u, v) = \int_\Omega \nabla u\, \overline{\nabla v} = -\langle \Delta u, \overline{v}\rangle = \langle \frac{f}{m}, \overline{v}\rangle = \int_\Omega f\, \frac{\overline{v}}{m} = \int_\Omega f\, \overline{j(v)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in V by [ArC], Proposition 3.2 it follows that $a(u, v) = \int_\Omega f\, \overline{j(v)}$ for all $v \in V$. Thus $w = j(u) \in D(A)$. This proves that $A = -(m\Delta m)$.

The operator A is self-adjoint since a is symmetric. It remains to show the invariance of the set C. The set C is closed and convex in $L_2(\Omega)$. Define $P \colon L_2(\Omega) \to C$ by $Pf = (\mathrm{Re}\, f) \wedge \frac{1}{m}$. Let $u \in V$. Define $w = (\mathrm{Re}\, u) \wedge 1 \in V$. Then $Pj(u) = j(w)$ and $\mathrm{Re}\, a(w, u - w) = 0$. Hence it follows from Proposition 2.9 that the set C is invariant under S. Since $f \leq 0$ if and only if $nf \in C$ for all $n \in \mathbf{N}$ the invariance of C also implies that the semigroup is positive. □

By a similarity transform we obtain two further kinds of multiplicative perturbations.

Proposition 4.11 *Let $\rho \colon \Omega \to (0, \infty)$ be such that $\frac{1}{\rho} \in L_{1,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(\rho\Delta)$ on $L_2(\Omega, \frac{1}{\rho}\, dx)$ by the following. Let $w, f \in L_2(\Omega, \frac{1}{\rho}\, dx)$. Then we define $w \in D((\rho\Delta))$ and $(\rho\Delta)\, w = f$ if and only if $w \in L_2(\Omega, \frac{1}{\rho}\, dx) \cap H^1_0(\Omega)$ and $\Delta w = \frac{f}{\rho}$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(\rho\Delta)$ is self-adjoint and generates a submarkovian semigroup.

Proof Let $m = \sqrt{\rho}$. Then $\frac{1}{m} \in L_{2,\mathrm{loc}}(\Omega)$. Define $U \colon L_2(\Omega, \frac{1}{\rho}\, dx) \to L_2(\Omega)$ by $Uf = \frac{f}{m}$. Then U is unitary and it is straightforward to verify that $(\rho\Delta) = U^{-1}(m\Delta m)U$. Therefore the operator $(\rho\Delta)$ is self-adjoint and generates a semigroup S. Let T be the semigroup generated by $(m\Delta m)$ on $L_2(\Omega)$. Then $S_t = U^{-1} T_t U$ for all $t > 0$. Let $C_1 = \{f \in L_2(\Omega, \frac{1}{\rho}\, dx) : f \leq 1\}$. Then $UC_1 = C$, where C is as in Proposition 4.10. Since T leaves C invariant, it follows that S leaves C_1 invariant and S is submarkovian. □

Proposition 4.12 *Let $\rho \colon \Omega \to (0, \infty)$ be such that $\frac{1}{\rho} \in L_{1,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(\Delta\rho)$ on $L_2(\Omega, \rho\, dx)$ by the following. Let $w, f \in L_2(\Omega, \rho\, dx)$. Then $w \in D((\Delta\rho))$ and $(\Delta\rho)\, w = f$ if and only if $\rho\, w \in H^1_0(\Omega)$ and $\Delta(\rho\, w) = f$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(\Delta\rho)$ is self-adjoint and generates a submarkovian semigroup.

Note that $L_2(\Omega, \rho\, dx) \subset L_{1,\mathrm{loc}}(\Omega) \subset \mathcal{D}(\Omega)'$.

Proof If $m = \sqrt{\rho}$ then the map $U \colon L_2(\Omega, \rho\, dx) \to L_2(\Omega)$ given by $U = m\, f$ is unitary and $U^{-1}(m\Delta m)U = (\Delta\rho)$. The rest is as in the proof of Proposition 4.11 □

4.3 Robin boundary conditions

Let $\Omega \subset \mathbf{R}^d$ be an open set with arbitrary boundary Γ. At first we consider an arbitrary Borel measure on Γ and then specialize to the $(d-1)$-dimensional Hausdorff measure.

For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega, \mathbf{C})$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x)\, \xi_i\, \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Let μ be a (positive) Borel measure on Γ such that $\mu(K) < \infty$ for every compact $K \subset \Gamma$. Define the form a by

$$D(a) = \{u \in H^1(\Omega) \cap C(\overline{\Omega}) : \int_\Gamma |u|^2 \, d\mu < \infty\}$$

and

$$a(u, v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u)\, a_{ij}\, \overline{\partial_j v} + \int_\Gamma u\, \overline{v}\, d\mu.$$

Then $C_c^\infty(\Omega) \subset D(a) \subset L_2(\Omega)$ and a is sectorial. In order to characterize the associated operator A we need to introduce two concepts and one more condition. First, define the Neumann form a_N by $D(a_N) = H^1(\Omega)$ and

$$a_N(u, v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u)\, a_{ij}\, \overline{\partial_j v}.$$

Throughout this subsection we suppose the form a_N is closable. Here we are more interested in the degeneracy caused by μ. If $u \in D(\overline{a_N})$ and $f \in L_2(\Omega)$ then we say that $\mathcal{A}u = f$ **weakly on** Ω if

$$\overline{a_N}(u, v) = \int_\Omega f\, \overline{v}$$

for all $v \in C_c^\infty(\Omega)$. If $u \in D(\overline{a_N})$ then we say that $\mathcal{A}u \in L_2(\Omega)$ **weakly on** Ω if there exists an $f \in L_2(\Omega)$ such that $\mathcal{A}u = f$ weakly on Ω. Clearly such a function f is unique, if it exists. Secondly, if $u \in D(\overline{a_N})$ and $\varphi \in L_2(\Gamma, \mu)$ then we say that φ is **an** a, μ**-trace of** u, or shortly, **a trace of** u, if there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = u$ in $D(\overline{a_N})$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma, \mu)$. Moreover, let $H^1_{a,\mu}(\Omega)$ be the set of all $u \in D(\overline{a_N})$ for which there exists a $\varphi \in L_2(\Gamma, \mu)$ such that φ is a trace of u. We emphasize that φ is not unique (almost everywhere) in general. Clearly $D(a) \subset H^1_{a,\mu}(\Omega)$. With the help of these definitions we can describe the operator A as follows.

Proposition 4.13 *Let $u, f \in L_2(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H^1_{a,\mu}(\Omega)$, $\mathcal{A}u = f$ weakly on Ω and there exists a $\varphi \in L_2(\Gamma, \mu)$ such that φ is a trace of u and*

$$\overline{a_N}(u, v) - \int_\Omega (\mathcal{A}u)\, \overline{v} = -\int_\Gamma \varphi\, \overline{v}\, d\mu \tag{17}$$

for all $v \in D(a)$.

If the conditions are valid, then the function φ is unique.

Proof '⇒'. There exists a Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that $\lim u_n = u$ in $L_2(\Omega)$ and $\lim a(u_n, v) = (f, v)_H$ for all $v \in D(a)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $D(\overline{a_N})$. Therefore $u \in D(\overline{a_N})$ and $\lim u_n = u$ in $D(\overline{a_N})$. Moreover, $u_1|_\Gamma, u_2|_\Gamma, \ldots$ is a Cauchy sequence in $L_2(\Gamma, \mu)$. Therefore $\varphi := \lim u_n|_\Gamma$ exists in $L_2(\Gamma, \mu)$. Then φ is a trace of u. Let $v \in D(a)$. Then

$$\overline{a_N}(u, v) + \int_\Gamma \varphi\, \overline{v}\, d\mu = \lim a(u_n, v) = (f, v)_H = \int_\Omega f\, \overline{v}.$$

Therefore if $v \in C_c^\infty(\Omega)$ then

$$\overline{a_N}(u,v) = \int_\Omega f\,\overline{v},$$

so $\mathcal{A}u = f$ weakly on Ω. Moreover,

$$\overline{a_N}(u,v) + \int_\Gamma \varphi\,\overline{v}\,d\mu = \int_\Omega (\mathcal{A}u)\,\overline{v}$$

for all $v \in D(a)$.

If also $\varphi' \in L_2(\Gamma,\mu)$ satisfies (17) then $\int_\Gamma (\varphi - \varphi')\,\overline{v}\,d\mu = 0$ for all $v \in D(a)$. But the space $\{v|_\Gamma : v \in H^1(\Omega) \cap C_c(\overline{\Omega})\}$ is a $*$-algebra which separates the points of Γ. Therefore it is dense in $C_0(\Gamma)$ and then it is also dense in $L_2(\Gamma,\mu)$. So $\varphi' = \varphi$.

'$\Leftarrow$'. There exist $\varphi \in L_2(\Gamma,\mu)$ and a sequence $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = u$ in $D(\overline{a_N})$, $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma,\mu)$ and (17) is valid for all $v \in D(a)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $D(a)$ and

$$\lim_{n\to\infty} a(u_n,v) = \overline{a_N}(u,v) + \int_\Gamma \varphi\,\overline{v}\,d\mu = \int_\Omega (\mathcal{A}u)\,\overline{v} = \int_\Omega f\,\overline{v}$$

for all $v \in D(a)$. So $u \in D(A)$ and $Au = f$. □

This proposition shows how our general results can be easily applied. It is worthwhile to consider closer the associated closed form since this is intimately related to the problem to define a trace in $L_2(\Gamma,\mu)$ of suitable functions in $H^1(\Omega)$.

Let

$$W = \{(u, u|_\Gamma) : u \in D(a)\}^-$$

where the closure is in $D(\overline{a_N}) \oplus L_2(\Gamma,\mu)$. Then the map $u \mapsto (u, u|_\Gamma)$ from $D(a)$ into W is an isometry and therefore it extends to a unitary map from the completion of $D(a)$ onto W. The form a closable if and only if the map $j\colon W \to L_2(\Omega)$ defined by $j(u,\varphi) = u$ is injective. This is not always the case, even if μ is the $(d-1)$-dimensional Hausdorff measure (see [ArW], Example 4.2). Note that if $\varphi \in L_2(\Gamma,\mu)$ then $(0,\varphi) \in W$ if and only if φ is a trace of 0.

The following lemma is due to Daners [Dan] Proposition 3.3 in the strongly elliptic case, but our proof is different.

Lemma 4.14 *There exists a Borel set $\Gamma_{a,\mu} \subset \Gamma$ such that*

$$\{\varphi \in L_2(\Gamma,\mu) : \varphi \textit{ is a trace of } 0\} = L_2(\Gamma \setminus \Gamma_{a,\mu}, \mu).$$

Proof Set $F = \{\varphi \in L_2(\Gamma,\mu) : (0,\varphi) \in W\}$. Then F is a closed subspace of $L_2(\Gamma,\mu)$.

First we show that $u\,\psi \in F$ for all $\psi \in F$ and $u \in D(a) \cap W^1_\infty(\mathbf{R}^d)$. Since $\psi \in F$ there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = 0$ in $D(\overline{a_N})$ and $\lim u_n|_\Gamma = \psi$ in $L_2(\Gamma,\mu)$. Then $u\,u_n \in D(a)$ for all $n \in \mathbf{N}$ and $\lim (u\,u_n)|_\Gamma = u\,\psi$ in $L_2(\Gamma,\mu)$. By the Leibniz rule one deduces that

$$(\Re\overline{a_N})(u\,u_n)^{1/2} \le \|u_n\|_2 \Big(\sum \Big\| \,|\frac{a_{ij} + \overline{a_{ji}}}{2}|\,|\partial_i u|\,|\partial_j u| \,\Big\|_\infty \Big)^{1/2} + \|u\|_\infty\,(\Re\overline{a_N})(u_n)^{1/2}$$

for all $n \in \mathbf{N}$ and $\lim u\,u_n = 0$ in $D(\overline{a_N})$. So $u\,\psi \in F$.

Secondly, let $P\colon L_2(\Gamma,\mu) \to F$ be the orthogonal projection. Let $\varphi \in L_2(\Gamma,\mu)$ and suppose that $\mu([\varphi \neq 0]) < \infty$. We shall prove that $P\varphi = 0$ a.e. on $[\varphi = 0]$. Let $A = [\varphi \neq 0]$. Since $\{u|_\Gamma : u \in H^1(\Omega) \cap C_c^\infty(\mathbf{R}^d)\}$ is dense in $L_2(\Gamma,\mu)$ there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C_c^\infty(\mathbf{R}^d)$ such that $\lim u_n|_\Gamma = \mathbb{1}_A$ in $L_2(\Gamma,\mu)$. Then also $\lim (0 \vee \operatorname{Re} u_n \wedge \mathbb{1})|_\Gamma = \mathbb{1}_A$ in $L_2(\Gamma,\mu)$, so we may assume that $u_n \in D(a) \cap W^1_\infty(\mathbf{R}^d)$ and $0 \leq u_n \leq \mathbb{1}$ for all $n \in \mathbf{N}$. Passing to a subsequence if necessary we may assume that $\lim u_n|_\Gamma = \mathbb{1}_A$ a.e. Therefore $\lim u_n \, P\varphi = \mathbb{1}_A \, P\varphi$ in $L_2(\Gamma,\mu)$. Since $u_n \, P\varphi \in F$ for all $n \in \mathbf{N}$ one deduces that $\mathbb{1}_A \, P\varphi \in F$. Then

$$\|\varphi - \mathbb{1}_A \, P\varphi\|_{L_2(\Gamma,\mu)} = \|\mathbb{1}_A(\varphi - P\varphi)\|_{L_2(\Gamma,\mu)} \leq \|\varphi - P\varphi\|_{L_2(\Gamma,\mu)}.$$

So $\mathbb{1}_A \, P\varphi = P\varphi$ and $P\varphi = 0$ a.e. on $A^{\rm c} = [\varphi = 0]$. Now the lemma easily follows from Zaanen's theorem [ArT] Proposition 1.7. □

Obvious the set $\Gamma_{a,\mu}$ in Lemma 4.14 is unique in the sense that $\mu(\Gamma_{a,\mu} \Delta \Gamma') = 0$ whenever $\Gamma' \subset \Gamma$ is another Borel set with this property. It is clear from the construction of $\Gamma_{a,\mu}$ and definition of $H^1_{a,\mu}(\Omega)$ that there exists a unique map $\operatorname{Tr}_{a,\mu}\colon H^1_{a,\mu}(\Omega) \to L_2(\Gamma_{a,\mu},\mu)$ in a natural way, which we call **trace**. Note that if $u \in H^1_{a,\mu}(\Omega)$ then $\operatorname{Tr}_{a,\mu} u$ is the unique $\varphi \in L_2(\Gamma_{a,\mu},\mu)$ such that φ is an a,μ-trace of u. In general, however, the map $\operatorname{Tr}_{a,\mu}$ is not continuous from $(H^1_{a,\mu}(\Omega), \|\cdot\|_{\overline{a_N}})$ into $L_2(\Gamma_{a,\mu},\mu)$. A counter example is in [Dan], Remark 3.5(f).

The map $u \mapsto (u, \operatorname{Tr}_{a,\mu} u)$ from $H^1_{a,\mu}(\Omega)$ into $D(\overline{a_N}) \oplus L_2(\Gamma_{a,\mu},\mu)$ is injective. Therefore one can define a norm on $H^1_{a,\mu}(\Omega)$ by

$$\|u\|^2_{H^1_{a,\mu}(\Omega)} = \|u\|^2_{D(\overline{a_N})} + \|\operatorname{Tr}_{a,\mu} u\|^2_{L_2(\Gamma_{a,\mu},\mu)}.$$

It is easy to verify that $H^1_{a,\mu}(\Omega)$ is a Hilbert space. Moreover, the map $\operatorname{Tr}_{a,\mu}\colon H^1_{a,\mu}(\Omega) \to L_2(\Gamma_{a,\mu},\mu)$ is a continuous linear operator with dense range.

It is now possible to reconsider the element $\varphi \in L_2(\Gamma,\mu)$ in Proposition 4.13.

Proposition 4.15 *Let $u, f \in L_2(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H^1_{a,\mu}(\Omega)$, $\mathcal{A}u = f$ weakly on Ω and*

$$\overline{a_N}(u,v) - \int_\Omega (\mathcal{A}u)\, \overline{v} = - \int_\Gamma \operatorname{Tr}_{a,\mu} u \, \overline{v} \, d\mu$$

for all $v \in D(a)$.

Proof Let $u \in D(A)$ and $\varphi \in L_2(\Gamma,\mu)$ be the corresponding unique element as in Proposition 4.13. If $\psi \in L_2(\Gamma \setminus \Gamma_{a,\mu},\mu) = F$ then there exists a sequence $v_1, v_2, \ldots \in D(a)$ such that $\lim v_n = 0$ in $D(\overline{a_N})$ and $\lim v_n|_\Gamma = \psi$ in $L_2(\Gamma,\mu)$. Substituting $v = v_n$ in (17) and taking the limit $n \to \infty$ one deduces that $\int_\Gamma \varphi \, \overline{\psi} \, d\mu = 0$. So $\varphi \in L_2(\Gamma_{a,\mu},\mu)$ and $\varphi = \operatorname{Tr}_{a,\mu} u$. □

We now consider the case where μ is the $(d-1)$-dimensional Hausdorff measure, which we denote by σ. In particular, we assume that $\sigma(K) < \infty$ for every compact $K \subset \Gamma$. Moreover, we write $\Gamma_a = \Gamma_{a,\sigma}$ and $\operatorname{Tr}_a = \operatorname{Tr}_{a,\sigma}$. The measure σ coincides with the usual surface measure if Ω is C^1. We continue to consider, however, the case where Ω is an arbitrary bounded open set. The only assumption that we make is that $\sigma(K) < \infty$ for every compact $K \subset \Gamma$. If Ω has a Lipschitz continuous boundary and the form a equals the Laplacian form l then $\Gamma_l = \Gamma_a = \Omega$ by the trace theorem (see [Neč] Théorème 2.4.2). By [ArW] Proposition 5.5 it follows that $\sigma(\Gamma_l) > 0$ if Ω is bounded, without any regularity condition on the boundary. (Note, however, that there exists an open connected subset $\Omega \subset \mathbf{R}^3$ such that $\sigma(\Gamma \setminus \Gamma_l) > 0$, see [ArW], Example 4.3). The embedding of $H^1_{l,\sigma}(\Omega)$ into $L_2(\Omega)$ is compact if Ω has finite measure, by [ArW] Corollary 5.2. This surprising phenomenon is a consequence of Maz'ya's inequality. It was Daners [Dan] who was the first to exploit this inequality to establish results for Robin boundary conditions on rough domains. Further results were given in [ArW] Section 5.

We conclude our remarks by considering $\mu = \beta\,\sigma$, where $\beta \in L_\infty(\Gamma, \mathbf{R})$ and $\beta \geq 0$ a.e. We define the weak normal derivative with respect to the matrix (a_{ij}). Let $\varphi \in L_2(\Gamma, \mu)$, $u \in D(\overline{a_N})$ and suppose that $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω. Then we say that φ is the (a_{ij})**-normal derivative of** u if

$$\overline{a_N}(u,v) - \int_\Omega (\mathcal{A}u)\,\overline{v} = \int_\Gamma \varphi\,\overline{v}\,d\sigma$$

for all $v \in D(a)$. If Ω is of class C^1, μ is the $(d-1)$-dimensional Hausdorff measure and $u \in C^1(\overline{\Omega})$ then our weak definition coincides with the classical definition by Green's theorem. We reformulate Proposition 4.13.

Proposition 4.16 *Let $u, f \in L_2(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H^1_{a,\beta\,\sigma}(\Omega)$, $\mathcal{A}u = f$ weakly on Ω and $-\beta \operatorname{Tr}_{a,\beta\sigma} u$ is the (a_{ij})-normal derivative of u.*

Note that if the matrix (a_{ij}) of coefficients if strongly elliptic and if $u \in D(A)$ and $Au = f$ then $u \in H^1(\Omega)$, $\mathcal{A}u = f$ weakly on Ω, u has a trace $\operatorname{Tr} u$ and $\nu \cdot a\nabla u = -\beta \operatorname{Tr} u$ weakly. Thus one recovers the classical statement.

4.4 The Dirichlet-to-Neumann operator

Let Ω be a bounded open subset of $\mathbf{R}^d$ with Lipschitz boundary Γ, provided with the $(d-1)$-dimensional Hausdorff measure. Let $\operatorname{Tr}: H^1(\Omega) \to L_2(\Gamma)$ be the trace map. We denote by Δ_D the Dirichlet Laplacian on Ω. If $\varphi \in L_2(\Gamma)$, $u \in H^1(\Omega)$ and $\Delta u \in L_2(\Omega)$ weakly on Ω then we say that $\frac{\partial u}{\partial \nu} = \varphi$ weakly if φ is the (a_{ij})-normal derivative of u, where $a_{ij} = \delta_{ij}$.

Let $\lambda \in \mathbf{R}$ and suppose that $\lambda \notin \sigma(-\Delta_D)$. The **Dirichlet-to-Neumann operator** D_λ on $L_2(\Gamma)$ is defined as follows. Let $\varphi, \psi \in L_2(\Gamma)$. Then we define $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$ if there exists a $u \in H^1(\Omega)$ such that $-\Delta u = \lambda u$ weakly on Ω, $\operatorname{Tr} u = \varphi$ and $\frac{\partial u}{\partial \nu} = \psi$ weakly. We next show that the Dirichlet-to-Neumann operator is an example of the m-sectorial operators obtained in Theorem 2.2.

Define the sesquilinear form $a: H^1(\Omega) \times H^1(\Omega) \to \mathbf{C}$ by

$$a(u,v) = \int_\Omega \nabla u\, \overline{\nabla v} - \lambda \int_\Omega u\,\overline{v}.$$

Moreover, define $j: H^1(\Omega) \to L_2(\Gamma)$ by $j(u) = \operatorname{Tr} u$. Clearly the form a is continuous, the map j is bounded and $j(H^1(\Omega))$ is dense in $L_2(\Gamma)$. Using the definitions one deduces that

$$V(a) = \{u \in H^1(\Omega) : -\Delta u = \lambda u \text{ weakly on } \Omega\}.$$

It follows from Step 1 in the proof of Proposition 3.3 in [ArM] that there exist $\omega \in \mathbf{R}$ and $\mu > 0$ such that

$$\operatorname{Re} a(u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2$$

for all $u \in V(a)$. So the conditions of Theorem 2.2 are satisfied.

Note that if λ_1 is the lowest eigenvalue of the operator $-\Delta_D$ on Ω and $u \in H_0^1(\Omega)$ is an eigenfunction with eigenvalue λ_1, then (2) is not valid if $\lambda > \lambda_1$. Therefore Theorem 2.1 is not applicable and this example is the reason why we used the space $V(a)$ in Theorem 2.2.

Let A be the operator associated with (a, j). We next show that $A = D_\lambda$. Let $\varphi, \psi \in L_2(\Gamma)$. Suppose $\varphi \in D(A)$ and $A\varphi = \psi$. Then there is a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $a(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in H^1(\Omega)$. For all $v \in H_0^1(\Omega)$ one has

$$\int_\Omega \nabla u \, \overline{\nabla v} - \lambda \int_\Omega u \, \overline{v} = a(u, v) = 0,$$

so $-\Delta u = \lambda u$ weakly on Ω. Then

$$\int_\Omega \nabla u \, \overline{\nabla v} + \int_\Omega (\Delta u) \, \overline{v} = a(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. So $\frac{\partial u}{\partial \nu} = \psi$ weakly. Therefore $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$. Conversely, suppose $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$. By definition there exists a $u \in H^1(\Omega)$ such that $-\Delta u = \lambda u$ weakly on Ω, $\operatorname{Tr} u = \varphi$ and $\frac{\partial u}{\partial \nu} = \psi$ weakly. Then

$$a(u, v) = \int_\Omega \nabla u \, \overline{\nabla v} - \lambda \int_\Omega u \, \overline{v} = \int_\Omega \nabla u \, \overline{\nabla v} + \int_\Omega (\Delta u) \, \overline{v} = \int_\Gamma \frac{\partial u}{\partial \nu} \, \overline{\operatorname{Tr} v} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. So $\varphi = j(u) \in D(A)$ and $\psi = Aj(u) = A\varphi$. Thus $D_\lambda = A$ is the operator associated with (a, j).

If S is the semigroup generated by $-D_\lambda$ then it follows as in the proof of Corollary 4.9 that S is real and positive. Moreover, if $\lambda \leq 0$ then S extends consistently to a continuous contraction semigroup on $L_p(\Omega)$ for all $p \in [1, \infty]$.

4.5 Wentzell boundary conditions

Let again Ω be an open subset of $\mathbf{R}^d$ with arbitrary boundary Γ and let σ be the $(d-1)$-dimensional Hausdorff measure on Γ. We assume that $\sigma(K) < \infty$ for every compact $K \subset \Gamma$. All L_p spaces on Γ are with respect to the measure σ, expect if written different explicitly. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum_{i,j=1}^d a_{ij}(x) \, \xi_i \, \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Define the form b by

$$D(b) = \{u \in H^1(\Omega) \cap C(\overline{\Omega}) : \int_\Gamma |u|^2 \, d\sigma < \infty\}$$

and

$$b(u, v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u) \, a_{ij} \, \overline{\partial_j v} + \int_\Gamma u \, \overline{v} \, d\sigma.$$

As in Subsection 4.3 we define the Neumann form b_N by $D(b_N) = H^1(\Omega)$ and

$$b_N(u, v) = \sum_{i,j=1}^d \int_\Omega (\partial_i u) \, a_{ij} \, \overline{\partial_j v}.$$

Throughout this subsection we assume that the form b_N is closable. Set $\widetilde{\Gamma} = \Gamma_{b,\sigma}$ and $\mathrm{Tr} = \mathrm{Tr}_{b,\sigma}$. Moreover, we assume that the map $\mathrm{Tr}\colon (H^1_{b,\sigma}(\Omega), \|\cdot\|_{\overline{b_N}}) \to L_2(\widetilde{\Gamma})$ is continuous.

Fix $\alpha \in L_\infty(\widetilde{\Gamma})$ and $B \in \mathcal{L}(L_2(\widetilde{\Gamma}))$. Throughout this subsection we assume that there exists an $\omega > 0$ such that

$$\omega \, \|B\varphi\|^2_{L_2(\widetilde{\Gamma})} + \int_{\widetilde{\Gamma}} \mathrm{Re}\, \alpha \, |\varphi|^2 \geq 0 \tag{18}$$

for all $\varphi \in L_2(\widetilde{\Gamma})$. In particular, if $\beta \in L_\infty(\Gamma)$ and B is the multiplication operator with β then we assume that

$$\omega \, |\beta|^2 + \mathrm{Re}\, \alpha \geq 0$$

for some $\omega > 0$.

Define the form a by

$$D(a) = H^1_{b,\sigma}(\Omega)$$

and

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) \, a_{ij} \, \overline{\partial_j v} + \int_{\widetilde{\Gamma}} \mathrm{Tr}\, u \, \overline{\mathrm{Tr}\, v} \, \alpha \, d\sigma.$$

Let H be the closure of the space $\{(u, B(\mathrm{Tr}\, u)) : u \in H^1_{b,\sigma}(\Omega)\}$ in the space $L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ with induced norm. Define the injective map $j\colon H^1_{b,\sigma}(\Omega) \to H$ by

$$j(u) = (u, B(\mathrm{Tr}\, u)).$$

If B has dense range, then $H = L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ since the space $\{(u, \mathrm{Tr}\, u) : u \in H^1_{b,\sigma}(\Omega)\}$ is dense in $L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ by Step a) in the proof of Theorem 2.3 in [AMPR]. Then the claim follows by the range condition on B. Note that the condition (18) together with the assumed continuity of $\mathrm{Tr}\colon (H^1_{b,\sigma}(\Omega), \|\cdot\|_{\overline{b_N}}) \to L_2(\widetilde{\Gamma})$ imply that a is j-sectorial. Let A be the operator associated with (a,j).

Proposition 4.17 *Let $x, y \in H$. Then $x \in D(A)$ and $Ax = y$ if and only if there exist $u \in H^1_{b,\sigma}(\Omega)$ and $\psi \in L_2(\widetilde{\Gamma})$ such that $x = (u, B(\mathrm{Tr}\, u))$, $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω, $(B^*\psi - \alpha \, \mathrm{Tr}\, u)$ is the (a_{ij})-normal derivative of u and $y = (\mathcal{A}u, \psi)$.*

Proof '$\Rightarrow$'. There exists a $u \in H^1_{b,\sigma}(\Omega)$ such that $x = j(u)$. Write $y = (f, \psi) \in H$. Then

$$\overline{b_N}(u,v) + \int_{\widetilde{\Gamma}} \mathrm{Tr}\, u \, \overline{\mathrm{Tr}\, v} \, \alpha \, d\sigma = (y, j(v))_H = \int_\Omega f \, \overline{v} + \int_{\widetilde{\Gamma}} \psi \, \overline{B(\mathrm{Tr}\, v)} \, d\sigma$$

for all $v \in H^1_{b,\sigma}(\Omega)$. Taking only $v \in C_c^\infty(\Omega)$ one deduces that $\mathcal{A}u = f$ weakly on Ω. In particular, $y = (f, \psi) = (\mathcal{A}u, \psi)$. Moreover,

$$\overline{b_N}(u,v) - \int_\Omega (\mathcal{A}u) \, \overline{v} = \int_{\widetilde{\Gamma}} (B^*\psi - \alpha \, \mathrm{Tr}\, u) \, \overline{\mathrm{Tr}\, v} \, d\sigma$$

for all $v \in H^1_{b,\sigma}(\Omega)$, which implies that $(B^*\psi - \alpha \, \mathrm{Tr}\, u)$ is the (a_{ij})-normal derivative of u.

'$\Leftarrow$'. Let $u \in H^1_{b,\sigma}(\Omega)$ and $\psi \in L_2(\widetilde{\Gamma})$ be such that $x = (u, B(\mathrm{Tr}\, u))$, $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω, $(B^*\psi - \alpha \, \mathrm{Tr}\, u)$ is the (a_{ij})-normal derivative of u and $y = (\mathcal{A}u, \psi)$. Then $x = j(u)$. Since $(B^*\psi - \alpha \, \mathrm{Tr}\, u)$ is the (a_{ij})-normal derivative of u one deduces that

$$\overline{b_N}(u,v) - \int_{\widetilde{\Gamma}} (\mathcal{A}u) \, \overline{v} = \int_{\widetilde{\Gamma}} (B^*\psi - \alpha \, \mathrm{Tr}\, u) \, \overline{\mathrm{Tr}\, v} \, d\sigma$$

for all $v \in H^1_{b,\sigma}(\Omega)$. So

$$a(u,v) = \overline{b_N}(u,v) + \int_{\widetilde{\Gamma}} \operatorname{Tr} u \, \overline{\operatorname{Tr} v} \, \alpha \, d\sigma$$

$$= \int_\Omega (\mathcal{A}u)\, \overline{v} + \int_{\widetilde{\Gamma}} \psi \, \overline{B(\operatorname{Tr} v)} \, d\sigma = (y, j(v))_H$$

for all $v \in H^1_{b,\sigma}(\Omega)$. Therefore $x \in D(A)$ and $Ax = y$. □

Suppose that B has dense range. Then H is isomorphic with $L_2(\Omega \sqcup \widetilde{\Gamma})$ in a natural way. We use this isomorphism to identify H with $L_2(\Omega \sqcup \widetilde{\Gamma})$. It is easy to verify as in the proof of Corollary 3.16(a) that S leaves $L_2(\Omega, \mathbf{R}) \oplus L_2(\widetilde{\Gamma}, \mathbf{R})$ invariant if the form b_N is real, α is real valued and B maps $L_2(\widetilde{\Gamma}; \mathbf{R})$ into itself. We next characterize positivity of S.

Proposition 4.18 *Suppose the form b_N is real, α is real valued and B maps $L_2(\widetilde{\Gamma}; \mathbf{R})$ densely into itself.*

(a) *The map B is a lattice homomorphism if and only if the semigroup S is positive.*

(b) *If $\sigma(\widetilde{\Gamma}) < \infty$, the map B is a lattice homomorphism, $\alpha \geq 0$ and there exists a $c \geq 1$ such that $\frac{1}{c} \mathbb{1} \leq B\mathbb{1} \leq c\,\mathbb{1}$, then S extends continuously to a bounded semigroup on $L_\infty(\Omega \sqcup \widetilde{\Gamma})$.*

Proof '(a)'. Let $C = \{(u, \varphi) \in H : u \geq 0 \text{ and } \varphi \geq 0\}$. Then C is closed and convex in H. Define $P \colon H \to C$ by $P(u,\varphi) = ((\operatorname{Re} u)^+, (\operatorname{Re} \varphi)^+)$. Then P is the orthogonal projection onto C.

'⇒'. Let $u \in H^1_{b,\sigma}(\Omega)$. Then $(\operatorname{Re} u)^+ \in H^1_{b,\sigma}(\Omega)$ and

$$j((\operatorname{Re} u)^+) = ((\operatorname{Re} u)^+, B(\operatorname{Tr}((\operatorname{Re} u)^+))) = ((\operatorname{Re} u)^+, (\operatorname{Re} B(\operatorname{Tr} u))^+) = Pj(u)$$

since B is a lattice homomorphism. Moreover,

$$\operatorname{Re} a((\operatorname{Re} u)^+, u - (\operatorname{Re} u)^+) = a((\operatorname{Re} u)^+, -(\operatorname{Re} u)^-) = 0.$$

So C is invariant under S by Proposition 2.9.

'⇐'. If S is positive then C is invariant under S. Let $u \in H^1_{b,\sigma}(\Omega)$. It follows from Proposition 2.9 that there exists a $w \in H^1_{b,\sigma}(\Omega)$ such that $Pj(u) = j(w)$. Then $((\operatorname{Re} u)^+, (\operatorname{Re} B(\operatorname{Tr} u))^+) = Pj(u) = j(w) = (w, B(\operatorname{Tr} w))$. Therefore $w = (\operatorname{Re} u)^+$ and

$$(\operatorname{Re} B(\operatorname{Tr} u))^+ = B(\operatorname{Tr} w) = B(\operatorname{Tr}((\operatorname{Re} u)^+)) = B((\operatorname{Re} \operatorname{Tr} u)^+).$$

This is for all $u \in H^1_{b,\sigma}(\Omega)$. Since $\operatorname{Tr} H^1_{b,\sigma}(\Omega)$ is dense in $L_2(\widetilde{\Gamma})$ one deduces that $(B\varphi)^+ = B(\varphi^+)$ for all $\varphi \in L_2(\widetilde{\Gamma}, \mathbf{R})$. So B is a lattice homomorphism.

'(b)'. Let $C = \{(u, \varphi) \in H : u \leq \mathbb{1} \text{ and } \varphi \leq B\mathbb{1}\}$. Then C is closed and convex. Define $P \colon H \to C$ by $P(u, \varphi) = ((\operatorname{Re} u) \wedge \mathbb{1}, (\operatorname{Re} \varphi) \wedge B\mathbb{1})$. Then P is the projection of H onto C. Let $u \in H^1_{b,\sigma}(\Omega)$. Define $w = (\operatorname{Re} u) \wedge \mathbb{1}$. Then $w \in H^1_{b,\sigma}(\Omega)$ and $Pj(u) = ((\operatorname{Re} u) \wedge \mathbb{1}, (\operatorname{Re} B(\operatorname{Tr} u)) \wedge B\mathbb{1}) = ((\operatorname{Re} u) \wedge \mathbb{1}, B(\operatorname{Tr}((\operatorname{Re} u) \wedge \mathbb{1}))) = j(w)$. Moreover,

$$\operatorname{Re} a(w, u - w) = \operatorname{Re} a((\operatorname{Re} u) \wedge \mathbb{1}, i \operatorname{Im} u + (\operatorname{Re} u - \mathbb{1})^+) = a((\operatorname{Re} u) \wedge \mathbb{1}, (\operatorname{Re} u - \mathbb{1})^+)$$

$$= \int_{\widetilde{\Gamma}} \alpha \operatorname{Tr}((\operatorname{Re} u) \wedge \mathbb{1}) \operatorname{Tr}((\operatorname{Re} u - \mathbb{1})^+) = \int_{\widetilde{\Gamma}} \alpha \operatorname{Tr}((\operatorname{Re} u - \mathbb{1})^+) \geq 0$$

So by Proposition 2.9 the set C is invariant under S.

Finally, let $(u, \varphi) \in H$ and suppose that $u \le \mathbb{1}$ and $\varphi \le \mathbb{1}$. Then $\frac{1}{c}\varphi \le B\mathbb{1}$ and $\frac{1}{c}(u, \varphi) \in C$. Let $t > 0$ and write $(v, \psi) = S_t(u, \varphi)$. Then $\frac{1}{c}(v, \psi) \in C$. Hence $v \le c\,\mathbb{1}$ and $\psi \le c\,B\mathbb{1} \le c^2\,\mathbb{1}$. So S extends to a continuous semigroup on L_∞ and $\|S_t\|_{\infty\to\infty} \le c^2$ for all $t > 0$. □

Using the operator A one can define another semigroup generator which looks different. If $u \in D(\overline{b_N})$ then we say that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ **weakly on** Ω if there exists an $f \in H^1_{b,\sigma}(\Omega)$ such that $\mathcal{A}u = f$ weakly on Ω. In the next proposition the part A_1 of A in the Sobolev space is a realization of the elliptic operator with Wentzell boundary conditions. This is another approach than the one used in [FGGR].

Proposition 4.19 *Define the operator A_1 on $H^1_{b,\sigma}(\Omega)$ by taking as domain $D(A_1)$ the set of all $u \in H^1_{b,\sigma}(\Omega)$ such that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ weakly on Ω and $(B^*B(\operatorname{Tr}\mathcal{A}u) - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u; and letting $A_1 u = \mathcal{A}u$ for all $u \in D(A_1)$. Then $-A_1$ generates a holomorphic semigroup on $H^1_{b,\sigma}(\Omega)$.*

Proof Let a_c be the classical form associated with (a, j) (see Theorem 2.5). Then A is associated with the closed sectorial form a_c. Define the operator A_0 in H by $D(A_0) = \{w \in D(A) : Aw \in D(a_c)\}$ and $A_0 w = Aw$ for all $w \in D(A_0)$. Then $-A_0$ generates a holomorphic semigroup in the Hilbert space $(D(a_c), \|\cdot\|_{a_c})$. The map $j\colon H^1_{b,\sigma}(\Omega) \to D(a_c)$ is a isomorphism of normed spaces. Hence the operator $-j^{-1}A_0 j$ generates a holomorphic semigroup on $H^1_{b,\sigma}(\Omega)$. Therefore it suffices to show that $A_1 = j^{-1}A_0 j$.

Let $u \in D(j^{-1}A_0 j)$. Then $j(u) \in D(A)$, $Aj(u) \in j(H^1_{b,\sigma}(\Omega))$ and $A_0 j(u) = Aj(u)$. It follows from Proposition 4.17 that $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω and there exists a $\psi \in L_2(\widetilde{\Gamma})$ such that $(B^*\psi - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u and $Aj(u) = (\mathcal{A}u, \psi)$. Since $Aj(u) \in j(H^1_{b,\sigma}(\Omega))$ one deduces that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ and $j(\mathcal{A}u) = (\mathcal{A}u, \psi) = Aj(u)$. In particular, $\psi = B(\operatorname{Tr}\mathcal{A}u)$. Therefore $(B^*B(\operatorname{Tr}\mathcal{A}u) - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u and $u \in D(A_1)$. Then $A_1 u = \mathcal{A}u = j^{-1}A_0 j(u)$. Conversely, suppose that $u \in D(A_1)$. Then $j(u) \in D(a_c)$ and it follows from Proposition 4.17 that $j(u) \in D(A)$ with $Aj(u) = (\mathcal{A}u, B\operatorname{Tr}\mathcal{A}u) = j(\mathcal{A}u)$. So $j(u) \in D(A_0)$ and $u \in D(j^{-1}A_0 j)$. □

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